A Case Analysis of one Stage Six Mathematics Student in a Tutoring Intervention

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Declaration

I declare that this research thesis is my own work. It has not been submitted in any form for any purpose at any institute of education. I confirm that any information derived from published or unpublished work was referenced in the text and in the reference section.

Xia Miao Zeng
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List of Abbreviations

Fig: Figure
IPP: Information processing process (IPP)
Ine: Inequation
Ins: Instruction
Res: Response of the student
Sp: Small problem solving carried out by the student.
Equ: Equation.
Ref: student’s reflection
SOI: Select-Organize-Integrate model
Author’s Publication in Refereed Journal

Abstract

I have been tutoring mathematics and physics to students from high schools. I found that many of the students didn’t understand the actual meaning of the mathematical content and were only concerned with the procedures of calculations.

The purpose of this research is therefore to develop an effective practical scheme for the teaching of mathematics that integrates the researcher's tutoring experience with features attained from the existing theories of learning.

For many years, researchers have argued about the advantages and disadvantage of the two apparently opposite pedagogical approaches, namely, that of minimally guided instruction and direct instructional guidance. Few discuss the possibility of incorporating the advantages from both approaches.

This research study, attempts to take a careful step toward this challenging goal by working out a practical scheme that uses features attained from the two approaches mentioned above. During the review of the literature, a practical scheme that describes how a new concept or topic should be taught has slowly emerged. For convenience, the practical scheme will be called Simple Steps.

The study will be carried out through the process of tutoring. Case studies will be used in this study making use of field notes and digital recorded sessions and interviews.

The process consists of three stages. The intervention involves a topic from the stage 6 mathematics syllabus, which students usually find difficult or confusing, namely the topic of logarithmic function:

\[ y = \log_a x \]

Stage 1 is the interview before the tutoring process. The purpose of the interview is to obtain information of the student’s understanding of mathematics. The researcher will ask the student mathematics questions related to the basic procedural understanding of the topic.

Stage 2 is the tutoring process. The researcher will use the principles of Simple Steps to improve the student’s conceptual knowledge and procedural knowledge of logarithms. Field notes will be kept and sessions recorded. It seeks to capture different aspects of the student’s engagement and her/his understanding of mathematics.

Stage 3 is the test after the tutoring followed by a feedback Interview, which will provide information about the effectiveness of Simple Steps in helping the student to understand mathematics concept.

Finally, the student will be given a questionnaire to fill in, which allows the student to give a self-assessment about her/his engagement during learning.
There are a number of different pedagogical approaches to the teaching of mathematics. However, considerable dispute exists as to which one is the best. Some of the aspects of these pedagogical ideas will be investigated during this research. It will be hoped that the findings of this study will have the potential to inform the teaching of mathematics in a tutoring context.
Chapter 1
Introduction

1.1 Research Background

1.1.1 The Quality of Mathematics Education
The researcher has been privately tutoring mathematics and physics to students at high schools in New South Wales (NSW) more than ten years. He found that many of the students did not possess many important mathematical understandings. They were only concerned with the procedures of calculations. This phenomenon may well be related to the fact that there are “a growing number of students who drop mathematics before or during Years 11 and 12” (Palmer 2006, p. 6). Palmer also comments that “there are many students in NSW for whom General Mathematics is much too difficult” (p. 6). Note that General Mathematics is the lowest level course on offer in NSW, and that stage 6 refers to the final 2 years of High School of NSW. The final examination is called the Higher School Certificate (HSC).

The standard of Mathematics education could affect the situation in other fields. According to Mack (2006), “Australia is facing a severe shortage of engineers and scientists, to say nothing of mathematicians and statisticians” (p. 1). Mack (2006) suggests that Stage 6 Mathematics should be designed to stimulate and encourage students to develop their mathematical ability for further use post the HSC. Mack (2006) emphasises the importance of deep conceptual learning in teaching and learning because it enables the internalising of concepts to occur.

Similarly, Stacey (2006) points out that without a strong understanding, students simply treat Mathematics as a meaningless set of routines and never realise the value of Mathematics learning. The consequence of not understanding may result in a loss of confidence and enjoyment, which could turn intellectual pleasure into rote learning (Stacey, 2006). This effect could be dramatic, as Michael Turtle, the Youth Affairs reporter of the ABC (2008) reported: “international testing of school students has found Australia is failing to keep pace with much of the developed world in science and maths.” He also said that: “in maths and science, Australia is now below many Asian countries like Japan and Singapore, as well as England, the US and Russia” and that “the Australia Science Teachers Association says more support is needed if Australian students are to keep up with their international counterparts.” (Turtle, 2008, p. 1). Geoff Masters, from the Australian Council for Educational Research
said that “the quality of teaching and the curriculum are the two biggest factors” for the urgent reform of the educational system (quoted from Turtle, 2008, p. 1). When discussing the changing structure of the Stage 6 Mathematics course, Mack (2006) comments that the teaching staff and available resources at the school level are the key factors.

1.1.2 The Purpose of the Research

Based on well-established pedagogical theories in the field of Mathematics Education, the researcher sought to devise an effective practical scheme in the context of Mathematics tutoring. The focus of this practical scheme is that the student obtains a deep conceptual understanding of the Mathematics. Links will be made with existing research, the researcher’s teaching experience, and the interaction between different pedagogical approaches and the learning process.

There are a number of different pedagogical approaches to the teaching of Mathematics and considerable dispute exists as to which one is the best. However, as Ormrod (2008) comments: “no single theory explains everything that researchers have discovered about learning” (p. 7). With this in mind, the research was focussed on the important aspects of different pedagogical approaches that help promote understanding, particularly in the context of tutoring.

There is a considerable quantity of research in regard to teaching in schools but little research in regard to interactions between the teacher and the student in a tutoring environment. Since most literature on pedagogy does not distinguish between learning in a classroom setting and learning in a one-on-one tutoring environment, tutoring will be treated as a process of teaching and learning (Rosé, Vanlehn, The Natural Language Tutoring Group Learning Research & Development Center, University of Pittsburgh, 2003).

1.1.3 Important Aspects of the Literature Related to the Study

The purpose of this section is to discuss some important aspects of pedagogy related to different approaches in the teaching of mathematics. These aspects will be considered by the researcher when devising a practical scheme for the teaching of mathematics in a one-on-one tutoring environment.
Considerable dispute exists as to which pedagogical approach is best for the teaching of mathematics. As Kirschner, Sweller and Clark (2006) point out, “Disputes about the impact of instructional guidance during teaching have been ongoing for at least the past half-century (Ausubel, 1964; Craig, 1956; Mayer, 2004; Shulman & Keisher, 1966)” (p. 75).

According to Huiit (2004), there are five major categories of learning theory namely, behavioural theories, constructivism theories, humanistic theories, social cognitive theories, and cognitive learning theories. Based on these theories, a wide variety of pedagogical approaches have been developed for various educational purposes. The most controversial approaches have been direct instruction or explicit instruction (Huiit, 2003) and discovery learning. Kirschner et al. (2006), classifies direct instruction as direct instructional guidance and discovery learning as minimally guided instruction.

Since the debate has involved many important ideas related to teaching and learning particularly in the field of Mathematics, this study began by focussing on ideas from discovery learning and direct Instruction within the theoretical frameworks of behaviourism, constructivism, and cognitive science.

1.2 Behaviourism and Direct Instruction

Direct instruction is a highly structured instructional approach. Some of its ideas are rooted in behaviourism (Huiit 2003; Burton, Moore, & Magliaro, 2004; Magliaro, Lockee & Burton, 2005).

1.2.1 Behaviourism

Behaviourism views learning as a process in which people are controlled or affected by their environment (Mcleod, 2007). It focuses on how environmental factors (called stimuli) affect learning through observable behaviour called responses (Mcleod, 2007; Ormrod, 2008). Behaviourism rejects introspection as a method of evaluation and believes that only objectively observable behaviour is scientifically measurable and quantifiable (Mcleod, 2007).
As Skinner (1968) stated “To think often means simply to behave” (p. 119), and “to think verbally or nonverbally, mathematically….means to behave with respect to stimuli” (Skinner, 1968, p. 119).

One of the learning theories developed by Skinner is operant conditioning. The basic principle of operant conditioning is that “there is a functional and interconnected relationship between the stimuli that preceded a response” (Burton, Moore, & Magliaro, 2004, p.10). This theory is closely related to other theories such as reinforcement scripted lessons that support the principle of direct instruction (Bocchino, 2000).

1.2.2 Direct Instruction
Direct instruction was initially developed by Engelmann and Bereiter in the 1960s (Magliaro et al., 2005; Bereiter & Engelmann, 1966). It uses a precise, step-by-step training strategy to teach and help students improve their understandings (Gersten & Carnine, 1986). It emphasises the role of discrimination to enable the precise acquisition of concepts and knowledge (Engelmann & Carnine, 1982; Kozloff & LaNunziata, 1999).

Also known as explicit teaching (Rosenshine, 1986), direct instruction is usually considered as a useful instructional method particularly in the field of mathematics and science teaching where the difficult concepts are involved. Some of its practical principles will be used as a reference to establish the practical scheme in this study, namely: Step by Step, Discrimination, and Guided practice as will be briefly discussed later (Rosenshine, 1986; Gersten & Carnine, 1986).

Advantages and Disadvantages of Direct Instruction
Direct instruction is usually considered an effective tool in that it allows the teacher to present material in a more efficient way than would be possible if another method were used (Saskatchewan, 2001).

But direct instruction is often criticised for not encouraging the student to gain a deep, comprehensive understanding of the material (Tweed, 2004), and to think reflectively (Kuhn, 2007). Another concern about direct instruction is its “teacher-centred” style. This highly structured approach could make the student passive in his learning (Kenny, 1980; Tweed, 2004).
In this study, the researcher does not want the tutor to be limited to just showing the student how to solve a difficult problem. He wants the student to be engaged in discovering new knowledge or concepts rather than just repairing old ones (Collins, 1988 as cited by McArthur et al., 1990). Thus the researcher looked to the field of constructivism and cognitive science where the emphasis is on what is happening in the human mind during learning.

1.3. Constructivism and Discovery learning

1.3.1 Constructivism and Learning
The main idea of constructivist learning is that people construct their understanding of the world within the framework of their own knowledge (Ishii, 2003). According to this perspective, knowledge cannot be simply passed on to the learner, but should be actively constructed by the learner. Learners build new knowledge and interpret information based on their past experiences and existing knowledge (Lesh & Doerr, 2003).

This study will mainly focus on the constructivist theories relating to how knowledge is constructed from the view of the three influential scholars namely, Jean Piaget, Lev Vygotsky, and Jerome Bruner (Learning Point Associates, 2011).

Their theories have provided some philosophical support for the establishment of the practical teaching scheme in this study. A brief discussion of these theories will be given in the next chapter.

One of the important pedagogical approaches based on the view of constructivist learning theory is discovery learning.

1.3.2 Discovery Learning
Discovery learning is a method of instruction based on constructivist learning theory. It encourages the student to use his/her existing knowledge to discover relationships and new materials to be learned (Learning Theories Knowledgebase, 2009).
It emphasizes the role of the learner in the process of problem solving, during which a learner is encouraged to discover the procedures or develop hypotheses. It is said that the active participation in the process of problem solving may allow the student to produce critical thinking and obtain better transfer and conceptual knowledge (Rittle-Johnson, 2006; De Jong & Van Joolingen, 1998; Darwen, 2001).

However, almost for the same reason, discovery learning received criticisms for its possibilities of inducing misconceptions during learning (Learning Theories Knowledgebase, 2009; Martin, 2000). Mayer (2004) argued that a learner could even fail to come into contact with the correct information during problem solving in the pure discovery learning environment.

From the point of view of mathematics tutoring, where clearing misconceptions and problem solving are equally important, the following questions arose: is it possible to use the method of discovery learning during the process of tutoring to help the student to tackle learning difficulties? Further discussion in the next chapter had lead to one of the research questions:

How can the problem-solving approach, proposed by the supporters of discovery learning, help the learner to understand a mathematical concept or topic while keeping the possibility of a misconception to a minimum?

As Rose et al. (2009) pointed out, a tutorial system should “provide a more effective or efficient means of instruction than an otherwise equivalent purely text based approach” (p. 4). The framework of discovery learning seems to be inadequate for building genuine knowledge in a short time frame. In order to obtain a deep theoretical understanding of a successful instructional approach in a tutoring environment (Shanaham, 1998), the researcher decided to investigate the cognitive learning process.

1.4 Theories of Cognitive Learning Relating to the Study

The cognitive theory of learning studies how knowledge is processed and stored (Huitt 2006; SIL International 1999). Some constructivists also contributed their ideas to the development of the theory of cognitive learning such as Piaget, Brumer, and Vygotsky (Huitt, 2006; Shuell 1986).
1.4.1 Basic Concepts Related to the Cognitive Theory of Learning

In searching theories of learning, the researcher has chosen the theory of cognitive learning as the basic framework to study and analyse the process of tutoring. The following sections will briefly discuss some basic concepts relating to this study. A more detailed discussion will be given in next chapter.

The Select-Organize-Integrate (SOI) Model and the Related Concepts

One of the important concepts in cognitive learning is information processing process. It explains how knowledge is processed and stored in memory. Gange described it as the mental process going on in any form of memory task (as cited in SIL International, 1999). There are different models of information processing approach to the process of learning (Huiit, 2003).

Instead of directly using these models, the current study will use the Select-Organize-Integrate (SOI) model introduced by Mayer (1996), as a basis to study the process of tutoring. The SOI model describes three processes in knowledge construction during learning in the human memory system, namely: (a) the process of selecting with prior knowledge incoming information; (b) the process of organising the selected materials; and (c) the process of integrating new information with the prior knowledge.

Network models of Knowledge

Another important idea is that knowledge is stored in memory in the form of a knowledge network. This view of the structure of knowledge provides a useful way to describe the state of understanding in terms of connection and representation. Briefly speaking, when the connection between information is weak, the understanding is fragile and when it is strengthened, understanding it increases (Hiebert et al., 1992).

Another useful term to describe the construction of knowledge is the concept of schema. According to Rumelhart and Norman (1976), schema can be thought of as memory unit. Schema is active because it is the record of our experiences or the representation of our knowledge of particular situations. One of the important issues related to learning is that schema can be incorrectly constructed and thus related to the term misconceptions (Thompson & Logue, 2006). Research has shown, however, that a deep-seated schema is difficult to change (SIL International, 1999), since they can sequentially built upon (Thompson et al., 2006).
Researchers also show that student misconceptions are related to topics which are difficult to learn (Miller, Streveler, Yang, Santiago & Roma, 2009; Chi, 2005). Thus, seeking methods that help the student to grasp difficult materials, identify misconceptions and repair them as they occur, becomes the primary issue of this study.

1.5 Discussion of the Research Questions

As stated earlier, the purpose of this research is to construct a practical one-to-one tutoring scheme for the teaching of Mathematics that integrates important features obtained from existing theories of learning. The goal of this tutoring scheme is that the student gains a deep understanding of the mathematics. As the result of searching the literature, such a scheme slowly emerged. This scheme for teaching a new mathematical concept or topic will be called Simple Steps.

In constructing Simple Steps, the cognitive theory of learning was used as the theoretical framework. At the same time, advantages of both direct instruction and discovery learning were incorporated. The five steps of Simple Steps are outlined below. After investigating a student’s response to this tutoring approach, it will be possible to evaluate these steps and describe them in detail.

Step 1 Review the concepts and notation that is related to the new topic.
Step 2 Introduce the new topic.
Step 3 Make generalisations based on concrete examples to establish formulae and conceptual understandings.
Step 4 Use the formulae and conceptual understandings to gain procedural knowledge.
Step 5 Solve basic problems and clear any misunderstandings.

In carrying out the tutoring process, a particular topic of the Stage 6 Mathematics curriculum which the participant finds confusing and difficult will be chosen as the focusing topic. The topic was the logarithmic function:

\[ y = \log_{a}x \]

The investigation of the tutoring process will be guided by the following research questions:
1) How can the principle of direct instruction, in the context of logarithms, be applied while keeping the learner actively engaged?

2) How can the problem-solving approach, in the context of logarithms, proposed by supporters of discovery learning, help the learner to understand a mathematical concept or topic while keeping the possibility of a misconception to a minimum?

1.6 Methodology

1.6.1 Significance of the Research

There is a considerable research regarding pedagogy in schools but little research regarding pedagogy in a one-to-one tutoring environment. This research involves the establishment of a practical scheme for the teaching of mathematics in a tutoring context. It also examines the significance of conceptual understanding in improving the performance of a student undertaking the NSW stage 6 Mathematics course.

For many years, researchers have argued about the advantages and disadvantages of the two apparently opposite pedagogical approaches of direct instruction and discovery learning (Kirschner, et al., 2006; Hmelo-Silver et al., 2006; Prince, 2004; Marer, 2004; Burton et al., 2004; Magliaro, et al., 2005; Lesh, Doerr, Carmona & Hjalmarson, 2003; Kuhn, 2007), but few discuss the possibility of incorporating advantages from both approaches. By working out a practical scheme that uses features attained from the two approaches, the study has the potential to make a contribution to the current literature. A report of the National Council of Teachers of Mathematics by Steen (1999) states:

“According to supporters, constructivism focuses education on the learner (what happens in students’ minds), on inquiry (seeking the right questions, not just the right answers), on relevance (questions of natural interest to children), and on activity (learning with both hand mind) … yet critics … contend that constructivist methods too easily slight the importance both of didactics (systematic instruction) and drill (systematic practice)” (p. 279).

1.6.2 The Process of Data Collection

The process of data collection consisted of three stages.
Stage 1 An interview before the tutoring process
The purpose of this interview was to obtain information concerning the student’s understanding of the mathematics topic. The researcher asked the student mathematics questions that required a basic procedural understanding of the topic. These questions were chosen from the final test for comparative purposes.

Stage 2 The tutoring process
The researcher used the principles of Simple Steps to improve the student’s conceptual knowledge and procedural knowledge of logarithms. Field notes were kept and sessions recorded. The notes sought to capture different aspects of the student’s engagement and his understanding of the mathematics.

Stage 3 A test and a feedback interview after the tutoring process
The student was given a written test after the tutoring process to provide information about the effectiveness of Simple Steps in helping him understand mathematics concepts within the topic of logarithmic functions.

After the test there was a feedback interview between the student and the researcher. The purpose of this interview was to investigate the student’s conceptual understanding of the topic.

Finally, the student was given a written questionnaire in which he was asked to give a self-assessment of his engagement during the tutoring process.

1.6.3. A Brief outline of the Chapters
This section gives the concluding remark outlining the chapters to come and their purpose.
Chapter 2 is the literature review (LR). The purpose of this chapter is to review literatures that inform pedagogy in the field of mathematics learning and teaching.

Chapter 3 discusses methodology that could help establish the practical teaching scheme. The significance of the research, the research question, and the ethical issues were also discussed. Chapter 4 presents the data and the initial analysis including the process of coding and theme identification.
Chapter 5 presents the major process of data analysis during which the method that leads to the answer of research question are also addressed.

A summary of the entire analysis will be presented in Chapter 6. The answers to the research questions from a more theoretical point of view will be given. The implication of the research and its limitations, suggestions for the future research will also be discussed.
Chapter 2
Literature Review

2.1 Introduction
The purpose of this chapter is to review literature which can be used to inform pedagogy in a tutoring context. In particular, the methods of discovery learning, direct instruction, and the cognitive learning process will be discussed within the theoretical frameworks of behaviourism, constructivism, and cognitive science. The aim is to categorise these principles and theories in the light of a tutoring context.

2.1.1 The Tutoring Context
Tutoring has had a long history as an educational variable (Bausell, Moody & Walzl, 1972) but “has largely escaped empirical scrutiny” (p. 591). Despite the fact that a number of studies have demonstrated that one-on-one tutoring or individualised instruction is one of the most effective methods of learning, (Bloom, 1984; McArthur, Stasz & Zmuidzinas, 1990), very few studies “offer a precise information-processing account of this skill” (McArthur et al., 1990, p. 197). Many of the studies of one-on-one tutoring have concentrated on comparing the effectiveness of human tutors and intelligent tutoring systems (Merrill, Reiser, Ranney & Trafton, 1992; McArthur et al., 1990; Rosé, Vanlehn, The Natural Language Tutoring Group Learning Research &. Development Center, University of Pittsburgh, 2003, Wood & wood, 1996). This research treats one-on-one tutoring as an educational variable (Bausell et al., 1972).

The tutoring process investigated in this study aims to achieve more than simply enabling the student to solve a difficult problem. It aims to engage the student with the tutor in discovering new knowledge or concepts rather than just repairing old ones (Collins, 1988 as cited by McArthur et al., 1990). Because much more reinforcement and encouragement is possible in a one-on-one tutoring context than is possible with intelligent tutoring systems, the structure of the human tutoring process can be more individualised (Bloom, 1984; Merrill et al., 1992).
2.2 Behaviourism and Direct Instruction

2.2.1 Behaviourism

Ivan Pavlov (1849-1936), John B. Watson (1878-1958), E. Thorndike (1874-1949), C. L. Hull (1884-1952), and B. F. Skinner (1904-1990) are important thinkers in the field of behaviourism (Wagner 2005). Skinner (1968) stated “‘To think’ often means simply to behave” (p. 119), and “to think verbally or nonverbally, mathematically….means to behave with respect to stimuli” (Skinner, 1968, p. 119). He stated that thinking processes such as learning, discriminating, generalising, and abstracting are “not behaviour but changes in behaviour” (p. 120). Thus, from the point of view of a behaviourist, “learning can be defined as the relatively permanent change in behaviour brought about as a result of experience or practice” (Huitt & Hummel, 2006). There are two important processes involved in the process of learning. They are called classical conditioning and operant conditioning.

Pavlov and Classical Conditioning

Early behaviourist theory started with the work of Russian physiologist Ivan Pavlov on classical conditioning (Schunk, 2008; Wagner, 2005). Pavlov studied the digestive process of dogs, during which he demonstrated the relation between unconditioned stimulus (which caused unconditioned response), neutral stimuli, and conditioned stimulus. He discovered that after training, presenting a conditioned stimulus could produce a conditioned response, a process called classical conditioning (Slavin, 2000). Classical conditioning has explained some phenomena in human learning and has established the basic concepts of stimulus, generalisation, discrimination, extinction, and conditioning (Driscoll, 2005; Ormrod, 2008).

Watson and his View of Psychology

Based on Pavlov’s work, Watson continued the study of classical conditioning (DeMar, 1989). He believed that “Psychology as the behaviourist views it is a purely objective experimental branch of natural science. Its theoretical goal is the prediction and control of behaviour” (Watson, 1913, p. 1). Watson rejected the dominant introspectionist approach in psychology since it was unquantifiable. Instead, he focused his investigation on quantifiable events and behaviours (Hauser, 2006). He believed that all behaviour can be reduced to the basic stimulus-response patterns of behaviour (Watson, 1999).
Watson’s research and his advocacy of psychology as an objective science has “led to a
behaviourist tradition that dominated psychological research” (Ormrod, 2008, p. 2) and his
emphasis on the environment that plays an important role in human behaviour can be seen
from the work of Skinner (Schunk, 2008; Ormrod, 2008).

**Skinner and Operant Conditioning**

Another influential behaviourist is B. F. Skinner. His philosophical thought about
behaviourism is expressed as “radical behaviourism” and is seen as the combination of
methodological, analytical, and psychological behaviourisms (Stanford encyclopaedia of
philosophy, 2009).

Skinner developed the concept of “operant conditioning”. He illustrated this concept by using
the example of a hungry rat placed in a box. If the rat presses a lever protruding from the wall
of the box, it moves downward, and “causes a food tray to swing into reach long enough for
that rat to grab a food pellet” (Ormrod, 2008, p. 51). Immediately after eating the delivered
food the rat begins to press the lever fairly rapidly. The behaviour is said to have been
strengthened or reinforced by a single consequence (Skinner, 1976; Schunk 2008). So
learning takes place as the result of a change in behaviour which is the consequence of
reinforcement given to the learner during the learning process (Driscoll, 2005; Atherton,
2010; Skinner, 1976).

### 2.2.2 Direct Instruction

Direct Instruction, according to Magliaro et al. (2005), “is an instruction model that focuses
on the interaction between teacher and students” (p. 41). Direct instruction was initially
developed by Engelmann and Bereiter in the 1960s (Magliaro et al., 2005; Bereiter &
Engelmann, 1966), and is now in its “third decade of influencing curriculum, instruction, and
research. It is also in its third decade of controversy” (Magliaro et al., 2005, p. 41).

Direct instruction also known as explicit teaching or explicit instruction (Rosenshine, 1986;
Hall, 2002) has its roots in the learning theory of behaviourism (Conway, 1997; Huit, 2003).
Engelmann (2005) explains: “Teaching is the act of changing the learner’s behaviour in
specified ways and achieving this behavioural change through the manipulation of
environmental variables only” (p. 25). Thus in order to reinforce the required response to
establish the change in behaviour, schedules of reinforcement have to be presented to direct
and maintain the desired change in behaviour (Driscoll, 2005). These teacher-directed approaches enable the teacher to construct scripts that are designed to strengthen the learning and minimise the confusion (Parsons & Polson, 2002).

Nowadays, a number of variations and models of direct instruction have been developed but it has maintained its core principles (Magliaro et al., 2005; Burton, Moore & Magliaro, 2004; Huit, 1996; Comway, 1997) that are important to the teaching of difficult subjects such as Mathematics and Physics. The core principles are:

**Step-by-step**
Step-by-step is “a systematic method for presenting material in small steps” (Rosenshine, 1986, p. 61). This becomes more important when presenting new or difficult material (Rosenshine, 1986). Rosenshine (1986) used the theory of information-processing to explain why step by step is a scientific strategy. He explains that the capacity of the working memory is limited and if too much information is present, the learner will be confused and skim material. This will be discussed in more detail in terms of cognitive load theory in latter sections.

**Guided practice**
According to Rosenshine (1995), the two findings from research “that are most relevant to teaching are (1) the importance of teaching in small steps and (2) the importance of guiding student practice” (p. 264). After presenting or demonstrating material in small steps, the teacher may ask students to perform exercises that illustrate what has been taught (Rosenshine, 1986; Gersten et al., 1986). During the guided practice, the student can learn how solutions are obtained using the step-by-step strategy and the teacher can check for the student’s understanding and provide corrective feedback immediately (Gersten et al., 1986).

The importance of the guided practice can be understood using the theory of cognitive information processing. According to this theory, new information must be processed in the working memory and reconstructed with the existing knowledge before it moves to the long term memory (Slavin, 2000; Rosenshine, 1986, 1995). During guided practice, the student has to answer questions and explain material. In doing so, he engages “in the cognitive processing activities of organizing, reviewing, rehearsing, summarizing, comparing and contrasting” (Rosenshine, 1995, p. 265).
Discrimination
In the process of learning, a student may face different aspects of characteristics related to a particular concept. This requires the student to discriminate between characteristics and be able to identify the concept (Kenny, 1980). Discriminative stimuli include mathematical symbols, scientific notations, letters, words, and numbers (Slavin, 2000). The core contribution of direct instruction to educational research is the analysis of discrimination in the procedure of studying a concept (Kenny, 1980).

A mathematical concept may possess some characteristics which are similar to other concepts making it difficult to grasp. Learning to discriminate means learning to recognise the “rules” and the “sameness” of the defining features for a given concept or knowledge and exclude irrelevant attributes (Engelmann et al., 1982; Kinder et al., 1991).

Advantages and Disadvantages of Teaching by Direct Instruction

Advantages of Direct Instruction
It is generally acknowledged that direct instruction helps students acquire essential concepts and propositions in carefully divided sequences of lessons providing an effective method for improving basic skills such as remedial reading and mathematics (Kozloff et al., 1999; Brinder et al., 1990). It allows the teacher to prepare the students with necessary tools and knowledge to move to a more advanced level with confidence and skills (Pazychodzin, Marchand-Martella, Martella & Azim, 2004; Conway, 1997).

Many researchers have shown that the systematic and explicit principle of direct instruction has made it very effective and useful in improving the learning rate among children with learning difficulties, particularly in special education and for middle school students at-risk of failure in Mathematics (Gujjar, 2007; Folres & Kaylor, 2009; Gersten & Carrmine, 1984).

Disadvantages of Direct Instruction
One concern about the direct instruction’s approach is that it does not encourage the student to gain deep understanding of the material comprehensively (Tweed, 2004), and to think reflectively (Kuhn, 2007). As McVittie (2008) mentioned, some students cannot solve problems with efficiency because they have not tried hard enough to explore the problems
themselves. This disadvantage cannot be overcome by frequent practice alone. It pertains to the student’s role in the learning process (Kenny, 1980; Tweed, 2004). The highly structured approach could make the learning passive (Tweed, 2004) and “inhibit the development of creativity” (Kenny, 1980, p. 3). Some studies suggest that an effective tutor should maintain a delicate balance that ensures the learner is actively engaged in overcoming obstacles so learning may occur (Merrill et al., 1992; Wood et al., 1996).

As discussed earlier, the role of a tutor is not limited to the repair of old knowledge for the learner. The tutor and the student need to work together to discover new knowledge (McArthur et al., 1990). It is necessary, therefore, for the tutor to understand the individual’s thinking. Behaviourism is unable to explain what happens to the conceptual framework of the learner during learning. What is happening in the human mind during learning is the concern of the fields of constructivism and cognitive science.

2.3 Constructivism and Discovery Learning

2.3.1 Constructivism and Learning
Constructivism is a term widely used within many different disciplines (Sjøberg, 2007). As Driscoll (2005) commented: “there is no single constructivist theory of instruction” (p. 386). Variation in the meaning of constructivism reflects people’s own understanding of different aspects of a constructivist theory (Gagnon & Collay, 1999; Driscoll, 2005). The main idea of constructivism is that people construct their understanding of the world within the framework of their own knowledge (Ishii, 2003). According to this perspective, knowledge cannot be simply passed on, but should be actively constructed by learners. They build new knowledge and interpret information based on their past experiences and existing knowledge (Lesh & Doerr, 2003).

Although there are several varieties of constructivism within the field of education (Sjøberg, 2007), there are two major strands (Kanselaar, 2002). They are: cognitive constructivism and social constructivism. Cognitive constructivism has an individualistic perspective of learning. It is concerned with the mental development of the individual, while social constructivism emphasises the collaborative nature of learning and understanding (Atherton, 2005, Teaching Resource Center, 2008). There are many theorists in the field of constructivism. Among them, John Dewey, Jean Piaget, Lev Vygotsky, David P. Ausubel, and Jerome Bruner
provide the theoretical underpinnings for the constructivist approach to teaching and learning (Lutz & Huiit, 2004; Huiit, 2003). While Piaget and Bruner are the chief theorists in the field of cognitive constructivism, Vygotsky is the major theorist in the field of social constructivism (Huiit, 2003; Read, 2006). The following section outlines some of the basic ideas of these three scholars in relation to cognitive development.

**Piaget’s Theory of Cognitive Development**

Jean Piaget (1896-1980) was a biologic philosopher and child psychologist (Driscoll, 2005). His work on cognitive development was one of the most influential theories that shaped the development of constructivism during the twentieth century (Confrey & Kazak, 2006). There are two main strands in Piaget’s theory of cognitive development: the basic mechanisms that account for cognitive development and the stages of development that each individual must pass (Gruber et al., 1977; Teaching resource centre, 2008; Huiit & Hummel, 2003).

**The Processes of Development**

The processes of development are the mechanisms that account for intellectual growth. According to Piaget, the factors involved are: biological maturation, experience with the physical and social environment, and equilibrium (Gruber et al., 1977; Schunk, 2008; Piaget, 1967). Equilibrium “pertains to the functioning of the system as a whole, and thus embraces the other three factors” (Gruber et al., 1977, p. 783). This review focuses on the processes that are most related to learning.

**Assimilation**

Assimilation is the process of adapting new information to the existing schema or cognitive structure (Slavin, 2000; Schunk, 2008; Huiit & Hummel 2003). For example, if a child sees an unfamiliar object, he may grasp it or bang it. He is using the resulting information about the object to incorporate it into existing schemata (Slavin, 2000). In other words, assimilation is the process that fits the external world into the existing cognitive structure (Salvin, 2000; Piaget, 1967).

**Accommodation**

There are situations where pre-existing knowledge and experience do not match the new information an individual receives. The individual needs to modify or change his cognitive structure in order to accept this information (Huiit et al., 2003; Kearsley, 2008). For example,
when the child bangs an egg, unexpected consequences occur and the child might modify his scheme to distinguish between banging hard objects and soft objects (Slavin, 2000). The process of changing or altering existing schema in the light of new information is known as accommodation. New schema may also be constructed in the process (Wagner, 2005).

**Equilibration**

When an event or situation cannot be explained or understood in terms of existing schemata, a state of disequilibrium is created (Sandwell, 1995; Slavin, 2000; Piaget, 1967). Disequilibrium is referred to as cognitive conflict or simply “conflict” (Schunk, 2008; Piaget, 1967). In order to re-establish equilibrium between a new factor and mental organisation, the individual needs to ‘assimilate’ the external information into the existing cognitive structure. The existing structure needs to be readjusted or even changed to ‘accommodate’ new information (Piaget, 1967). The process of regulating and achieving balance between assimilation and accommodation is called equilibration (Piaget, 1967; Elkind, 1967; Wagner, 2005).

Learning occurs when the individual engages in cognitive conflict and achieves equilibration through assimilation and accommodation (Schunk, 2008; Piaget, 1967; Piaget, 1977). However, equilibration is not merely an automatic state. It is more appropriate to describe it as a process corresponding to a tendency to continuously search for greater equilibrium and coherence between existing knowledge and new knowledge gained from experience (Piaget, 1977; Piaget, 1967).

Piaget’s theory has greatly influenced the philosophy and psychology of Mathematics education (Ernest, 1993; Leman, 1998), but it is sometimes considered as lacking in social context for the development of knowledge construction (Leman, 1998; Devries, 1997). Since learning involves the acquisition of knowledge through both individual and social activity, it is necessary to also consider the philosophy and psychology of the learning process from the aspect of social interaction (Bruning et al., 2004; Confrey, 1995).

**Vygotsky and the Theory of the Zone of Proximal Development**

Vygotsky (2004) emphasised that the process of thinking cannot simply be considered as a simple transition of external data into the human mind. Rather, the whole content is reformed
and reconstructed in conjunction with the individual’s maturation and mastery of his environment.

Lev Semenovich Vygotsky (1896-1943) was a Russian psychologist and scholar (Bruner, 2004). Vygotsky believed that the social instruction of the culture in which a child grows is crucial in determining his cognitive skills and patterns of thinking (Schütz, 2004). Vygotsky’s idea of a zone of proximal development (ZPD) reveals some aspects of the social nature of the learning process by which children acquire knowledge (Schütz, 2004; Vygotsky, 1978).

**Zone of Proximal Development**

According to Vygotsky, there are two levels at which an individual can solve a given problem: actual developmental level (i.e. the individual can solve the problem independently) and potential developmental level (i.e. the individual is capable of solving the problem only under guidance or in collaboration with peers). The potential developmental level is also called the zone of proximal development (Vygotsky, 1978; Schütz, 2004).

According to Vygotsky’s theory (1978), a person can only imitate within his developmental level. Consequently, the level at which a Mathematics lesson is given or the method of solving a problem should not exceed the learner’s level of potential development. On the other hand, if the teaching is oriented toward the level already reached, it is ineffective, because “it does not aim for a new stage of the development process” (Vygotsky, 1978 p. 88). Based on this analysis, Vygotsky (1978) proposed “that an essential feature of learning is that it creates the zone of proximal development” (p. 88). A learner’s zone of proximal development is the level at which learning takes place and cognitive change occurs through interaction with people or peers in his environment (Schunk, 2008; Teaching Resource, 2008; Vygotsky, 1978).

Vygotsky’s theory implies that when a student works together with a tutor in the zone of proximal development, he can accomplish a more difficult task than would be possible if he worked alone (Ormrod, 2008; Bruning et al., 2004). This theory provides philosophical support for a tutoring system.
**Bruner’s Theory of Learning**

Jerome Bruner (1915- ) is one of the most influential psychologists of the twentieth century. He made a profound contribution to the study of pedagogy and the theory of cognitive growth (Smith, 2002; Schunk, 2008). Bruner agrees with Piaget’s notion of progressive stages in cognitive development but disagrees with Piaget’s idea that all individuals will progress in the same way regardless of their cultural and social difference (Lutz et al., 2004; Driscoll, 2005). Bruner’s theories focus on the environmental and experiential factors which influence the developmental pattern of each individual and thus affect intellectual ability through all stages (Smith, 2002; Gale Encyclopaedia of psychology, 2001; Bruner, 1965). In the following, some of Bruner’s theories will be discussed, including his model of human thought.

**Modes of Representations**

Bruner believed that an individual’s ability to interpret new information based on existing mental structures is critical to learning and cognitive growth (Lutz et al., 2004). This ability requires the mastery of techniques that enable a person to represent and manage the knowledge they construct as they interact with the complex environments around them (Bruner, 1964).

Bruner (1964) suggested that people use three systems to structure their knowledge and understanding of the world: action, imagery and perception, and language and reason (Bruner, 1964; Driscoll, 2005). These three systems are called: enactive representation, iconic representation, and symbolic representation. They are not structures, but are different forms of cognitive processing (Schunk, 2008; Bruner, 1964). They are present at all stages of development and are developed “through the building and establishment of progressively more sophisticated and specific mental schemes or structures (Driscoll, 2000)” (Lutz et al., 2004, p. 7).

**Enactive representation**

In this mode, people demonstrate their thinking (Lutz et al., 2004) and their experience of past events (Bruner, 1964) through appropriate motor activity. For example, a young boy may take you to a nearby store but he may not be able to describe the exact path to the store. (Driscoll, 2005).
Iconic representation
In this mode, past experience and knowledge is represented by spatial, temporal and qualitative structures or is characterised by a set of images that represent concepts (Kristinsdottir, 2001; Lutz et al., 2004). For example, a boy may express his knowledge about the location of a store by drawing a map indicating the route from his house to that store (Driscoll, 2005).

Symbolic representation
In this mode, a system of symbols is used to encode knowledge. Language is an example of a symbol system used by humans for encoding and representing experience (Driscoll, 2005; Schunk, 2008). It also provides a means of transforming experience (Bruner, 1964). The idea of people using different modes to represent knowledge has important implications for the teaching of Mathematics (Bruner, 1966). It suggests that the instruction should contain a mix of modes that optimise the learning (Bruner, 1966).

In regard to the fact that students have different intellectual backgrounds, Bruner suggested that the curriculum be organised so that students can build their knowledge continuously upon what they have learned. This pedagogical curriculum is called a spiral curriculum (Kearsley, 2008; Bruner, 1965; Bruner, 1973).

Spiral Curriculum
Bruner (1965) observed that many schools were wasting time by postponing the teaching of many important concepts because of their difficulty. He recommended a “spiral curriculum”. The same concept can be understood from different perspectives using different modes of representation. For example, a child can gain some understanding of the principles of topology and set theory by playing certain games. Any concept in Science and Mathematics can be approached by starting from a basic form of thought and moving to a more complex form (Bruner, 1965). He suggested that an early start program should emphasise both the scrupulous intellectual honesty and the intuition students possess at some simple level. This would then be revisited, reconstructed, and carried to higher levels of rigorousness, abstraction and comprehensiveness (Bruner, 1965; Bruner, 1960).

Bruner sees teaching as a process of prompting the intellectual quality (Schunk, 2008; Bruner, 1965). To him, the intellectual activity of what a scientist does at his desk is the same...
as what anybody does when he is engaged in like activities. This view reflects the idea of discovery learning (Raimi, 2004; Bruner, 1965).

Bruner’s theory has important implications in the context of tutoring especially in the teaching of Mathematics. According to Bruner, the teaching should depend on the thinking technique the individual is using (Schunk, 2008; Bruner, 1966). Thus, for example, when introducing a difficult concept, the tutor may try to represent the concept using an iconic mode before the symbolic mode is introduced. As Anderson (2004) stated, people will remember a sentence better if they develop a picture that represents the content of the sentence. In summary, Bruner’s theory implies that a tutor should try to raise the imagination of his student.

2.3.2 Discovery Learning

Discovery learning “has a long history in education innovation” (Slavin, 2000; p. 259). The concept of discovery learning has emerged as the educational and philosophical idea of many scholars such as Rousseau, Petalozzi, Dewey, Piaget, Bruner and Palzert (Clark, 1997; Conway, 1997). It became “the subject of study in cognitive and instructional science since the work by Bruner (1961)” (Van Joolingen & Jong, 1997).

Bruner was probably the leading proponent of discovery learning in Mathematics education (Kristinsdóttir, 2008). He defined discovery as “all forms of obtaining knowledge for oneself by the use of one’s own mind” (1961, p. 22 as cited by Driscoll, 2005). The role of the teacher is then to encourage the students to construct hypotheses and make decisions, and allow the students to discover new principles of their own accord (Kearsley, 2008). From the view of an educator, “discovery” does not mean to find out something which was unknown before (Bruner, 1961). Rather, it is a process involving reorganising of previously known information in order to assimilate or understand the newly encountered event (Kristinsdóttir, 2001).

Many models are based upon the discovery learning model. Holms and Hoffman (2000) summarised the three main attributes of discovery as

1. Instead of accepting information passively, students create, integrate, and generalise knowledge through exploring and problem solving. This view of learning has
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essentially changed the roles of students and teachers in traditional pedagogical practice (Holms & Hoffman, 2000).

2. The process of learning is driven by interest-based activities in which students determine the sequence and frequency with which learning occurs. (Holms et al., 2000). “Learning is not a static progression of lessons and activities”.

3. Activities are designed to encourage integration of new knowledge with the learner’s existing knowledge (Holms et al., 2000).

Different models have been recommended based upon the discovery learning model. Holms et al. (2000), for example, proposed that there are five main architectures which create the environment for discovery learning: case-based learning, incidental learning, learning by exploring/conversing, learning by reflection, and simulation-based learning. Learning by reflection will be discussed later, because it is an important process in the teaching of Mathematics.

Kirschner et al. (2006) has classified approaches that are pedagogically equivalent to discovery learning as minimally guided instructional approaches and has defined direct instructional guidance as “providing information that fully explains the concepts and procedures that students are required to learn as well as learning strategy support that is compatible with human cognitive architecture” (p. 75). However, Hmelo-Silver, Duncan, and Chihn (2006) argued that Kirschner et al. had mistakenly conflated problem-based learning and inquiry learning with discovery learning. This review adopts the definition suggested by Kirschner and his colleagues which places discovery learning in the category of minimally guided instruction and direct instruction in the category of direct instructional guidance.

Although discovery learning has led to a few positive swings in the educational curriculum, it has never enjoyed an overwhelming acceptance (Clark, 2000). However, in spite of various criticisms of discovery learning, many scientific and mathematical educators consider it indispensable if teaching is to be a success (Harris & Taylor, 1983; Perry, 2004; McDaniel et al., 1990). As Perry (2004) argued, the traditional approach to mathematics education “is effective for some students, but surely not the best approach for all” (p. 694). It is necessary, therefore, to discuss some of the important concepts of discovery learning in relation to the teaching of Mathematics.
Problem solving

In discovery learning, problem solving is a situation “designed to require discovery on the part of the learner” (Gagne, 1966, p. 147). Gagne (1966) argues that problem-solving situations are designed for the learner to make a discovery; otherwise it would not be called problem solving. Davis (1966) realises the importance of the element of discovery. He said, for example, that if a quadratic equation:

\[ A \times A - 5 \times A + 6 = 0 \]

was given to a student to determine the value of A and the only method that was available to him was trial and error, then that student would probably make no discoveries.

Brumbaugh, Ashe, Ashe, and Rock (1997) commented that “even when the students are in a discovery setting, most of the time they are eventually told what should have been discovered” (p. 182). This comment opens up the timeless debate about discovery learning, namely “how much a teacher should help a student and how much the student should help himself” (Snelbecker, 1974, p. 425, as quoted by Martin, 2000). Instead, this research will focus on the searching for answers to the following question: How can co-operative problem solving help students to understand a mathematical concept in the context of tutoring? In connection to this question, two important features related to problem solving will be discussed: learning by reflection, and recognition of connection.

Learning by reflection

The notion of reflection differs substantially in the literature (Mewborn, 1999). Daudelin (1996) defines reflection as “a personal cognitive activity that requires stepping back from an experience to carefully and persistently think about its meaning to the person through the creation of inferences” (p. 36). This description suggests that learning takes place when the individual makes sense of past experience in an effort to understand future experience (Daudelin, 1996).

Learning by reflection is one of the architectures that support discovery learning in that individuals learn to apply higher-level cognitive skill through caring out sophisticated analyses and learning how to ask better questions when exploring a problem or issue (Bicknell-Holmes et al., 2000; Castronova, 2001). Daudelin (1996) commented that posing and answering questions can increase the learning power of reflection and that a thoughtful question is often better than concise answers. Wheatly’s study (1992) also suggested that...
being asked to justify a method of solution such as “will that work?” will often promote reflection. Wheatly (1992) explains that in the process of reflection, schemas of schemas are formed, that is, a second order construction is established.

Recognition of connection
Bruner (1961) states that discovery does not mean to find out something which was unknown before. Rather, it is a process involving the reorganisation of previously known information in order to assimilate or understand a newly encountered event (Kristinsdóttir, 2001). This idea coincides with the concept of a knowledge network in that a mathematical object is understood only when its role as a component of the global structure is recognised (Godino, 1996; Hiebert et al., 1986). This idea led the researcher to realise the relationships that exist between discovery learning and the theory of cognitive learning. These relationships will be discussed in Chapters 5 and 6.

Support and Criticism of Discovery Learning
Similar to direct instruction, discovery learning has been the focus of disputes in the field of education (Kirschner et al., 2006; Marer, 2004; Prince, 2004). Those arguments that are important to the teaching of Science and Mathematics will be briefly discussed.

Advantages of Discovery Learning
One of the remarkable advantages of discovery learning is that it actively engages the learner in the process of learning (Martin, 2000). As a result of active engagement, it has the capacity to motivate the learners in the process of learning (Martin, 2000; Castronova, 2001).

Discovery learning also provides an opportunity for the development of problem solving and intellectual thinking skills (Martin, 2000; Slavin, 2000; Raimi, 2004). It involves participants in higher levels of cognitive processing such as reflection, intuition evaluation, and innovation (Bicknell-Holmes et al., 2000; Raimi, 2004; Perry, 2004; Harvey, 1998; Bruner, 1965).

The controversy surrounding discovery learning relates to different aspects of teaching and psychological studies of learning (De Jong & Van Joolingen, 1998; Keislar & Shulman 1966) are discussed in the following sections.
Disadvantages of Discovery Learning

One criticism of discovery learning is associated with the theory of cognitive load (Tuovinen & Sweller, 1999). According to this theory, all knowledge is stored in the long term memory and can only be processed in the working memory (Sweller, Van Merrienboer & Paas, 1988). However, the capacity of the working memory is limited in the number of items of information it can hold simultaneously (Sweller, 2004). When material is inappropriate presented, a heavy cognitive load is imposed so the material cannot be understood (Tuovinen et al., 1999; Marcus, Cooper & Sweller, 1996).

Discovery learning has also been criticised for potentially enabling incorrect representations of concepts to develop during the learning process (Learning Theories Knowledgebase, 2009; Martin, 2000). In discovery learning environments, the learner is required to draw on his prior knowledge (Clark, 2000), and generate hypotheses in order to construct his own knowledge (Van Joolingen, 1999). Studies show that when students’ knowledge is inadequate, some choose to stick to their incorrect hypotheses (De Jong et al., 1998; Smith et al., 1993).

If problem solving is considered to be “the major vehicle for introducing important issues” (Schoenfeld, 1994; p. 16) and an effective means of engaging students in learning (Reiser et al., 1998), then an important question is:

   How can the problem-solving approach, advocated by the supporters of discovery learning, help the learner to understand a mathematical concept or topic while keeping the possibility of a misconception to a minimum?

This is one of the research questions of this study.

2.3.3 Conclusion

As mentioned previously, the researcher’s goal is to review learning theories that are applicable to the process of tutoring, because tutoring is an activity involving more directive instruction than the usual classroom (Merrill et al., 1992; Rose et al., 2009). The learning environment supported by discovery learning does not adequately describe the tutoring situation because it lacks the fine detail of the processes that make up the construction of learning during tutoring. It was decided, therefore, to look at cognitive learning processes which are “highly compatible with findings from brain research” (Schunk, 2008, p. 516).
2.4 Cognitive Learning Process
Cognitive learning is the process of acquiring knowledge and intellectual skills as a result of mental or cognitive processes (Huitt, 2004; Harris, 1995). Theories of cognitive learning are supported by cognitive science, a much broader field “that attempts to integrate research efforts from psychology, philosophy, linguistics, neuroscience, and AI” (Anderson, 2005, p. 11), where AI refers to artificial intelligence. Cognitive theories involve the concept of information processing process (IPP) and require knowledge of the formation of networks in memory (Schunk, 2008).

2.4.1 The Memory System
Knowledge is stored in memory. The study of memory is an important part of cognitive psychology (Huitt, 2003). Memory has three components: the sensory memory, the short-term memory and the long-term memory (Schraw, 2006; Schunk, 2008).

Sensory memory
Incoming information first arrives at the sensory memory. A limited amount of it will then be processed and forwarded to the working memory (Schraw, 2006). Only information to which we pay our attention can be moved into working memory. The rest will decay rapidly (Ormrod, 2008).

Working memory
Working memory, also called short-term memory, is the place where the mind operates and processes the information (Slavin, 2000). The processed information is then sent to the long-term memory where it stays permanently (Schraw, 2006). Working memory can hold information coming from the sensory memory and the long-term memory simultaneously. However, the information’s duration in the working memory is short, and the capacity of it is very limited (Kirschner et al., 2006, Sweller, 1988). It acts as a bottleneck through which only a limited amount of the sensory information can pass and reach the long-term memory (Schraw, 2006).

Long-term memory
Any information that needs to be remembered must be stored in long-term memory (Driscoll, 2005). Studies also show that the capacity of long-term memory is unlimited (Ormrod, 2008).
Cognitive learning theory relies on how knowledge is constructed and stored, it is necessary, therefore to discuss the structure of knowledge in human memory and schema.

Network Model of Knowledge
There are different types of knowledge stored in the long-term memory: declarative knowledge, procedural knowledge and self-regulatory knowledge. Researchers have proposed different network models to describe knowledge acquisition and representation. Network models will be used in this study as the major tool for describing learning (Schraw, 2006; Ormrod, 2008).

According to network models, pieces of information are stored in different units called nodes. Each node stores information related to a specific concept. When nodes with different properties are connected via relational links, more complex facts and concepts are formed (Schraw, 2006; Ormrod, 2008). The idea that specific information is organised into nodes stored in memory is very important. As Schraw (2006) points out “Almost all other models of knowledge representation incorporate the idea of nodes, although what a node includes varies from model to model (Anderson, 2000).” (p. 253).

Schema
According to Schunk (2008) “A schema (plural schemata) is a structure that organizes large amounts of information into a meaningful system” (p. 155). Ormrod (2008) explains that a schema is a set of ideas including concepts that are related to some specific object or event. According to this definition, a set of separated information will not give us a complete knowledge about a specific object or event. However, connections between information can be formed through practice or experience, resulting in the construction of a schema (Anderson, 2008). The process of constructing a schema is called the acquisition of a schema. Schemata are stored in long-term memory (Sweller, 1998).

Cognitive load
As discussed, working memory only holds a limited amount of information at a time (Driscoll, 2005). Cognitive load refers to the amount of information that needs to be processed in working memory (Driscoll, 2005). During learning, the learner uses his existing knowledge to assimilate the new information. In other words, the new information is being digested and incorporated into automated schemas. If the information can be absorbed into an
existing schema, the cognitive load will be low and the material will be understood (Marcus, Cooper & Sweller, 1996). If, on the other hand, the existing schemata fail to interpret new information efficiently, the amount of information needed will be in excess of the capacity of the working memory and the material will not be understood (Marcus et al., 1996).

This idea has shed light on the design of a pedagogical approach (Sweller, 2004). According to Sweller et al. (1998), a huge array of elements in the long-term memory can be organised into a schema which can be retrieved easily as a single entity and held in the working memory. According to this principle, the teaching of Mathematics should be designed to allow the learner to construct schemata at different levels of knowledge. These schemata can then be incorporated into a higher-order schema that can be held easily in working memory. This is a fundamental condition for understanding difficult material (Sweller, 2004; Marcus et al., 1996).

2.4.2 The SOI Model of Learning
The Select-Organize-Integrate (SOI) model was introduced by Mayer (1996). It describes three basic cognitive stages in a learning process. According to Mayer (1996), the constructivist’s view of learning involves more than just knowledge acquisition. It also involves the interpretation of the learner’s existing knowledge and skills. Mayer and Wittrock (2006) suggested that meaningful learning involves three cognitive processes of knowledge construction, namely: selecting, organising and integrating.

The selecting process
When information is presented to a learner in some form (such as a lecture or a picture), it is represented as images in the learner’s sensory memory, some of which are transferred to the working memory.

The organising process
The selected information in the working memory is organised into a coherent structure within the working memory.

The integrating process
The newly constructed knowledge in the working memory is connected to relevant knowledge retrieved from the long-term memory. This process is called the integrating
process because it involves building connections between the new knowledge and existing knowledge. The result of the integration “is a coherent cognitive structure that itself can be stored in long-term memory for future use.” (Mayer, 1996).

The cognitive process takes place in the working memory and must occur for meaningful learning to take place (Mayer et al., 2006). It becomes critically important when a new concept or topic is introduced in a one-on-one tutoring environment as discussed below.

2.5. Some Important Conceptions Related to Mathematics Learning

In this section, some important aspects of learning will be discussed, based on the cognitive learning theory. It is important to note that debate about the roles of conceptual knowledge and procedural knowledge in mathematics learning exists (Rittle-Johnson & Alibali). This controversy is not within the scope of this review. This review concentrates on concepts related to Mathematics learning.

2.5.1 Teaching and Learning Mathematics with Understanding

Promoting learning with understanding has been the goal of much research in Mathematics education (Hiebert & Carpenter, 1992; Silver, Mesa, Morris, Star & Benken, 2009). Hiebert et al. (1992) comment that many researchers in the field of cognitive science “can be interpreted as recognising that understanding is a fundamental aspect of learning and models of learning must attend to the issue of understanding” (p. 65).

According to Hiebert et al. (1992) “the mathematics is understood if its mental representation is part of a network of representations” (p. 67). Thus, when new information is appropriately assimilated into the internal network, understanding grows. When links between elements of knowledge are strengthened by reinforcing experiences, the level of understanding increases resulting in the accessibility of knowledge at a later time (de Jong & Ferguson-Hessler, 1986 as quoted by Prawat, 1989; Parwat, 1989; Hiebert et al., 1992).

According to constructivists however, knowledge cannot be just passively passed to the learner. A learner can only clearly understand what he has constructed (Can, 2006). A good teacher, therefore, should be able to present knowledge in a way that students can incorporate the new information into their existing knowledge network (Prawat, 1989; Gerace, 1992).
The teacher should also realise that when information is stored as discrete pieces in the memory, it is incoherent and not easy to retrieve later on (Ormrod, 2008; Hiebert, 1986). For a new concept to become part of the knowledge of the learner, appropriate connectedness between different pieces of information is required. This leads to the issue of conceptual and procedural knowledge.

**Conceptual and Procedural Knowledge**

In cognitive psychology a concept is considered to be a class of similar events or objects (Ormrod, 2008; Sigler & Samm, 2006). In this sense, our understanding about factual knowledge, concepts, theories or relationships among events is conceptual understanding (Glaesser, Gott & Roberts, 2009; Ormrod, 2008).

Conceptual knowledge is defined “as knowledge that is rich in relationships” (Hiebert & Lefevre, 1986, p. 3). It can be thought of as a connected network (Hiebert, et al., 1992). Thus, for a mathematical object to be a part of a conceptual system, its role as a component of the global structure must be recognised (Godino, 1996; Hiebert et al., 1986).

Another form of knowledge about how to do things ranging from simple to a sequence of actions is called procedural knowledge (Schraw, 2006; Hiebert et al., 1992). Procedural knowledge is knowledge that will “permit description of a specific relationship” (Hiebert, 1992, p. 78) related to our understanding about concepts, events, and procedures particularly in the field of Mathematics (Ormrod, 2008; Prawat, 1989; Hiebert et al., 1992).

One of the debates about conceptual knowledge and procedural knowledge in Science and Mathematics education is their relative importance in the context of instructional practice, or “skills verses understanding” (McCormick, 1997; Hiebert et al., 1992). A question that is debated is: Is it necessary to emphasise conceptual knowledge at the introductory level of a subject, or can a mathematical concept be learned via the acquisition of procedural skill? (Devlin, 2009; McCormick, 1997; Godino, 1996).

The issue of “understanding” underpins this research. The goal of the practical tutoring scheme is conceptual understanding, while procedural knowledge is seen as: a) the basic knowledge the tutee must acquire, and b) an important tool that helps the learner to improve his conceptual understandings (Glaesser et al., 2009; Devlin, 2009).
Understanding Instructions

Another aspect of understanding is the relationship between instructions and the acquisition of schema within the context of cognitive load theory (Marcus et al., 1996).

According to Marcus et al. (1996): “understanding is defined as the ability to follow instructions successfully and readily” (p. 50). Marcus et al. (1996) and Sweller (2004) identify two general factors that influence the level of this understanding: “the intrinsic complexity of the information and the manner in which the information is presented” (Marcus et al., 1996, p. 49). The first one refers to the inherent properties of the material which “is unalterable other than by schema acquisition” (Bruning, Schraw, Norby & Ronning, 2004, p. 30). The second factor is associated with extraneous cognitive load, already discussed in section 2.5.5.

Learning “involves automation” (Marcus et al., 1996, p. 50). Any cognitive activity that is performed in an automatic fashion is termed automaticity (Bruning et al., 2004; Ormrod, 2008; Sweller, 1998; Driscoll, 2005). Automated schemata are formed after a particular cognitive schema is repeatedly and successfully applied or when sources of information become habitual (Mayer, 2005; Sweller et al., 1998; Driscoll, 2005). Marcus et al. (1996) and Sweller (2004) comment that if knowledge acquisition requires a large number of unrelated cognitive processes, an inadequacy of schemas could result in an excessive cognitive load and understanding may fail. If, on the other hand, many elements of the information “can be incorporated into a single automated schema…the material will be easy to understand” (Marcus et al., 1996, p. 50).

2.6 Summary

In searching for a theoretical framework and instructional techniques that support Mathematics teaching in a tutoring context, the researcher has examined the learning theories of behaviourism, constructivism, and cognitive science. From these theories, several pedagogical approaches are identified and reviewed. They are: direct instruction, discovery learning, and the SOI model. Some of the important terminology are briefly summarised here:

1) Behaviourism and Direction
Behaviourism views learning as a process in which people are controlled or affected by their environment (Mcleod, 2007). Some of the important concepts are:

- Classical conditioning and operant conditioning.

The associated pedagogical approach is:

- Direct instruction. Important concepts of direct instruction are:
  - “Step-by-Step”, “guided practice”, and “discrimination”.

2) Constructivism and Discovery Learning

According to this perspective, knowledge cannot be simply passed on, but should be actively constructed by learners. Some of important concepts are:

- Piaget’s theory of Cognitive Development.
- Vygotsky’s theory of the Zone of Proximal Development
- Bruner’s Theory of Learning. Some of his ideas are:
  - “modes of representations” and “the spiral curriculum”.

The associate pedagogical approach is:

- Discovery learning. Important concepts of discovery learning are:
  - “problem solving”, “learning by reflection” and “recognition of connection”.

A discussion of advantages and disadvantages of discovery learning relates to one of the two research questions (Section 2.3.2). The other research question will be discussed in Chapter 3.

3) The Cognitive Learning Process

Cognitive learning is the acquisition of knowledge and intellectual skills as a result of mental or cognitive processes. Some of its important concepts are:

- The memory system, the network model of knowledge and schemata.
- Cognitive load

The model that describes the process of learning is:

- The Select-Organize-Integrate (SOI) model.

Some important concepts related to Mathematics learning are:

- Conceptual understanding: for a mathematical object to be a part of a conceptual system, its role as a component of the global structure must be recognised.
- Procedural knowledge: the basic knowledge the learner must acquire, and the important tools that help the learner to improve his level of understanding.
2.6.1 Conclusions

Constructivist theories and cognitive learning theories “are cognitive in nature” (Schunk, 2008, p. 516). Constructivism places greater emphasis on learners’ construction of knowledge and beliefs” (Schunk, 2008, p. 516), while cognitive learning pays more attention to how people acquire, interpret, and remember information (Ormrod, 2008; Schunk, 2008). The researcher has chosen cognitive learning theory as the basic pedagogical principle for developing a tutoring system because it meets the condition for a systematic, meaningful introduction of mathematical concepts in the tutoring context. Furthermore, the process of acquisition of schema as viewed from cognitive learning theory agrees with the constructivist view of learning (Bruning et al., 2004) and at the same time, it provides theoretical support for the explicit and systematic features of direct instruction (Kirschner et al., 2006; Rittle-Johnson, 2006; Sweller, 2004). Cognitive learning theory therefore allows the researcher the possibility of constructing a practical scheme that incorporates both of the advantages of direct instructional guidance approaches and discovery learning in the teaching of Mathematics within a tutoring environment.
Chapter 3
Methodology

3.1 Research Background
As stated in Section 1.1.2, the purpose of this research is to devise a practical scheme for the teaching of mathematics that integrates the researcher’s tutoring experience with features attained from well-established theories of learning. The focus of this practical scheme is that the student develops a deep conceptual understanding of the mathematics.

As a result of reviewing the literature, a practical scheme for teaching a new concept or topic slowly emerged. This scheme is called Simple Steps.

3.1.1 The Idea of Simple Steps
Simple Steps is based upon the cognitive theory of learning. It views the process of learning and teaching as the process of interaction between the tutor and the student. At the same time, it tries to incorporate principles from both discovery learning and direct instructional guidance within a tutoring environment.

When devising the Simple Steps tutoring scheme, two important aspects of understanding were considered. The first one is mentioned by Hiebert et al. (1992): “the mathematics is understood if its mental representation is part of a network of representations” (p. 67). According to constructivist theory, learning takes place when the learner uses his existing knowledge to assimilate new information. A growth in understanding results in the growth of networks (Hiebert et al., 1992). Thus, understanding can be defined in terms of the networking of knowledge.

The second aspect of understanding is “the ability to follow instructions successfully and readily” (Marcus, et al., 1996, p.50) as discussed in section 2.5.1. This condition implies that the absence of appropriate schemas could result in a failure to understand (Marcus, et al., 1996).

The following is an outline of the Simple Steps tutoring scheme that became the object of investigation in this study.
Step 1 Review the Concepts and Notation that is Related to the New Topic.
The purpose of this step is to bring the learner to the prerequisite level. According to the constructivist perception of learning, the learner will interpret new information in terms of her/his pre-existing knowledge. If this knowledge is incorrectly constructed, the student will have misconceptions stored in his long term memory (Murphy, et al., 2006). To replace these misconceptions, teaching is required in a step by step manner. The student will then have the schemata necessary for making connections with the new topic.

Step 2 Introduce the New Topic.
A new definition or concept is introduced directly in a simple fashion. The teacher gives a simple illustration of the concept by presenting a concrete or an obvious example so the learner obtains a clear picture of it.

Step 3 Generalise the Topic from Concrete to Abstract
Topics introduced in the previous two steps are discussed at a more advanced level. Elements or arithmetic expressions in the simple illustration are replaced by more general and abstract forms. The student is guided to understand the meaning of the mathematical symbols and the relationships that link and govern them.

Step 4 Obtain Useful Formulae and Information Based on a Conceptual Understanding to Gain Procedural Knowledge.
The purpose of this step is to learn basic procedures. This may include some useful formulae based on the concepts introduced. For example, after introducing the concept of \( y = \sin x, y = \cos x, \) and \( y = \tan x, \) the student will be required to learn some important formulae such as: \( \sin^2 x + \cos^2 x = 1 \) or \( \frac{\sin x}{\cos x} = \tan x. \) These formulae are the basis for a procedural knowledge of trigonometric functions. As Hiebert et al. (1986) pointed out, procedural knowledge could be learned by rote memorising. The important purpose of this step is therefore to seek methods that allow the student to understand how the formulae are conceptually related.
Step 5 Solve Basic Problems and Clear any Misunderstandings.
In this stage, the learner learns how to solve a variety of questions involving different aspects of the topic. At the same time, it aims to repair any possible misconceptions induced during learning.

Topic Selection
In preparing the tutoring lessons, I chose the topic of the logarithmic function:

\[ y = \log_a x \]

This is a confusing topic in which students have many misconceptions. Kaput (1989) (as quoted by Kieran, 2005) states that difficulties in algebra are due to (a) difficulties inherent in the concise and implicit syntax of algebraic symbols and (b) a lack of linkages to other representations. The logarithmic function \( y = \log_a x \) possesses can cause both these difficulties. Therefore it appeared to be an ideal topic to choose. The Simple Steps scheme was used to improve the student’s conceptual and procedural knowledge of logarithms.

3.1.2 The Research Questions
The two main research questions that are to be addressed through investigating the application of the Simple Steps tutoring scheme now become:

3) How can the principle of direct instruction, in the context of logarithms, be applied while keeping the learner actively engaged?
4) How can the problem-solving approach, in the context of logarithms, proposed by supporters of discovery learning, help the learner to understand a mathematical concept or topic while keeping the possibility of a misconception to a minimum?

The second question arose from the “Comments on Disadvantages of Discovery Learning” (Section 2.3.2).

3.1.3 Significance of the Research
It is hoped that this research will have implications for the teaching of senior high school students in Mathematics (stage 6), in particular, the teaching of Mathematics in the context of one-to-one tutoring. There has been little research of teaching and learning in this context compared to teaching and learning in the classroom.
This study also has the potential to contribute to the current debate in the literature concerning the advantages and disadvantages of two apparently opposite pedagogical approaches, namely, that of minimally guided instruction and direct instructional guidance (Kirschner, et al., 2006; Hmelo-Silver et al., 2006; Prince, 2004; Marer, 2004; Burton et al., 2004; Magliaro, et al., 2005; Lesh, Doerr, Carmona & Hjalmarsom, 2003; Kuhn, 2007). Few researchers discuss the possibility of incorporating advantages from both approaches.

3.2 An Overview of the Research Method

This is a case study in which the data is primarily qualitative. Data will be collected through interviewing, observing, and taking field notes. The researcher will also rely on “the views of participants” (Creswell, 2008 p. 46).

3.2.1 Identifying the Case

The “case” in this study is a senior male high school student who is tutored using the Simple Steps scheme, the aim of the tutor being that the student obtains a conceptual understanding of a topic that is reasonably new to him.

3.2.2 The Study Procedure

The case study took place over a two week period in which the student was taught a topic from the NSW Stage 6 Mathematics syllabus. It was composed of three main stages: an interview before the tutoring process, the tutoring process, and a test after the tutoring process. This stage includes a feedback interview and the use of questionnaire. The questionnaire is not only a source to measure the student’s engagement, but also provides opportunities that encourage the student to summarize what he has been doing in the learning process (Munns, et al., 2006). The final test was immediately followed by a feedback interview. These stages are illustrated in Figure 3.1:
3.2.3 Selecting the Participant and Topic Selection

Creswell (2008) states that in a qualitative inquiry, the researcher purposefully or intentionally selects individuals or sites in order to develop a detailed understanding of a phenomenon. Since the purpose of this study is that the student attains a conceptual understanding, a topic was chosen which students usually find difficult or confusing and therefore only learn procedurally.

Potential participants and their parent or guardian were sent a letter inviting them to take part in the study. The letter briefly described the process of the study and the participant’s role (See appendix C). Written consent was obtained from a parent of the chosen participant.

3.3. Stage 1 – The Initial Interview

3.3.1 General Principles of the Interview

Fontana and Frey (2005) comment that interviewing is not merely an exchange of questions and answers; it is “a collaborative effort” (p. 698). According to Creswell (2008), Hancock (2006) and Kavale (1996), there are several general steps for conducting a successful interview. The interview was guided by these steps, namely:

- Identify the interviewee.
- Determine the type of interview, develop an interview guide or interview protocol. The protocol should identify open ended questions that allow the researcher to gain insights into the research questions (Hancock, 2006).
- Obtain consent from the interviewee and tell the interviewee the purpose of the interview and the time the interview will take.
- Make brief notes during the interview and record the notes on an interview protocol. Recognise that the notes taken may be incomplete in some situations. An abbreviated form of note taking may be needed to speed up the process. (Creswell, 2008).
- Do not disregard the legal and ethical protections (Hancock, 2006).
- Be well mannered when the interview is complete and understand that the information obtained from the interview is confidential (Hancock, 2006; Creswell, 2008).
3.3.2 Conducting the Interview

The main purpose of the initial interview was to obtain information concerning the student’s learning background and its relationship to his engagement in lessons at school. A second purpose was to obtain information about the student’s level of understanding of Mathematics and in particular, his understanding of logarithms. This was achieved by inserting Mathematics questions into the conversation in an easy-going fashion.

A pre-test was not given because such a test is often stressful and might lead to an unwanted negative response. If the student does not understand the mathematical concept very well, she/he may feel worried, embarrassed, or even nervous. Under such pressure, the test will not give a clear measure of the student’s understanding (Hong, Sas & Sas, 2006). Many students believe that because they have memorised formulae, they understand (Hong, et al., 2006). Simple Steps tries to prevent students from thinking in this way. In Simple Steps, problems are constructed in a way that encourages the student to understand conceptually. Because the student did not have a strong conceptual understanding at this stage, some of the test questions were not asked. It was a semi-structured interview because the questions asked depended on the interviewer’s assessment of the student’s level of understanding.

In carrying out the interview, the researcher mainly followed the methods advocated by Creswell (2008), Hancock (2006), and Kvale (1996) outlined above.

The Interview Questions

The interview aimed to determine the student’s level of engagement with Mathematics at school and determine his understanding of logarithms. The interview questions could be classified into two groups accordingly. Pre-determined questions were asked about the student’s level of engagement. The questions used to probe the student’s level of understanding were more spontaneous.

Questions Related to the Student’s Engagement

This group of questions were asked to obtain information about the student’s enjoyment and level of participation in classroom activities.

- Do you like Mathematics at school?
Do you enjoy the school classroom learning environment?
What do you do when you encounter difficulties in following your school teacher’s explanation during learning?
How active are you in your school mathematics classroom?
The response to these questions provided an initial measure of the student’s engagement in classroom activities, thus enabling the researcher to compare this level of engagement with his level of engagement in the tutoring process.

This question was asked to obtain information that a tutor needs to know:

What do you need help with in mathematics at the moment?

Questions Related to the Second Research Question
These questions are Mathematics questions given during the interview to test the student’s level of understanding in the field of logarithms. These questions had two purposes.

Firstly, they enabled the tutor to understand the level of the student’s prior knowledge so the tutor could present the material at a level that did not exceed the zone of proximal development of the student.

Secondly, the questions given in the interview were the same as those given in the test after the tutoring. This allowed the researcher to compare both results in order to exam the effectiveness of Simple Steps. However, as discussed earlier, these questions were only asked in such a way that the student would not feel stressed. The questions were therefore at a simple numerical level, for example:

What is the logarithm of 9 to base 3?

During the “interview test”, if the student did not understand a question, the researcher stopped asking any similar questions and would move on to other questions. The concept not understood was then taught during the tutoring process. If the student could answer a numerical question, the researcher asked the student a related question that was more difficult, for example:

Can you tell me what the meaning of a logarithm of $b$ to base $a$?

The following question probed the student’s understanding of related concepts:
Can you tell me, what other concepts do you need in order to define the logarithmic function?

Since the concept of logarithms is related to the concept of exponential functions, the answer to the question above showed whether the student treated logarithms as a piece of isolated information or as information linked to a knowledge network.

The style of the interview questions could give information concerning the student’s conceptual understanding of logarithms and also allow the researcher to identify the student’s misconceptions in related topics. As Porter and Masingila (2000) state: “it is not unusual to find students who use mathematical procedures with little or no understanding of concepts behind these procedures” (p. 165). Research shows that a deficiency in a student’s written or oral language can be interpreted “as a lack of understanding of the subject” (Porter et al., 2000).

**Analysis of the Interview Data**

In case studies, data collection and analysis can be carried out concurrently (Baxter et al., 2008). In this study, transcription of the recording started immediately after the interview. The interview was recorded digitally and converted to text on a computer.

To analyse the text, the researcher used methods introduced by Kvale (1996) and Creswell (2008). This method is summarised below.

The first stage was to convert the recordings into text. This included transcribing all words, then highlighting the questions and marking the corresponding answers. Next, text segments were identified by a code and a phrase was attached to each segment to describe its meaning.

Similar codes were then grouped together to form themes. These themes are also called categories (Creswell, 2008). Kvale (1996) emphasises that the purpose of obtaining themes during the analysis of the interview analysis is to reduce the large interview texts into briefer and more succinct expressions. In doing so, the researcher needs to be able to identify the essence of the text.
After these steps, the interview becomes a manageable text which can be treated as a transcription with themes and can be combined with other data from different sources for further analysis. The method for analysing transcriptions is discussed in section 3.5.1 and the Data Analysis Section.

3.4 Stage 2 - The Tutoring Process

After the first interview, the tutoring process began. During this process, the researcher made observations and field notes. Observations were made keeping the research questions in mind. In this study the researcher was a participant observer. A participant observer is an observer who may take part in the events being studied (Creswell, 2008; Yin, 2003).

3.4.1 Ethical Issues

Ethics and validity are important concepts in qualitative research. Validity will be discussed in section 3.6.2.

At the University of West Sydney, research involving human participants must have prior ethics approval and acknowledge sources. Approval for this research was obtained from the Human Research Ethics Committee, under the guidance of the researcher’s supervisors A/Professor Allan White (Principal Supervisor) and Dr Dacheng Zhao (Associate Supervisor). Approval was obtained for the interviews, making observations during tutoring and making field notes during tutoring using an audiotape.

Approval was also obtained for the process used to select an appropriate participant student. The researcher obtained consent from the student’s parent for his participation and the parent was told the level of risk to the participant (Hancock, 2006; Crewell, 2008). The potential risks included feelings of embarrassment or anxiety. At any time, the student had the right to redraw and the researcher would assure the student that he would not be upset.

3.4.2 Conducting Observations while Tutoring

This section describes how the researcher conducted the tutoring process while also observing, and how the processes of Simple Steps were incorporated.
Understanding the Material
In order for the student to understand difficult material, Simple Steps uses the technique of dividing the teaching processes into several small steps. This is a principle of direct instruction and it is supported by the theory of cognitive load. According to the definition of understanding discussed in section 3.1., instructions should be designed to ameliorate the effect of limited working memory. The tutor should ensure that the material he presents during Simple Steps does not exceed the capacity of working memory so that it “can be incorporated into automated schemas” (Marcus et al., 1996, p. 50). The tutor did this by asking the student to explain concepts and their meaning after each small step. He also recorded his observations during or after each small step of the tutoring process. These field notes were both descriptive and reflective (Creswell, 2008). They were used as the basis of the researcher’s “qualitative interpretations” of the student’s state of understanding (Denzin & Lincoln, 2005).

Engaging Students Actively in the Construction of Knowledge
This section is related to the first research question.

If all knowledge is taught in a direct instructional manner, creativity may be inhibited (Kenny, 1980) as discussed in the literature review. It may also lead to students possessing only procedural knowledge and understanding. To overcome the disadvantages of a highly structural approach, Simple Steps encourages the student to think by requiring them to give suggestions or solutions to complete some mathematical process. This requirement follows each small step, if necessary. The student may also be asked to do some exercises immediately after the introduction of a concept or a procedure. This task is related to the newly introduced concept and is not difficult. This approach is a modification of the minimal guidance approach. However, Simple Steps is different from the minimal guidance approach in that the teacher and the student work out the answers together. The teacher makes sure that the student understands the concept before the solution is provided.

As well as ensuring that the student is engaged, the researcher hypothesises that the chance of generating student’s misconceptions will be minimised when the teacher and the student are working together to explore the nature of the Mathematics.
Student Engagement

The important effect of student engagement on student’s knowledge acquisition is recognised by educators (Appleton, Christenson & Furlong, 2008; Elaine, 2003). This study uses Munns and Woodward’s definition of engagement (Munns & Woodward, 2006), namely that an engaged student is a student who:

- is reflectively involved in deep understanding and expertise (high cognition).
- genuinely values what they are doing (high emotion).
- actively participates in school and classroom activities (high behaviour).

According to Munns et al. (2006), “this definition captures student engagement as a substantive sense of satisfaction with, and a psychological investment in, the classroom work being undertaken” (p. 194).

Assessment of the Student’s Engagement

Munns et al. (2006) state that a student’s self-assessment can provide “critical feedback to teachers about whether students are engaged. Perhaps this is the only way that teachers will know” (p. 195). Elaine (2003) sees self-report as one method of measuring a student’s engagement and points out that “no one instrument is likely to be able to comprehensively assess student engagement on the entire construct dimensions listed” (pp. 3-4).

In this study, self-assessment provided by the student was the method used to measure the student’s engagement. The student was given a written questionnaire after the tutoring session. The questions were designed and modified according to the framework provided by Munns et al. (2006) to capture the “significant factor in encouraging productive student engagement with school as a place and education as a resource” (Munns et al., 2006).

Helme et al. (2001) found that self-reports could be made more reliable by combining them with other sources of data such as recorded observations. The researcher’s observations were recorded as field notes during and after the tutoring process. These field notes captured additional aspects of the student’s engagement and his understanding of the Mathematics. They will be compared and combined with the student’s self-assessment.
**Writing Field Notes during the Tutoring Process**

Since the tutoring process and interviews were digitally recorded, the purpose of writing field notes was not simply to obtain a record of everything. Rather, it was to quickly grasp important observations (Tjora, 2006; Creswell, 2008) and write reflectively about them at any moment.

In order to take field notes efficiently, the researcher chose a method of classification introduced by Tjora (2006). Tjora (2006) classified different types of observations into 10 modes: “naively describing, generalizing, interpreting, wondering, explaining, quantifying, dramatizing, experimenting, reacting and reflection, and assessing” (p. 437), each of them associated with a type of field note. In this study, five of these modes were used: naively describing, generalising, interpreting, explaining, and assessing. The use of this terminology enabled the function of each field note to be identified and classified, thus making the data easier to analyse.

Field notes associated with naively describing
Naive descriptions record what the researcher has seen. When there was a critical incident that the researcher found difficult to interpret, the native descriptions might be used, as they may provide additional information about the circumstance (Tjora, 2006).

Field notes associated with generalising descriptions
Under some circumstances, the researcher writes down a generalised description of what he sees. A generalised description is one that describes a normal or typical event (Tjora, 2006).

Field notes associated with interpreting
In this type of observation, the researcher suggests something about the student’s intentions (Tjora, 2006). He gives a direct description of what he believes to be the impact of (i.e. the effectiveness or the inadequacy) of the pedagogy he observed. For example, at sometime during the tutoring, the student showed great enthusiasm about what was being taught. The researcher interpreted this as the result of the instructional method.

Field notes of this type are particularly useful when analysing data that address the questions such as “Is the student actively engaged?” or “Is the student getting confused?”
Field notes associated with explaining
These field notes are associated with a causal process (Tjora, 2006). They explain how the student’s behaviour changes as a result of contextual changes. During the tutoring or the interviewing processes, the student may react in some way which requires the tutor to think reflectively in order to give an appropriate explanation. This provides an opportunity for the researcher to improve the pedagogical method.

Field notes associated with assessing
These field notes assess the student’s behaviour and assess different aspects of the student’s engagement. They also allow the tutor to reflect on his experience (Tjora, 2006) during the course of the tutoring or interviewing processes. They provided a rich resource in identifying the effectiveness of Simple Steps.

Organising the Field Notes
As mentioned earlier, the field notes were converted to computer text. Then text segments were identified and assigned a code or phrase to describe them. The “observation mode” attached to the code of each segment allowed the underlying context and the meaning of the field notes to be easily recognised.

Next, the codes were examined to find those that were similar. Similar codes were grouped to form “themes”. Codes that were very similar to existing codes were made redundant. The researcher also paid attention to emerging meanings from the codes and added some new codes if necessary.

The researcher then started to identify underlying answers to the research questions. These themes were called “basic themes.” The basic themes were used for further thematic analysis to be discussed in the Data Analysis Section (Attride – Stirling, 2001; Creswell, 2008; Zorn & Ruccio, 1998).

Discussion
The labelling and classification of the field notes enabled the researcher to sort this information to address the research questions. This was particularly helpful when the field notes were compared with other sources of data in a process of triangulation. This process will be discussed in Section 3.7.
3.5 Stage 3 - The Post-test and Feedback Interview

3.5.1 The Post-test

As mentioned previously, the student was tested after the tutoring process. The test was constructed to assess the student’s conceptual understanding of the topic. There were several criteria that guided the construction of the test.

The test needed to serve as: a) a measure of the student’s understanding and b) a tool to help the student improve his conceptual understanding. Therefore attention was paid to the points where misconceptions could easily occur. The test needed to allow the tutor to recognise how well the student had grasped the materials so he could discuss this with the student in the feedback interview.

The problems in the test were arranged in a sequential manner, with each basic procedure being supported by concepts. It was hoped that this sequencing of problems could help the student improve his conceptual understanding and at the same time, help him develop a solid knowledge of the symbols that carry the context.

An Example of a Problem

Given the numbers 2, 10 and 100:

a) Within the field of integers, use the operator ‘+’ ‘-’ ‘x’ or ‘÷’ to give a relationship between 10 and 100 (e.g. 10+90=100), and 2 and 10.
   Possible answers: 10 X 10=100, 10-8=2. (Other answers are possible.)

b) Within the field of exponential functions, use an equation of the form: \( y = a^x \) to give a relationship between 2, 10 and 100. Identify the name of each quantity.
   Possible answer: 100=10^2 where 2 is called the index or exponent and 10 is called the base.

c) Within the field of logarithms, give an equation to express the relationship between 2, 10 and 100. Identify the name of each quantity.
   Possible answer: \( 2=log_{10}100 \) where 2 is the logarithm, 10 is the base, and 100 is the exponential.
Questions of this type, linking conceptual knowledge and procedural knowledge, were included in the test. The key idea is that “the development of conceptual knowledge is achieved by the connection of relationships between pieces of information” (Hiebert, 1986, p. 4). When knowledge is stored as isolated item, it is not part of the conceptual network. However, when existing knowledge (such as $y = a^x$) is connected to new information (such as $x = \log_a y$), a new level of abstraction is formed and the isolated information becomes conceptual (Hiebert, 1986).

3.5.2 Data Obtained from the Feedback Interview

After the test there was a feedback interview between the student and the researcher. In preparing the interview, the researcher first marked the test but did not reveal the mark to the student. The purpose of the feedback interview was to investigate the student’s conceptual understanding of the topic. If a mistake was found in the written test, the researcher would try to identify the possible misconception and then discuss it with the student during the feedback interview.

Indeed, as the National Council of Teaching of Mathematics (2000) pointed out, an effective mathematical discourse involves observing and listening to student’s ideas and explanations about Mathematics (Walshaw & Anthony, 2008). Such a teacher-student interaction can best be created in a co-operative environment which is a “coherent world in its own right” (Denzin, 2001. p. 25). From the student’s explanation of her/his knowledge, the researcher can determine any inadequacies of the pedagogical approach used in the tutoring session. More importantly, the conversational environment provides an opportunity for the student to develop a deeper understanding of the topic because he has to make connections between pieces of information (Hiebert & Carpenter, 1992), impose meaning into procedures and make a reasonable interpretation of his solutions to the problems (Stein, Grober & Henningsen, 1996). In this way, the feedback interview became an important component of the tutoring process as well as providing information about the effectiveness of Simple Steps.

3.6 Data Analysis

This section describes the strategy used to analyse information collected from multiple sources in the process of the research and thereby interpret the data in relation to the research questions (Hancock, 2006).
There were six sources of data, namely: the initial interview, field notes made during the tutoring, the audio recording of the tutoring process, the post-test, the feedback interview, and the questionnaire.

As Cohen, Manion and Morrison (2007) pointed out: “There is no one single or correct way to analyse and present qualitative data” (p. 4). In carrying out the data analysis, the researcher combined several methods from different sources of literature, some of them overlapping. These methods were: thematic analysis, general methods for case studies; and triangulation. Their use will be discussed briefly in the following sections.

3.6.1 An Outline of the Data Analysis

The data analysis were comprised of five sequential processes, called parts. Figure 3.2 illustrates the analysis procedure. Part 1 and part 2 were parts of the initial analysis, with further analysis in part 3 and part 4. Part 5 was a triangulation. Only some of the data was chosen for the triangulation. In the triangulation process data from difference sources were compared for the purpose of establishing internal validity (Meijer et al., 2002).

In the following, the main processes will be briefly discussed. Greater detail will be given in Chapters 4 and 5. Field notes will be used to give an example of the process of analysis that was used.
Figure 3.2: The process of analysis

**Initial Analysis**

Part 1 and part 2 comprised the coding and identification of the basic themes as described by Attride-Stirling (2001) and Creswell (2008).

Part 1: Coding the Material
Transcribed data from different sources were divided into segments. A code was attached to each segment denoting the initial meaning of the segment (Creswell, 2008; Attride-Stirling, 2001).

Part 2: Identifying the Basic Themes
With the research questions in mind, the researcher went through the text segments corresponding to each code to abstract themes. This procedure enabled the researcher to identify the underlying patterns or evidence that did or did not support the hypothesis.
Further Analysis
Part 3: Categorization of Themes
Themes obtained from part 2 were refined if necessary, then arranged in groups according to certain guidelines (Attride-Stirling, 2001). Part 3 provided an opportunity for the researcher to focus attention on the conceptual detail of the data and interpret it, allowing unit 4 to be carried out more effectively (Attride-Stirling, 2001).

Part 4: Basic Descriptions
Based on the outcomes of part 3, the researcher explored the themes in full detail. As a result, explicit patterns emerged. Theoretical descriptions were also given if necessary. The results of this analysis were used to support an interpretive discussion of answers to the research questions (Farmer, et al., 2006; Creswell, 2008; Attride-Stirling, 2001).

Part 5: Triangulation
The main purpose of this part was to exam the effectiveness of Simple Steps and to provide the internal validity of the study. A comparison of data from different sources was used to identify to what extent they agreed with each other. For example, the test before the tutoring process and after the tutoring was compared to demonstrate the effectiveness of Simple Steps.

3.6.2 Validity and Limitation of the Study
According to Creswell (2008) and Golafshani (2003), validity means that the data obtained from the various instruments are meaningful and enable the researcher to draw useful conclusions. Of major concern were the biases related to the role of the researcher who could bring his own predispositions to the findings or results (Hancock, 2006). One method of reducing this effect is “member checking” (Creswell, 2008). The researcher brings the finding or the transcription of the interview back to the participant student to ask him “about the accuracy of the report” (Creswell, 2008, p. 267).

Furthermore, since most part of the process of tutoring was audio-recorded, the research supervisors could exam the recording to verify that the researcher had transcribed and reported the research interviews and observations correctly.
In order to enhance the internal validity of this research, the method of triangulation was also used (Meijer, Verloop & Beijaard, 2002). If these data from different sources agree with each other, their convergence can be interpreted as a measure of validity (Moran-Ellis et al., 2006; Golafshani, 2003).

Another concern about this study is the tutoring context. This context places a limit on the time that the tutor can allow to fully explore every aspect of the learning and to regularly check the student understands. There should also be more time and space for the student to explore situations to which the concepts of logarithms could be applied. This could result in a deeper understanding of the subject.
Chapter 4

Results of Data Collection and Initial Analysis

This chapter presents the results of the data collection phase and the initial analysis of the data. As mentioned earlier, the study is divided into three stages, namely: an interview before the tutoring process, the tutoring process, and a test after the tutoring process together with an interview with the student about his performance in this test.

There were six sources of data from the study. These are described in Table 4.1. A preliminary analysis of these data will be made in this chapter. A more thorough and detailed analysis and evaluation will be made in Chapter 5.

Table 4.1
Sources of Data

<table>
<thead>
<tr>
<th>Source of Data</th>
<th>Collection Stage</th>
<th>Main purpose</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interview transcription</td>
<td>Before the tutoring</td>
<td>To obtain information about the student’s initial understanding of the mathematics and the student’s understanding after the tutoring process</td>
</tr>
<tr>
<td></td>
<td>After the tutoring</td>
<td></td>
</tr>
<tr>
<td>Field notes</td>
<td>Obtained mainly during and after the tutoring sessions</td>
<td>To monitor the interaction between the tutor and the student</td>
</tr>
<tr>
<td>Audio recording of the tutoring process</td>
<td>During tutoring sessions</td>
<td>To record the conversations during the tutoring process</td>
</tr>
<tr>
<td>Written test</td>
<td>After tutoring sessions</td>
<td>To examine the student’s understanding of the mathematics as a result of the tutoring</td>
</tr>
<tr>
<td>Interview transcription</td>
<td>After the test</td>
<td>To understand how the student perceived his performance in the test</td>
</tr>
<tr>
<td>Questionnaire</td>
<td>After feedback interview</td>
<td>To assess the students’ engagement during the tutoring sessions</td>
</tr>
</tbody>
</table>

4.1 Data Collected at Stage 1: The Initial Analysis of the Data

4.1.1 A Brief Discussion

In this chapter, the transcriptions of interviews and field notes will be presented in the form of tables to make a first step toward the data analysis. This style of approach is a modification
of an approach described by Creswell (2008). It allows the researcher to present the data with the initial analysis simultaneously.

As discussed in section 3.6.1, the process of data analysis is divided into five parts. The first part is to code the data. The second part is to identify the basic themes. These steps were discussed in section 3.7.1. The method of coding is that described by Creswell (2008) and Attride-Stirling (2001). The purpose here is to dissect the text data into segments. A theme describing the meaning of the text is then attached to each code (Creswell, 2008). The codes are written in the left column of each table and the themes are written in the right column as shown in Table 4.2. The same codes will be used in next chapter to identify the original source of the text, thereby maintaining transparency as well as being tools for further analysis.

The process of abstracting meaning from text segments and identifying salient episodes is guided by the codes. In carrying out the process, the text is reread. Ideas, patterns, and structures are initially identified as themes or word groups (Creswell, 2008; Attride-stirling, 2001; Bernard et al., 2010). These themes or ideas provide the initial descriptions of the data but they are not fixed and will be further refined in the latter steps of analysis if necessary (Creswell, 2008; Attride-Stirling, 2001; Bernard et al., 2010).

4.1.2 The Interview before the Tutoring Process

As discussed in section 3.3.2, the interview between the researcher and student was conducted before the tutoring process. The purpose of the interview was to obtain information about the student’s understanding of mathematics and his engagement in classroom activities at school.

The interview was conducted in an elucidated manner by posing predetermined questions. It was recorded and transcribed and was mainly analysed using the thematic approach. This analysis will be presented in the next chapter.

The interview conducted before the Simple Steps tutoring process consisted of two sections. In the first section the student was questioned about his engagement in school classroom activities. The second section consisted of several mathematics questions that assessed the student’s understanding of the topic of logarithms. These questions were chosen from the
prepared test that would be given after the tutoring process. The results of the test therefore measured the effectiveness of the tutoring process.

**Interview Segments: Section 1**

The data presented in the centre of Table 4.2 are segments selected from the first part of the interview. Segments of the interview which were repeated or irrelevant have been excluded. As previously mentioned, coding was carried out first. The codes are presented in the left column of the table. The code “In1.1 Liking of math” for example, means the text is the first segment from interview 1 and the comment is a summary of the key idea of the statement. The basic theme and/or ideas about this interview segment are given in the right column.

**Table 4.2**

Interview segments: coding and basic theme identifying

<table>
<thead>
<tr>
<th>Code</th>
<th>Interview segment</th>
<th>Basic themes and ideas</th>
</tr>
</thead>
<tbody>
<tr>
<td>In1.1 Liking of maths</td>
<td>Tutor: Do you like mathematics at school?</td>
<td>Finds maths interesting</td>
</tr>
<tr>
<td></td>
<td>Student: Yes.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Tutor: Why?</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Student: It is interesting and engaging. It helps to</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>relate math to society life.</td>
</tr>
<tr>
<td>In1.2 Desire to learn</td>
<td>Tutor: What is your goal for coming to this tutoring?</td>
<td>Wants to know more about logarithms</td>
</tr>
<tr>
<td></td>
<td>Student: To understand more about logarithm.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Tutor: In your environment, does your teacher</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>emphasis concept or procedure?</td>
</tr>
<tr>
<td></td>
<td>Student: Ah, no, not quite. (Here the student didn’t</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>answer the question clearly; I should have</td>
</tr>
<tr>
<td></td>
<td></td>
<td>asked him to further clarify this question.</td>
</tr>
<tr>
<td>In1.3 Classroom environment</td>
<td>Tutor: Do you enjoy the classroom learning environment in school?</td>
<td>Does not enjoy classroom learning environment</td>
</tr>
<tr>
<td></td>
<td>Student: Ah, not, not quite.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Tutor: Can you tell me why?</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Student: It’s too small, lack of equipment, like the</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>blackboard whiteboard is not big enough</td>
</tr>
<tr>
<td></td>
<td></td>
<td>for a big example ... and poorly</td>
</tr>
<tr>
<td></td>
<td></td>
<td>renovated.</td>
</tr>
<tr>
<td>In1.4 Need for help</td>
<td>Tutor: What do you need help with in mathematics at the moment?</td>
<td>Concentrates but does not ask questions</td>
</tr>
<tr>
<td></td>
<td>Student: Understanding of the concepts of particular</td>
<td></td>
</tr>
<tr>
<td></td>
<td>topics.</td>
<td></td>
</tr>
</tbody>
</table>
**Interview Segments: Section 2**

The second part of the interview was a simple mathematics test. The questions were given orally rather than as a formal test. This interview is transcribed in Table 4.3. Note that the
parts of the interview involving detailed calculations have been replaced by descriptions to clarify the results.

Table 4.3
Interview segments: coding and basic theme identifying

<table>
<thead>
<tr>
<th>Code</th>
<th>Interview segment</th>
<th>Basic theme and ideas</th>
</tr>
</thead>
</table>
| In1.6 Exponential functions | Tutor: The following questions are about your knowledge of mathematics, in particular about current understanding of logarithm. (Pointing to the graph of \( y = 2^x \)) Do you know the name of this function?  
Student: No.                                                                 | Does not realise the connection between logarithms and exponential functions          |
| In1.7 Small test   | Tutor: This is exponential function. You should have learned it. What textbook are you using? (In order for the student not to be frustrated, I gave a very brief explanation of exponential functions, then the test continued)  
Student: Yes. Er, I’ve heard of it before.  
Tutor: But you told me you have had some advanced lessons on logarithms.  
Student: Yes.  
Tutor: Do you remember how to do it? Try.  
Student: I can try.  
Tutor: So we start at... \( 9 = 3^2 \), can you calculate the logarithm of 9 to base 3?  
Student: Er, yes. (The student calculated it correctly). \( \log_3{9} = 2 \)  
Tutor: Good, so you understand them. That is good. | Can solve simple questions  
Notation change causes difficulty  
Can calculate logarithms correctly |
| In1.8 Misunderstanding | The next question was: Within the field of logarithms, give an equation to express the relationship between 2, 10 and 100, identify the names for each quantity.  
The student incorrectly wrote: \( \log_2{100} = 10 \)  
He could calculate the first question, which was basically the same as this one but failed to solve the other | Feel confused  
Failed to solve when presentation changed. |
only because the presentation has changed. I concluded that he was guessing rather than calculating.
So I said :
So you confused here.

Student: Yes.

Next the student attempted question 2 which contained parts a), b), and c) as illustrated bellow.

a) Express $9 \times 27$ in the form $a^{n \cdot m}$ using 3 as the base. Show you working.

b) Using the result from part a), find a possible relationship between 243, 3 and 5 within the field of logarithms.

c) Using the result from a) and b) to show

$$\log_3(9 \times 27) = \log_3 9 + \log_3 27$$

Show your working.
The student can calculate a), but not b) and c).
When I pointed to part c), and asked:

Tutor: Do you have any idea to simplify this question?

Student: Em, no. not like that.

Tutor: OK, we stop here.

The conversational segments above show information that was collected at the beginning of the study and a preliminary analysis of this data. In order to link and develop patterns and ideas coherently and reliably throughout the analysis, notes were made (Coyne & Cowley, 2006; Bernhard, et al., 2010). They were labeled, for example, In#Note1, where “In” refers to “interview”.

Notes and Ideas (In#Note1)
The student came to the tutoring session to gain a better understanding of logarithms. According to him, he was not actively engaged in the classroom learning (see In1.5). Despite the fact that he had concentrated on the teacher’s explanation, he had no chance to ask questions until the end of the lessons (see In1.5). As a result, he was passively taught. This can be seen from the fact that he did his test by guessing rather than understanding the underpinning mathematical meaning. (See In1.8)
Having noted this, the tutoring process started with revision to enable the student to link new to existing knowledge, rather than launch directly into the topic of logarithms.

4.2 Data Collected at Stage 2: the Tutoring Process

The tutoring process began at stage 2. The process used is known as the Simple Steps process. It is an experimental pedagogical approach and is divided into five steps. During the tutoring process, two major types of data were collected: an audio tape of the tutoring process and field notes. Only a brief description of these data is given here. They will be fully explored in the next chapter.

4.2.1 Recording of the Tutoring Process

The audio recording and the tutor's field notes will be analysed together because these data complement each other.

4.2.2 Field notes of the Tutoring Process

Field notes were made during and after each step of the Simple Steps tutoring process. The field notes are summarised in the following tables.

Field Notes: Step 1 of the Simple Steps

The purpose of this step is to bring the learner to the prerequisite level. Table 4.4 describes this step. Here “F1.1 Exponential function” is the code for field notes of step 1 of Simple Steps, number 1”.

<table>
<thead>
<tr>
<th>Codes</th>
<th>Field notes</th>
<th>Themes and ideas</th>
</tr>
</thead>
<tbody>
<tr>
<td>F1.1 Exponential function</td>
<td>The topic to be introduced is the logarithmic function $y = \log_a x$. The revision started with the exponential function $y = a^x$. Since the logarithmic function is the inverse function of the exponential function. When I talked to the student, I found that his conception about exponential functions was quite unclear. So I started the tutoring from the properties of exponential functions. I pointed to the graph of $y = a^x$. Tutor: To understand logarithms, must understand the</td>
<td>Conception about logarithms and exponential functions unclear</td>
</tr>
</tbody>
</table>
concept of exponential function $y = a^x$.

I then asked him to talk about the condition of “$a$”.

Student: Ah, “$a$” cannot be...

I pointed to the graph of $y = a^x$, allowing the student to recall the conditions for $a$ from the graph. Then we said the same words almost at the same time: “$a$ must be larger than 0. That is the condition”.

In fact, there are two condition for “$a$”, namely: $0 < a < 1$, and $a > 1$. But I had to state them step by step.

Why “$a$” must be larger than zero is not easy for a beginner to understand. It is reasonable to take it as a problem for the student to think about. However, it would be unwise to introduce so many concepts at the beginning because of the large cognitive load. Accepting the principle from direct instruction, I input a small amount of information into the student’s memory in a step by step manner and allowed the student to “discover” the reason through guiding questions.

Tutor: So $a > 0$ is one of the conditions, what is the other?

(18:33)

Student: $0 < a < 1$.

Tutor: How can we plot $y = a^x$ for $a > 0$?

Student: Em.

F1.2 Sketching a graph

I realised that it would be difficult for a student to handle an abstract question without knowing the graph of the function. To understand the graph, I suggested the student replace “$a$” with a number, say “2” in $y = a^x$. The student sketched the graph of $y = 2^x$ for $a > 1$ successfully as shown in figure 4.1:

![Graph of $y = 2^x$](image)

Figure 4.1

F1.3 Simple illustration

Tutor: Now, what happen if $0 < a < 1$?

Student: Em, not sure.

I realised it would be helpful for the student to learn to Convert an abstract concept into a...
convert an abstract concept into numbers that can be graphed.
Tutor: Why not replace “a” with a number? For example, we choose \( a = \frac{1}{2} \).
After replacing “a” with \( \frac{1}{2} \), the student could sketch the graph correctly and could discuss the fundamental difference between \( y = ax \) and \( y = (\frac{1}{a})^x \).
Student: The graph is symmetric about the Y–axis…
He said confidently. To me, this was the first step toward understanding. The simple illustration \( y = 2^x \) and \( y = \left(\frac{1}{2}\right)^x \) had helped him to grasp the concept of an exponential function.

| F1.4 Reasoning difficulty | After the revision of exponential functions, we discussed the concept of an inverse function: Tutor: Have you learned the concept of inverse? Student: Yes.
Tutor: What is it?
Student: Em, the opposite, the opposite of…
To me, opposite was an ambiguous description of inverse, an ambiguity could probably be the sign of misconception. When I talked to the student I found that he was confused with the notions \( y = ax \) and \( x = ay \). This misconception had blocked his understanding. |
| F1.5 Misconception | I considered one of the best ways to remove the student’s misconception was to provide an opportunity for the student to reflect on what she/he had done. Instead of explaining theoretically, I asked: Tutor: What do you know about inverse? Give me an example.
The student could not give any example. So I suggested:
Tutor: OK, give me an example.
I pointed to the graph of \( y = 2^x \) for \( a>1 \) and asked:
Tutor: What is the graph of its inverse?
Student: Em, flip.
The student was silent. Later on I found that he could not even work out the inverse of a very basic form of a given function.
It seems to me that the unfamiliarity with the concept of function and its graph \( y = f(x) \) might have resulted in the uncertainties of the concepts of “symmetry” (see F1.3), “opposite” (see F1.4), and “flip”. |
| F1.6 Visualize concepts | Even though I could not recognise the exact misconception, I used a simpler function to enable the student to integrate the concept of functions, inverses, and graph |
their graphs as illustrated here. Firstly, the function should be as simple as possible. So I used the function

$$y = 2x \quad (1)$$

which was compared with the meaning of a function:

$$y = 2x \Rightarrow y = f(x) \quad (2)$$

Where \(x\), usually used as the independent variable, is the input of the function into \(f(x)\), and \(y\), usually denoted as the dependent variable, will be the output of the function. I also emphasised that the independent roles between \(x\) and \(y\) could exchange depending on what situation we were interested.

Figure 4.2 was the summary by the student about the concept of function using \(x\) as independent variable and \(y\) as dependent variable.

![Figure 4.2](image)

Note that he had used a special statement (See A) “\(y\) is \(f(x)\)” to remind himself that \(y\) was an output of the function \(f(x)\). Simple as this summary could be, it could be a sign of understanding as can be seen in the following section.

<table>
<thead>
<tr>
<th>F1.7 Crucial Step</th>
<th>After explaining the meaning of function, I asked:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tutor: We say (y) is a dependent variable or the function of (x) while (x) is called the independent variable. Now, what is an inverse?</td>
<td></td>
</tr>
<tr>
<td>Student: (x) is the function of (y)?</td>
<td></td>
</tr>
<tr>
<td>Tutor: Yes. The student started to understand the concept of “inverse function”.</td>
<td></td>
</tr>
<tr>
<td>Next, the student was asked to solve equation (1), expressing (x) as a function of (y):</td>
<td></td>
</tr>
<tr>
<td>(x = \frac{1}{2}y) \quad (2)</td>
<td></td>
</tr>
<tr>
<td>Now the crucial steps came in: In order to study how (x)</td>
<td></td>
</tr>
</tbody>
</table>

Vehicle that carries targeted concept

Started to understand the concept

Understand the meaning of notions

F1.7 Crucial Step
varies as a function of $y$, the positions of $x$ and $y$ are exchanged so that $x$ becomes the dependent variable instead of $y$. So equation (2) was changed to: (by the student)

$$y = \frac{1}{2}x$$

(3)

I told the student that this was the inverse of $y = 2x$. His expression gave me the impression that he had never before thought the meaning of inverse to be so rich. I considered this as a crucial step towards a deep understanding of the concepts of inverse.

**F1.8 True reinforcement**

In order to achieve automated schema acquisition, the student was always required to conduct problem solving after any new concept was taught. This time, the problem was to find the inverse of

$$y = 2x - 1$$

(4)

which was a bid harder than equation (1). He worked out the solution with great easy and I was really happy, I said: T: *wow, too quick! ... Yes that is it!* I could not help praising his progress I considered it would be a perfect time for the student to reflect on what he had learned out the moment of successfully attempting a problem solving, I said: T: Do you need to write out something? S: OK (he wrote some summaries) T: Are you sure you understand? S: Yes!

His voice was full of confidence. This was the most confident response from him since the tutoring process has begun. I believed that this was the result of his progression in the path toward understanding as well as the happy feedback from the tutor, the true reinforcement and encouragement received by the learner that could actively engage the student in learning (Mayer, 1993) as can be seen in the following section.

**F1.9 Plotting**

Since the student was in a moment of enjoyment of study, I decided to push his level of understanding one level up. I asked him to investigate the graphical difference between the function $y = 2x$ and its inverse by plotting values for $x$ and $y$. The student carried out the task confidently. I was very happy:

Tutor: Do you know what you are doing?
Student: Finding the inverse about…
Tutor: *You haven’t done this in detail before?*
Student: Not much, not in full detail.
Tutor: You should understand it at the beginning of year 11.
Student: Maybe we forgot.
The graph plotted by the student is shown in figure 4.3.

Figure 4.3

Tutor: Now, what can you say about these? You said symmetry about ...
Student: Symmetry about the straight line $y = x$.
Working with a simpler function led to understanding.

<table>
<thead>
<tr>
<th>F1.10 Higher abstraction</th>
<th>Introduction of abstract notion after grasping basic ideas</th>
</tr>
</thead>
<tbody>
<tr>
<td>Next, we started the discussion of function and inverse function using the more abstract notion $y = f(x)$ and $y = f^{-1}(x)$. The student was able to answer my questions correctly and I realised that he had already grasped the basic concept of inverse function. The simple function $y = 2x$ had helped him to overcome his reasoning difficulties to reach an understanding of the definition of an inverse function.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>F1.11 Problem Solving</th>
<th>Chance of automated schema construction</th>
</tr>
</thead>
<tbody>
<tr>
<td>In order to strengthen the newly learned knowledge, the student was required to plot other functions and their inverses. This was to help the student develop an automated schema construction. One such problem was to plot the graph of the inverse of the exponential function $y = 2^x$. The student sketched the graph successfully even though he didn’t realize that it was the graph of a logarithm! This will be discussed again in step 3.</td>
<td></td>
</tr>
<tr>
<td>Can plot a harder graph</td>
<td></td>
</tr>
</tbody>
</table>

Note and Ideas (F1#Note 1)
If the concepts of logarithms and exponential functions are not seen as interrelated information, they cannot be fully understood (see F1.1). Therefore the revision should start at the weak point of the student’s understanding of the topic. Conversation had provided good information about the student’s mathematical background (see In1.6 and In1.7). One of the most important purposes of revision is to clear misconceptions that could block the student
from understanding (see F1.4). One of the ways that might help the tutor detect the misconception is by setting up a simple problem then working on it together with the student who will be required to answer questions verbally or through calculations (see F1.7 and F1.8). When the student uses knowledge which was incorrectly constructed to answer questions, mistakes occur. This provides opportunities for the tutor to recognise the misconception and help the student to overcome it (see F1.4, F1.6, and F1.7).

A question that arises here: is it possible to develop a practical model to guide the student along the right track of understanding by starting with a numerical example?

**Field Notes: Step 2 of the Simple Steps**

After the revision, a new definition or concept is introduced but in a simple fashion. Table 4.6 illustrates this process.

<table>
<thead>
<tr>
<th>Codes</th>
<th>Field notes</th>
<th>Basic themes and ideas</th>
</tr>
</thead>
<tbody>
<tr>
<td>F2.1 Numeric</td>
<td>Instead of directly introducing the general notion of logarithms ( y = \log_a x ), I first used a simple illustration of an exponential ( (8 = 2^3) ) written as a logarithm. I thought it would be helpful to use a numeric example to explain the meaning of an algebraic structure in a step by step manner before it was abstracted.</td>
<td>Creating a numeric example before abstract concepts</td>
</tr>
</tbody>
</table>
| Simple         | For example, to illustrate how \( 8 = 2^3 \) is related to the idea of logarithm, I explained: Tutor: We say that \( 3 \) is the logarithm of \( 8 \) with base \( 2 \), and is written: \[
\begin{align*}
3 &= \log_2 8 \\
1000 &= 10^3
\end{align*}
\]
To ensure the student had understood the idea, I asked the student to express as a logarithm, the relation \( 1000 = 10^3 \).

The purpose of this type of question is to automate the knowledge. The student wrote down the answer effortlessly:

\[
3 = \log_{10} 1000
\]

This was the same question he failed to solve in the initial test. This example shows that he is now reasoning correctly. | Using simple expression to introduce concepts                        |
| illustration   |                                                                                           | Good example showing correct reasoning                       |
| F2.3           | To connect the newly understood definition of a                                         | Connecting new                                               |
Connecting knowledge to the concept of inverse functions previously learned, I said:

Tutor: So the value of logarithms is, in fact, the exponential. The inverse of exponential is log, and the inverse of log is ...

Student: Exponential

His answer was correct. Just through one simple illustration, he got the idea! He was ready for a higher level of learning of the topic.

Note and Ideas (F2#note1)

Using a simple illustration is a direct application of Bruner’s (1964) theory of cognitive growth. It is helpful to use a simple illustration to explain a complicated concept or topic in a step by step manner before it is abstracted (see F2.2). If a simple illustration is properly designed such as \(8 = 2^3\) and \(3 = \log_2 8\), it actively engages the learner because the learner understands it.

Field Notes: Step 3 of the Simple Steps

Now the student is ready to move to a more abstract and advanced understanding of logarithms. Table 4.7 illustrates this process.

Table 4.6

Field Notes for Step 3

<table>
<thead>
<tr>
<th>Codes</th>
<th>Field Notes</th>
<th>Basic themes and Ideas</th>
</tr>
</thead>
<tbody>
<tr>
<td>F3.1 Abstraction</td>
<td>In these steps, all concepts and topics introduced in the previous steps will be discussed at a more advanced level. Elements or expressions in the previous steps will be replaced by a more general and abstract form. After some basic review, the lesson began.</td>
<td>Replacing numbers with letters</td>
</tr>
<tr>
<td>F3.2 Gradual change</td>
<td>The approach here was to define logarithms as the inverse of (y = a^x). This could have been done easily by presenting (x = a^y) and asking the student to express the relation as a logarithm as illustrated in step 2. The answer is: (y = \log_a x) (1) as I had planned to do earlier. But I changed my idea during this study. To get an in-depth understanding of logarithms, the student should be involved in an active learning process. When I asked the student how to generalise logarithms</td>
<td></td>
</tr>
</tbody>
</table>
from cases of numbers by using a more abstract notion, the student answered:

Student: Em, generalise to a function? He seemed to understand what I meant but had no idea what to do. I decided to replace the numbers with an abstract symbol gradually. The purpose was twofold: Firstly it was hoped that the gradual change of the presentation could highlight the effect of different numbers in an expression and thus enable the student to think actively about what was going on. Secondly, it was also hoped that the gradual changing of the representation using simple illustrations could provide opportunities for the student to see how a mathematical concept could be analyzed.

In order to visualize how the function $y = a^x$ was related to the logarithm $y = \log_a x$, I first replaced $a$ in $y = a^x$ with 2 to get

$$y = 2^x \quad (2)$$

This allowed the student to work on it directly.

Next, the student was asked to find its inverse function. But he seemed not to know how to carry out at first. Thus, the student was required to review what he had learned in step 1 and connect the new information with the existing knowledge. He looked at the graph of $y = 2^x$ and its inverse he had plotted in step 1. The inverse function was not called a logarithm at that time as mentioned earlier. He looked and thought. Then something interesting happened. Firstly, he exchanged the position of $x$ and $y$ in $y = 2^x$ and wrote $x = 2^y$ near the right side of the graph as shown in figure 4.4a (See A).

![Figure 4.4a](image)

**F3.3 Articulation**

- Replace numbers with symbols gradually.
- To think actively on what was going on
- To see how a mathematical concept could be analyzed
- Finding connections between knowledge
- Visualize the relationships between related functions
Note that the expression on the right hand side of the graph, as shown in Figure 4.4b.

![Figure 4.4b](image)

Using example to think about the properties of logarithms

Probing the student to reflect

Please note that the question was written by me, the student was probed to reflect on what he had done, what was that meant?

He looked at what he wrote previously (See figure 4.3a B):

- The graph of $y = f(x)$ and the inverse $y = f^{-1}(x)$ is symmetry about the straight line $y = x$

Then he remembered that to get the inverse function of $y = f(x)$, $y$ and $x$ must exchange their position so he start from

$$y = 2^x$$  \hspace{1cm} (4)

then changed to

$$x = 2^y$$  \hspace{1cm} (5)

As indicated in figure 4.3b. It was interesting to note that equation (5) had in fact given the correct name of the graph being investigated.

F3.4

This was an amazing discovery to me, since the expression $x = 2^y$ had illustrated the exact meaning of the logarithm of $x$ to the base 2: where the sign “$\Rightarrow$” at “$x$” emphasises that it is the input of the inverse and the sign “$\Leftarrow$” at “$y$” indicates that it is the output of the inverse function, even though the student didn’t realise this.

Secondly, by observing the simple illustration

$$8 = 2^3 \Rightarrow 3 = \log_2 8$$  \hspace{1cm} (3)

and

$$1000 = 10^3 \Rightarrow 3 = \log_{10} 1000$$  \hspace{1cm} (4)

He was able to identify

$$x = 2^y \text{ as } y = \log_2 x$$  \hspace{1cm} (5)

This is successful reasoning using analogy.

Next step is to generalise the expression $y = \log_2 x$ to a general form. The discussion started at the exponential function again

$$y = a^x$$  \hspace{1cm} (6)
The student obviously understood what I wanted him to do. So he exchanged the position of \( x \) and \( y \) in equation (6) to get
\[
x = a^y
\tag{7}
\]
The next step was to express this relationship in terms of logarithms but he made a mistake. He wrote:
\[
x = \log_a y
\tag{8}
\]
This shows that even after a concept is correctly learned, it could still be very fragile and weak (Heibert et al., 1992).

| \textbf{F3.5} | \textbf{Immediate feedback} | I told him that the answer was incorrect, and guided him to compare the equations (3), (4), (5), with (6), (7), (8) and Figure 4.3a; the student recognised the mistake in equation (8), and wrote it correctly: \[
y = \log_a x
\tag{9}
\] |
| \textbf{F3.6} | \textbf{Another condition} | To give a complete definition of the logarithmic function, another condition \[
y = \log_a x \text{ for } 0 < a < 1.
\] was needed to be addressed. The method used was similar to the case when \( a > 1 \). After guided practice using several numerical examples, the introduction to the general notion of logarithms was completed. Simple numerical illustrations had been used as tools to allow the student to articulate the notion of the topic. It may be difficult to identify where the misconception comes from, but it is still possible to notice and correct it when it occurs. |

Note and Ideas (F3#note 1)

In teaching the notion of logarithms, I could have started directly from the relationship between \( 8 = 2^3 \) and \( 3 = \log_2 8 \) then moved to \( x = a^y \) and \( y = \log_a x \) as I had planned before the tutoring process. But I did not choose to do so, since the understanding of a mathematical structure does not mean that the concept is conceptually learned. According to Hiebert et al. (1992) and Ausble (1968) (as quoted by Hiebert et al., 1992), a student’s prior knowledge will fundamentally influence “what they learn and how they perform” (Hiebert et al., 1992, p. 80). From this point of view, a properly designed simple illustration can help the student to connect a basic concept previously learned (inverse functions) to a relevant topic being studied (see F3.3, F3.4 and F3.5).
The fact that the student used simple numerical illustrations to help himself reason (see F3.4) had inspired me to consider: is it possible to develop the idea of simple numerical illustrations as a “pedagogical strategy”? When reasoning difficulties or misconceptions are encountered in algebra, a student might use a numerical illustration to remind himself of some aspects of a concept and thereby reduce the chance of forming misconceptions (see F3.3, F3.4). From the point of view of a tutor, it would be easy for the tutor to explain what was wrong using a simple numerical illustration already familiar to the student (see F3.5).

(F3#note 2). In this step together with step 2 of Simple Steps, the principle of direct instruction and discovery learning had been used. Bruner’s theory of knowledge representation had also been directly applied to the conceptualisation of logarithms.

**Field Notes: Step 4 of the Simple Steps**

The purpose of this step is to study the basic calculation procedure related to logarithms. This is the domain referred to as procedural knowledge which is “knowing how” or “the knowledge of the steps required attaining various goals” (Byrnes & Wasik, 1991, p. 777). One of the critical considerations is how to introduce complicated formulas while avoiding passive learning.

Table 4.7

<table>
<thead>
<tr>
<th>Codes</th>
<th>Field Notes</th>
<th>Themes and Ideas</th>
</tr>
</thead>
<tbody>
<tr>
<td>F4.1</td>
<td>The first formula to be taught was:</td>
<td>Presence of x caused confusion</td>
</tr>
<tr>
<td>Presenting</td>
<td>[ \log_a xy = \log_a x + \log_a y ] (1)</td>
<td></td>
</tr>
<tr>
<td>formula</td>
<td>Instead of directly teaching the formula, I first asked the student to prove</td>
<td></td>
</tr>
<tr>
<td></td>
<td>the following expression: [ \log_3(9 \times 27) = \log_3 9 + \log_3 27 ] (2)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>The student started the problem solving by constructing an equation:</td>
<td></td>
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<tr>
<td></td>
<td>[ x = \log_3 243 ] (3)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>in an attempt to interpret the meaning of the equation (2).</td>
<td></td>
</tr>
<tr>
<td></td>
<td>The idea was great because he could have finished the proof in no more than</td>
<td></td>
</tr>
<tr>
<td></td>
<td>three steps, but he stopped and thought hard. I realised that the presence</td>
<td></td>
</tr>
<tr>
<td></td>
<td>of x had confused him so he could not recognise the conceptual relationship</td>
<td></td>
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<tr>
<td></td>
<td>between the quantities in the expressions.</td>
<td></td>
</tr>
</tbody>
</table>

| F4.2      | After I suggested the student concentrate on numbers, he made great progress. | Recognise similar patterns        |
| Cognitive | By comparing equation (2) with \[ \log_2 8 = 3 \]                           |                                   |
| conflict  |                                                                            |                                   |
he realised that the \((9 \times 27)\) could be treated in the same way as the \(8\) so he wrote:
\[
\log_3(9 \times 27) = \log_3(3^2 \times 3^3) = \log_3 3^5 = 5
\]
(4)
as indicated in figure 4.4. Note that he had used intuition to obtain “5” in the last step, but then he stopped, wondering how to carry out the next step.

Figure 4.5

Notice how close a simple numerical illustration could result in correct reasoning at an abstract level:
Letting \(x = a^n\) and \(y = a^m\), by definition, we have
\[
\log_a x = n \quad \text{and} \quad \log_a y = m
\]
Now:
\[
x y = a^{n+m}
\]
(5)
By definition
\[
\log_a xy = n + m = \log_a x + \log_a y
\]
(6)
Done!

Note that also the equation (3) given by the student was basically equation (5), since \(243 = 9 \times 27\), from equation (3), we had:
\[
9 \times 27 = 3^x
\]
and equation (4) was also basically equation (6).
Thus, if the teaching of equation (1) had started from equation (5), the student could have learned the formula more directly but would have missed the opportunity of experiencing the process of disequilibrium or cognitive conflict, or the process of problem solving in this case: a “must” for the development and acquisition of schema as discussed in the literature review.
In his calculations above, \(\log_3 3^5 = 5\) is correct but the notation he used (see Figure 4.5)
\[
\log_{3^5} = 5
\]
(7)
was ambiguous. So I pointed to it and said:
T: Yes, correct, but notation …
S: Um
T: Confused?
S: Confused, I’m not sure how to take this (\(\log_{3^5}\)) to that (5).
T: Use definitions.
In fact there is an expression
\[
\log_a a^x = x
\]
that solves the puzzle, but I really wanted him to obtain it by himself. Because he almost touched it.
In order to teach this formula, I first showed that from the fact that $8 = 2^3$, one could directly obtain 3 using definition: $\log_2 2^3 = \log_2 8 = 3$.

Then I asked the student to explain why he could have

$$\log_3 3^5 = 5 \quad (8)$$

I expected he would answer the question using similar argument but he used another approach as indicated in the following. From what he got before

$$x = \log_3 243 = \log_3 3^5 \quad (9)$$

and using the definition of logarithms he got:

$$243 = 3^5 = 3^x \quad (10)$$

So

$$x = 5$$

Thus, the student could explain why $\log_3 3^5 = 5$ logically. This case showed that a teacher should not be just content with what a student gets, (such as $\log_3 3^5 = 5$ ) but also be concerned with how the answer was obtained. It is hoped that this cautiousness could reduce the chance of misconceptions.

Inspired by what the student got, I immediately set up several questions for the student to guide him towards obtaining the general formula as well as for him to acquire the schema. These questions were: to find the values of $\log_{10} 1000$ and $\log_6 216$

The student answered these questions easily:

$$\log_{10} 1000 = 3 \quad \text{and} \quad \log_6 216 = \log_6 3^3 = 3$$

According to these answers, the student was able to summarize them as a general formula:

$$\log_a a^x = x \quad (11)$$

As shown in figure 4.6 (see A):
equation (2) and continued the guided practice:
\[ \log_3 9 = \log_3 3^2 \]
\[ \log_3 27 = \log_3 3^3 \]
And thus finished the proof:
\[ \log_3 (9 \times 27) = \log_3 9 + \log_3 27 = 2 + 3 = 5 \]
When the student was asked to generalise the result of equation (2), he replaced 9 and 27 with \( x \) and \( y \) to get:
\[ \log_a xy = \log_a x + \log_a y \quad (1) \]
Here, I did not approach the formula using abstract notation because of the mathematical background of the student. I thought it was important to allow the learning of complex topics to progress in steps that were accessible to the learner. Since in stage 6 mathematics many topics usually involve several basic formulas, teachers need to treat the topic flexibly.

Approach the formula via steps accessible to the learner

F4.5 Skills of manipulating pattern

Once obtaining equation (1), we were going to make equation (11) \( \log_a a^x = x \) more general:
\[ \log_a x^n = n \log_a x \quad (12) \]
by continuing the study of equations (7), (8), (9), (10) and (11).
After my suggestion of using the numeric relation \( 8 = 2^3 \) the student replaced “3” with \( x \) and constructed a set of patterns in an attempt to articulate the meaning of algebraic expressions:
\[ 8 = 2^3 \rightarrow 8 = 2^x \quad (13) \]
\[ \log_2 2^3 = 3 \rightarrow \log_2 8 = x \log_2 \quad (14) \]
Note that he used \( \log \) instead of using \( \log_2 \), indicating that he was interested in the general ideas of logarithms. (It should be noted in mathematics that \( \log \) assumes a base of 10). However, he should have used “x” to replace “3” rather than “2” in “\( 8 = 2^3 \)”. Then he could make the reasoning easier.
So I told him that what he needed was to identify the corresponding variable correctly when using numeric illustrations to visualise abstract expressions.
Next, I and the student worked together following his track of reasoning. During the process, the student was encouraged to express his thinking using algebraic expressions which also enabled the tutor to monitor whether the student was on the right track of calculation.
Further discussion then brought us back to the equation
\[ y = \log_a x \quad (15) \]
and
\[ a^y = x \quad (16) \]
taking both sides to the power of \( n \), he obtained
\[ x^n = a^{yn} \quad (17) \]
I was thinking about how to carry out the next step in an easy way when the student found the idea. To my surprise, he proved the identity in just three steps.
Step 1: \[ x = a^y \] (18)

Step 2: Taking both sides to the power of \( n \):
\[ x^n = a^{yn} \] (19)

Step 3: Applying the definition of logarithms to equation (19) and using equation (15):
\[ \log_a x^n = yn = n \log_a x \]

obtaining:
\[ \log_a x^n = n \log_a x \] (20)

as shown in Figure 4.7

Figure 4.7

When \( x = a \), equation (20) became:
\[ \log_a a^n = n \log_a a = n \]

Which was just equation (11). \( \log_a a^x = x \).

What was important to me was that the skill of manipulating algebraic quantities of the student had been improved during the process of articulating simple illustrations.

It occurred to me that an effective way of promoting a student’s understanding of mathematics during procedural learning was to encourage the student to express ideas using simple illustrations which in turn allowed the tutor to monitor the learning.

Several important formulas were introduced in a similar fashion. The student’s understanding of logarithms was gradually improving. Finally we arrived at the last formula to be proved, perhaps the most complex one:
\[ \log_b x = \frac{\log_a x}{\log_a b} \] (21)

This proof was difficult. It seemed to require more procedural skills than other proofs. I was not sure when the student could eventually manage to start making this proof.

On the other hand I wanted to see whether he could use the idea of a simple numeric illustration, so I suggested he use \( 3 = \log_2 8 \) as a guide. I then left him to think independently.

His first attempt to construct a simple illustration was:
But he recognised that the structure was wrong. After a short time, he said.

Student: OK, I think I get it.
Tutor: You get it?
Student: I think so.

He then proceeded to write:

\[ y = \log_b x \quad (23) \]

Student: So I say, \(3 = \log_2 8\)
He confidently carried on from this. I was really surprised by what he did as illustrated in the following:

<table>
<thead>
<tr>
<th>F4.7 Pattern recognition</th>
<th>By observing: ( \log_b x = \frac{\log_a x}{\log_a b} ) the student used (3 = \log_2 8) to construct a pattern that matched the desired result:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>[ \log_b x = \frac{\log_a x}{\log_a b} \quad \log_2 8 = \frac{\log_{10} 8}{\log_{10} 2} \quad (24) ]</td>
</tr>
</tbody>
</table>

Note that he had introduced a random number “10” as the new base. The student had constructed a simple numerical illustration to help him to understand the meaning of the algebraic symbols. Next he wrote:

\[ 8 = 2^3 \]

Taking logarithms to base 10 of both sides:

\[ \log_{10} 8 = 3\log_{10} 2 \]

Dividing by \(\log_{10} 2\) on both sides

\[ 3 = \frac{\log_{10} 8}{\log_{10} 2} \]

Then, since \(3 = \log_2 8\), we have

\[ \log_2 8 = \frac{\log_{10} 8}{\log_{10} 2} \quad (25) \]

He was done! I thought he would have stopped there, but he didn’t. “So from this”, he pointed to the equation

\[ y = \log_b x \quad (\text{similar to } 3 = \log_2 8) \]

and wrote:

\[ x = b^y \quad (\text{similar to } 8 = 2^3) \]

Taking logarithms to base \(a\) on both sides:

\[ \log_a x = \log_a b^y \]

or:

\[ \log_a x = y\log_a b \quad (\text{similar to } \log_{10} 8 = 3\log_{10} 2) \]

Making \(y\) the subject:

\[ y = \frac{\log_a x}{\log_a b} \]

or:

\[ \log_b x = \frac{\log_a x}{\log_a b} \quad (26) \]

What a discovery! In fact, he had just used several steps to reach the goal as shown in Figure 4.7.
Figure 4.8

From figure 4.8, note that the student had started the proof using numbers (see A). Then he moved to abstract notation $y = \log_b x$ (see B). To connect logarithms with exponentials, he again used numbers (see B) to articulate the algebraic nature. Once he arrived at the result “3”, he seemed to recognise the “code” hidden inside the logarithmic expression, and immediately went back to $y = \log_b x$. From this he could now make a connection to new information (see C) and start the normal proving. Articulating a simple numerical illustration triggered his understanding of the logarithm notation. The student’s presentation has shed light on the relationship between the simple numeric illustration and the process of understanding via schemata formation and analogical reasoning. This relationship will be discussed later. It should be pointed out that the proving of formulas in this study is usually followed by a series of demonstrations and guided practice to clear any misconceptions and enable the acquisition of an automated schema. This will be discussed in step 5.

Note and Ideas (F4#Note 1)

According to Star (2005), research on procedural knowledge has not received enough attention since the “current characterization of conceptual and procedural knowledge reflects limiting assumptions about how procedures are known” (p. 404). Most classroom presentations focus on the computational model of procedural knowledge rather than on the study of deep meaning inherent in it (Hostos, Hostos & Bronx, 2004; Star, 2005).
It would be convenient to use the principle of direct instruction to present the mathematical formulas involved in a series of procedural calculations. In order to actively engage the student in learning, Simple Steps encourages the learner to construct simple numeric illustrations and look for algebraic patterns (see equation 9) to identify the mathematical relationship that a formula is expressing.

Simple Steps divides the learning of formulas into a sequence of connected procedures if the situation allows this to be done. For example, the learning of equation (1) was first presented in the context of numbers via an equation (2). The student then got stuck at equation (4) where a misconception occurred. This allowed the tutor to follow the student’s train of thought and necessitated the study of the equation $\log_a a^x = x$. This study then engaged the student in the exploration of a new formula, since he was puzzled over the meaning of the simple numeric illustration he had constructed i.e. $\log_3 3^5 = 5$. Finally, conceptual progression led to the construction of a general formula using algebra.

**Field Notes for Step 5 of Simple Steps**

This step has two purposes. The first purpose is help the student obtain a deeper understanding of a topic through learning and practising procedural skills. The second purpose is to clear any misunderstandings resulting from previous learning by introducing the student to specific types of problems which cover a variety of different aspects of the topic.

### Table 4.8
Field Notes for Step 5

<table>
<thead>
<tr>
<th>Codes</th>
<th>Field Notes</th>
<th>Themes and ideas</th>
</tr>
</thead>
<tbody>
<tr>
<td>F5.1 Conceptual link</td>
<td>When I presented a group of questions to the student in Step 1, he could not work on them confidently. I realised that even though all prerequisite formulae had been studied, the student needed the tutor’s guidance to connect his conceptual knowledge with procedural knowledge. Step 5 could be described as the process of learning mathematics through problem solving (Lubien ski, 2007). This process may also be seen as the first step toward establishing a connection between conceptual and procedural knowledge.</td>
<td>Learning mathematics through problem solving</td>
</tr>
<tr>
<td>F5.2 Involving a deep concept</td>
<td>In this step, different types of problems have been used to show how various concepts are connected to symbolic representations (Star, 2005). The concept discussed here</td>
<td>Requiring a level</td>
</tr>
</tbody>
</table>
is that of increasing and decreasing functions. For the
study covered in other areas see the test in Appendix A.
This concept has been chosen to explore the Simple Steps
approach to teaching procedural skills. Rigorous
discussion of a concept such as this usually requires a
level of abstraction that is well beyond the accessibility of
students at the senior high school level. As a result, the
computations involved are usually passively taught.

<table>
<thead>
<tr>
<th>F5.3 Challenge</th>
</tr>
</thead>
<tbody>
<tr>
<td>The challenge is to avoid complicated theoretical reasoning when teaching difficult new concepts and use a more intuitive approach that lies within the zone of proximal development of the learner. This task is discussed in the following.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>F5.4 Analogy and work out example</th>
</tr>
</thead>
<tbody>
<tr>
<td>It should be pointed out that Step 5 is different from the previous steps in that it unavoidably requires the introduction of knowledge and concepts outside the topic of logarithms for the student to handle in his problem solving with logarithms. Since so many researchers recommend the presentation of worked examples to enhance schema acquisition (Star, 2001; Sweller, 2003), I decided to teach problem solving explicitly using a worked example but with a modified perspective as discussed in the following:</td>
</tr>
<tr>
<td>a. Firstly, the method of analogical reasoning will be used. According to many researchers, analogical reasoning enhances schema acquisition (Chen, 1999; Cooper et al., 1987; Glick et al., 1983; Sweller et al., 1985). It was therefore hoped that a simple illustration would facilitate schema acquisition and help the student to engage in the learning actively. This was seen in Section F3.4 of table 4.6 where the simple illustration $8 = 2^3 \rightarrow 3 = \log_2 8$ and $1000 = 10^3 \rightarrow 3 = \log_{10} 1000$ was used as scaffolding to help the student understand the more general notion $x = a^y \rightarrow y = \log_a x$. See also Section F5.5.</td>
</tr>
<tr>
<td>b. As far as worked examples are concerned, Simple Steps is quite different from other pedagogical approaches. It tries not to allow the students to just sit silently watching the tutor’s illustration. Instead, the tutor always tries to actively involve the student so they work out the solutions together, even in the case where a new definition is just introduced (as seen in Section F5.5).</td>
</tr>
</tbody>
</table>

F5.5 When I asked the student whether he had learnt the
concept of increasing and decreasing functions, he answered “No”. So I started to introduce the concept at a basic level using the language of logarithms. This introduction included giving simple numeric examples and graphing functions. The examples were given by the tutor, however the student was required to take part in the process of constructing the algebraic patterns, graphing and finding solutions.

For example, I wanted to use the graph of \( y = \log_a x \) (for \( a > 1 \)) to illustrate the properties of an increasing function, so the student was asked to work out where the graph would cut the \( x \)-axis. The student could not answer the question at first, I then reminded him of using the illustration

\[ 8 = 2^3 \iff 3 = \log_2 8 \quad (1) \]

By analogy, he got:

\[ y = 0 \iff 0 = \log_a x \quad (2) \]

So

\[ a^0 = x \iff x = 1 \quad (3) \]

The result is shown in Figure 4.9 (See A).

Equation (2) and (3) indicated that the student was thinking mathematically rather than just formula copying. This shows the student’s procedural skill being improved through co-operation with the tutor in the process of working out the solutions to examples.

From now on, I will refer to an example that was worked out by both the teacher and student as a TS example. (Section F5.8 gives another example of TS example).

Next, the student was asked to investigate the properties of a decreasing function using \( y = \log_a x \) (for \( 0 < a < \))
solving 1). I call this a “small problem”.
In the whole tutoring process, there was always a “small
problem solving” following the introduction of a concept.
It was meant to provide an opportunity for the student to
reflect on what he had just learned.
However, up until this step (Step 5) I only thought “small
problem solving” could be used as a simple illustration to
help the student relate his existing knowledge to new
information.
As mentioned previously, “small problems” can also
enable a gradual improvement. Since they are
generalisations from familiar examples, they should be
easy for the student to solve using analogical reasoning.
However, if some of the elements in the structure of the
problems change, the student has to pay attention to this
change in order to work consistently with existing
concepts. “Small problems” also provide an opportunity
for the tutor to see whether the student has really
understood the concept. For example, following the above
problem, I asked the student to plot the graph of \( y = \log_a x \) (for \( 0 < a < 1 \)) but this time he needed to find out
the \( x \) co-ordinate when \( y = 1 \). This required a gradual
change (See point A in Figure 4.10). The purpose was to
remind the student of the logarithmic meaning of the
algebra.
To work out the solution, the student had to look back to
what he learned previously as shown in Figure 4.9.

\[ x = a^0 \quad (4) \quad (\text{See Figure 4.10 B}) \]

Using a small problem to detect

Note that even in this case, the student had made a
mistake by letting
\[ x = a^0 \]
instead of letting \( y = 1 \).
I believed that he had been confused again by the
equations
When he found his mistake by comparing the pattern in equation (1):

\[ y = a^x \quad \text{and} \quad x = a^y \quad (5) \]

and Figure 4.9, he wrote:

\[ 8 = 2^3 \iff 3 = \log_2 8 \]

as shown in Figure 4.10 B. This was a written reflection and served as a reminder to remind himself that it was “y” that corresponded to “1” in equation (6). So he got:

\[ y = a^1 \quad (6) \]

So, the figure previously drawn served as an illustration. I considered equation (6) as an illustration resulted from the small problem solving that served as a reminder. Thus, a “small problem” provides a gradual process during which the tutor can identify errors or misconceptions.

The results of a “small problem” could also serve as a simple illustration. Note also that small problems are sometimes the same things as TS examples (see F5.5), and depending on what is to be emphasised:

For example, if the solution of a small problem is obtained through the co-operation of the tutor and the student, then it can later be referred to as a TS example.

F5.7
Written reflection

After the brief introduction, the student was required to express his thinking using both verbal and written language.

The following brief summary was given by the student under the guidance of the tutor:

If \( y = f(x) \), if \( x_2 > x_1 \), and \( f(x_2) > f(x_1) \),
then \( f(x) \) is increasing in that domain \( (8) \)

This is shown in Figure 4.11.

Note that the \( x \)-axis (see A) was written by the student, but he could not manage to plot the corresponding graph.
of $y = f(x)$. I believed that reasoning difficulty had occurred (See next sections).

He did not state the situation for a function decreasing because he thought that he already understood the concept.

The purpose of this teaching was twofold. Firstly, it was considered to be some guided practice to help solidify new knowledge (Kline & Ishii, 2008). Secondly, it was hoped that the notes would serve as a reminder to help the student retrieve knowledge when solving further problems.

<table>
<thead>
<tr>
<th>F5.8 Summarize</th>
<th>Next, I wanted to give an example about how mathematical ideas could be related to pictures. Instead of directly drawing a picture, I asked the student to summarise all the concepts as indicated in Figures 4.9 and 4.10, using both graphs and algebraic symbols under my guidance. This was a TS example (See F5.5 and F5.6). The purpose was to provide an opportunity for the student to reflect on how mathematical thought such as the statement in (8), could be written in symbolic language. The following conversation showed how the tutor guided the student to express mathematical ideas and at the same time, identify possible errors or misconceptions. The conversation was about the increasing and decreasing properties of logarithms. We talked about the graph of $y = \log_a x$ (for $a &gt; 1$). T: Since $y_2 &gt; y_1$, the function is increasing. While I was speaking, the student wrote down what he thought using symbolic language (See label A in Figure 4.12). This graph was much improved than Figure 4.11.</th>
</tr>
</thead>
<tbody>
<tr>
<td>T:</td>
<td>Now what happens if $0 &lt; a &lt; 1$?</td>
</tr>
<tr>
<td>S:</td>
<td><em>decreasing</em>...</td>
</tr>
<tr>
<td>Then he added hesitantly:</td>
<td></td>
</tr>
<tr>
<td>S:</td>
<td>...In the case of an exponential function we have $y = a^x$...</td>
</tr>
<tr>
<td>This answer indicated that he was not sure about the case of $y = \log_a x$ ($0 &lt; a &lt; 1$). But he realised he could start</td>
<td></td>
</tr>
</tbody>
</table>

![Figure 4.12](image-url)
the reasoning from \( y = a^x \). This showed that the student was thinking rigorously. Following what he thought, I asked,

\[
T: \text{Now, if } a > 1, \text{ and } x_2 > x_1, \text{ what can you say?}
S: a^{x_2} > a^{x_1}.
\]

\[
T: \text{and if } 0 < a < 1, \text{ what can you say (for } x_2 > x_1). \text{ The student made mistake at first:}
\]

\[
a^{x_1} < a^{x_2}
\]

which was incorrect. So I gave him an insinuation, he had thought for a while then corrected it to

\[
a^{x_2} < a^{x_1}.
\]

T: Good.

Next, I used the properties of the exponential function to guide the student to study the case of logarithms (for \( 0 < a < 1 \)). The result is shown in Figure 4.13.

![Figure 4.13](image)

The purpose of presenting this example was to draw the graphs of increasing and decreasing functions and describe them.

This example would be used as a guide to explore deeper problems later in the tutoring process. Since these pictures were constructed by the student himself, they would remind him of their algebraic meaning, as seen in the following examples.

<table>
<thead>
<tr>
<th>F5.9 Another example From p35</th>
<th>After introducing the concept of increasing and decreasing functions, the student was asked to solve the following problems:</th>
</tr>
</thead>
</table>
|                              | 1. Compare the value of \( \log_5 6 \) and \( \log_6 5 \).  
                              | 2. See Appendix B.  
                              | The second question and its solution is given in Appendix B. The first one will be discussed in the following.  
                              | At first the student seemed to be confused by the roles of “5” and “6”. Several minutes passed before he asked:  
                              | Student: Change to the same base?  
                              | I understood that he wanted to change the base so that he could conduct the reasoning with ease.  
                              | T: Change to the same base?  
                              | S: Do I have to? |

<table>
<thead>
<tr>
<th></th>
<th>Giving insinuation to guide reasoning</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Raise the level of abstraction</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Learning how to express mathematical relations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Illustrations constructed by the student remind him of their algebraic meaning</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Discussion of misconception and mistake</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Using concepts to analyse a problem</td>
</tr>
</tbody>
</table>
T: No, you don’t have to. Think of using concept ...draw a graph.
In order to help him to connect his prior knowledge to new information, I referred him to the graph he previously sketched. (See Figure 4.9).
Since problems like this were new to him, I directly pointed to $\log_5 6$ and told him that in this expression $a > 0$, in the hope that he would think of the case $a > 1$. But he could not figure out which point he should refer to.

F5.10 Making progress
To help the student choose the correct graph, I pointed to the point C in Figure 9 and asked him to noticed that when $x = 1$ then $y = 0$  
(9)
This time he made good progress.
He seemed to remember the idea of using simple numeric illustrations and immediately sketched the graph shown in Figure 4.14. He pointed to the “5” in $\log_5 6$ and said:
S: It is equals to "a" so $\log_5 6$ ...
He finally realised that “5” was equivalent to “a”!
T: So this ($\log_5 6$) is larger than one …
I was so happy that I forgot to wait for the student to say the result.
It was phenomenally interesting that he took a long time to realise that “5” was equivalent to “a” in the general expression $\log_a x$. So “6” was of course equivalent to “x”!
This correspondence could have quickly been identified by someone else as soon as the problem was given. Misconceptions or reasoning difficulties can be different for different individuals and this might be the situation where one to one tutoring is needed.

F5.11 Solved the problem
To push him one level up, I said:
T: What about the second (I meant $\log_6 5$)?
S: Smaller than one.
T: [Now] work out.
To my surprise, he had found the connection between the formula:
$$\log_a x^n = n \log_a x$$ \hspace{1cm} (10)
learned in step 4 and the knowledge learned in this step as illustrated in the following:
First, he noticed that in equation (10), by taking $a = 5$, $x = 5$, and $n = 1$ he got 
$$\log_5 5 = 1$$
I consider this to be the construction of a simple illustration. Using $\log_5 5 = 1$ as a benchmark, he had 
$$\log_5 5 < \log_5 6$$
Since the function $\log_5 x$ is increasing. So 
$$1 = \log_5 5 < \log_5 6$$ \hspace{1cm} (11)
Similarly, he introduced another numerical value \( \log_6 6 = 1 \) as a second benchmark to get 
\[
\log_6 5 < \log_6 6 = 1
\]
so
\[
\log_6 5 < 1 \quad (12)
\]
Now from (11) and (12), he found
\[
\log_6 5 < \log_5 6 \quad (13)
\]
His proof is shown in Figure 4.14.

Note that this graph focussed the student on the right track of reasoning.

<table>
<thead>
<tr>
<th>Critical factors</th>
<th>Reflecting on this process, I summarised as follows:</th>
</tr>
</thead>
<tbody>
<tr>
<td>F5.12</td>
<td>Firstly there are two critical factors involved in this problem:</td>
</tr>
<tr>
<td></td>
<td>Factor 1) If ( x \leq a ), then ( \log_a x \leq 1 ) (for ( a &gt; 1 )) \quad (14)</td>
</tr>
<tr>
<td></td>
<td>Factor 2) If ( x \leq 1 ) then ( \log_a x \leq 0 ) (for ( a &gt; 1 )) \quad (15)</td>
</tr>
<tr>
<td></td>
<td>Factor 1 implied the critical benchmark (for example ( \log_5 5 = 1 )), while Factor 2 indicated the importance of the point A (see Figure 4.14), which divided the graph into two parts. I believe that when the student drew this graph these two factors triggered his understanding.</td>
</tr>
<tr>
<td></td>
<td>While it might be easy for someone to discriminate between these two factors, it could be confusing to distinguish between the two algebraic structures.</td>
</tr>
<tr>
<td></td>
<td>In the process of problem solving, these two factors might have been entangled with each other resulting in an obstacle to understanding.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Critical factors</th>
<th>Eliminate possible obstacles to the individual learner’s understanding</th>
</tr>
</thead>
<tbody>
<tr>
<td>F5.13</td>
<td>Since individuals face different difficulties depending on their misconceptions (as can be seen in F5.6 and F5.7), the critical factors could vary. If this is the case, one to one tutoring may shed light on how to help an individual learner overcome learning difficulties.</td>
</tr>
<tr>
<td>direct Instruction</td>
<td>Discernment between algebraic structures.</td>
</tr>
</tbody>
</table>
For example, even though a tutor might not know exactly what is going on in the learner’s mind (see F5.9), he could still try to eliminate the possible obstacles by using simple illustrations that clarify the algebraic structure at that moment (see equation 9). The above discussion illustrates the necessity of direct teaching to enable the learner to discriminate between “the unique aspects of seemingly similar situations” (Snowman, Dobozy, SCEVAK, BRYER, BARTLETT, BIEGLER, 2009, p. 233). Teaching is needed to enable discrimination.

| F5.14 More problem solving | “Small problem solving” as a preparation (discussed in F5.6) has been used to serve as the link between existing knowledge and the procedure being explored. In order to reduce the difficulty of a problem, a simple illustration associated with a certain concept could be constructed by the tutor or the student before the problem is explored. Whenever the student was in doubt or confused during problem solving, he would use the idea of a simple illustration to ensure his thinking was correct or to correct his mistake. For example, when he calculated the value of:  
\[ y = 10^{\frac{\log_{10}a}{a}} \]  (16)  he made a mistake as shown in Figure 4.15 |

The linkage between existing knowledge and the procedure
Problems accompanied by simple illustrations
Figure 4.15

His first answer was 10 (see point A), but he found it wasn’t quite right. So he immediately wrote the illustration $3 = \log_2 8$ and $8 = 2^3$ (see point B) in order to keep track of his calculations. He did the calculation again. This time, he carefully compared each quantity in $y = 10^{\frac{\log_{10} b^a}{a}}$ with that of $3 = \log_2 8$ and obtained the correct answer (see point C).

F5.15 Discovery learning

During the process of problem solving, the mathematical concepts or content were not passively taught: the student was encouraged to reason with the help of simple illustrations including a verbal summary and a graphical representation. This process enabled the student to create mathematical representations according to his own understanding of the concepts. This practice is consistent with the philosophical perception of discovery learning as Hammer (1997) pointed out: “discovery learning has taken on a range of meanings, but most often it refers to a form of curriculum in which students are exposed to particular questions and experiences in such a way that...”
they ‘discover’ for themselves the intended concepts” (p. 4).

Notes and Ideas (F5#Notel)
This step may be treated as the first step in establishing a connection between conceptual and procedural knowledge. In order to actively engage the learner, the idea of “small problem solving” is used to provide opportunity for the student to reflect on what he has just learned (See F5.6). This “small problem solving” could also be used to provide simple illustrations, constructed by the tutor or the learners himself that help him relate existing knowledge to new information (See F5.6).

There may be some critical factors causing a learner to have difficulty with particular problems. In the process of problem solving, algebraic structures might have been confused with each other resulting in misconceptions (See F5.11 and F5.12). In this situation, the tutor can help by explicitly pointing out the differences between these structures.

4.3 Data Collected at Stage 3: Data Collected after the Tutoring Process

4.3.1 The Test and the Feedback Interview
After the tutoring process, there was a test followed by a feedback interview. The test was designed to measure the student’s understanding of logarithmic functions. If any problems or mistakes were found, they would be discussed in the feedback interview to ensure the student had understood the topic.

The test contained ten questions. Some of the questions consisted of three sub-questions.

These questions were the same as those questions given in the interview before the tutoring process but this time the test would not be stopped if the student encountered difficulties with it. This time, the student was required to attempt all questions. This design allowed me to compare the results of the two tests.

Data Obtained from the Test
The test had ten questions. Six of the questions contained three sub questions. Most of these questions required a clear understanding of the concept of a logarithm. Some tested more complicated procedural skills. The whole test is in Appendix A.
The student performed quite well in the final test. He attempted all the questions and only made three mistakes. These mistakes were in sub questions. One was a careless error and the other two were due to unfamiliarity with the notation, as will be discussed in the following. Table 4.9 shows the mistakes the student made. Here “Test aft 1” means the code in the test after the tutoring process, number 1. The themes are brief summaries of the test and initial ideas about the data.

Table 4.9
Student errors in the test after tutoring process

<table>
<thead>
<tr>
<th>Codes</th>
<th>Field Notes</th>
<th>Themes and ideas</th>
</tr>
</thead>
</table>
| Test aft 1. Selected solutions | The question is a part of a main question: Show $125 = 5^{\log_5 125}$, then Given $y = a^x$, express $y$ in terms of logarithm in a structure similar to $125 = 5^{\log_5 125}$. The student attempted the first one correctly. For the second one, he wrote: $y = a^{\log_a x}$.

When he compared this answer with the first one, he corrected it to: $y = a^{\log_a y}$, which is correct.

The Second question is to simplify the expression $y = 10^{\left(\frac{\log_{10} b^a}{a}\right)}$. His answer was “b” which is correct but his notation was not quite clear as shown:

$$y = 10^{\left(\frac{\log_{10} b^a}{a}\right)}$$

$$y = 10^{\frac{\log_{10} b^a}{a}}$$

$$= b^{\frac{a}{a}}$$

$$= b$$

The other question is:

Show $\log_3 (9 \times 27) = \log_3 9 + \log_3 27$. Show your working.

His answer was $\log_3 (243) = 5 = 2 + 3 = \log_3 (9 \times 27)$

He did not express the relationship clearly. | Mistake due to careless reading of the question                                                                 | Did not express clearly                                                                          |
Test aft 2.
Attempted correctly
Most answers were correct, such as the following questions:
Simplify the following expression
\[(\log_{10} 20)(\log_{10} 81)\]
Simplify the following expressions
\[5^{\log_{5} 8+\log_{5} 9}\]

These answers are given in Appendix A. It is worth pointing out that during the test, the student did not simply immerse himself in calculations. He also paid attention to the conceptual meaning of each problem. He used simple illustrations to help obtain correct answers. For example, when he solved the following question:
\[3^{-\log_{2} x} = \frac{1}{\sqrt{27}}\]
he encountered the following expression
\[-\log_{2} x = -\frac{3}{2} \quad (9)\]
This confused him, so he used the simple illustration:
\[
\begin{align*}
\log_{2} 8 &= 3 \\
\log_{2} x &= \frac{3}{2}
\end{align*}
\]
obtaining
\[2^{\frac{3}{2}} = x,\]
giving
\[\sqrt{8} = x, \text{ so } x = 2\sqrt{2}.
\]

Test aft 3
Summary
On the whole, the test result was very good as some of the questions were quite difficult. The student only made three minor mistakes: one due to carelessness and the other two due to his use of notation. During the test, the student used the idea of a simple illustration to help him reason.

Notes and Ideas (Test aft Note)
This test was designed to test the student’ understanding of logarithm. The questions were the same as those given to the student before the tutoring process. However before the tutoring process, the student had attempted only the first few questions (as discussed in In#Note1). After the tutoring process he attempted all the questions and his result was very good. His test paper showed that he had tried to keep track of the conceptual meaning of a logarithm by using simple illustrations. (See Test aft 2).

Data Collected from the Feedback Interview
One of the purposes of this interview is to help the student to overcome the difficulties probably encountered in the test. But the result of the test turned out to be very good, and the
minor mistake the student has made was discussed immediately after the test as listed in Table 4.10. So the purpose of the feedback interview is mainly to provide an opportunity for the tutor and the student to exchange their ideas about the tutoring process and every study. Table 4.10 presents some interview segments related to these aspects but some sections which were repeated or were irrelevant have been taken out. Here “In2.1 suggestion” is the code for interview 2, section 1” and “suggestion” is a brief summary of the segment.

Table 4.10
Data selected from the feedback interview

<table>
<thead>
<tr>
<th>Code</th>
<th>Interview Segment</th>
<th>Themes and Ideas</th>
</tr>
</thead>
<tbody>
<tr>
<td>In2.1</td>
<td>Tutor: Who do you know who would find this learning helpful?</td>
<td>Help student overcome difficulties</td>
</tr>
<tr>
<td>Suggestion</td>
<td>Student: Students that is struggling in math and struggling into understanding mathematical topics concepts.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Tutor: Where do you think somewhere else can you use the information obtained in the tutoring, if there is? Do you think it is useful for physics, for example?</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Student: Yes, on other subjects such as science, and maybe economics, graph sketching.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Tutor: What other positive feeling would you want to express?</td>
<td>May useful for science economics, sketching</td>
</tr>
<tr>
<td></td>
<td>Student: Uh, learning, uh during the tutoring process it is easy to follows and understands.</td>
<td></td>
</tr>
<tr>
<td>In2.2</td>
<td>Tutor: Do you think the concept is important?</td>
<td>Procedure can be worked out from the concepts</td>
</tr>
<tr>
<td>Important arguments</td>
<td>Student: Yes.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Tutor: <em>When you understand the concept...?</em></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Student: The procedure can be worked out from the concepts, but the concept cannot be worked out from the procedure.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Tutor: Uh, that is one of the greatest arguments today, thank you!</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Tutor: What advice would you give me for the future tutoring?</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Student: <em>Uh, a longer time period to learn more...</em></td>
<td>Should have a longer time to learn so much</td>
</tr>
<tr>
<td></td>
<td>Tutor: Yes, so to have longer time.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Student: Time is too short to learn so much.</td>
<td></td>
</tr>
<tr>
<td>In2.3</td>
<td>Tutor: Do you think you are actively engaged in this learning?</td>
<td>Actively engaged in the learning</td>
</tr>
<tr>
<td>Enjoy learning</td>
<td>Student: Yes.</td>
<td></td>
</tr>
<tr>
<td>Tutor:</td>
<td>Do you enjoy the learning?</td>
<td>Student:</td>
</tr>
<tr>
<td>-------</td>
<td>---------------------------</td>
<td>----------</td>
</tr>
<tr>
<td>Tutor:</td>
<td>Were you pressed to learn?</td>
<td>Student:</td>
</tr>
<tr>
<td>Tutor:</td>
<td>Do you think you have actively learned or passively taught?</td>
<td>Student:</td>
</tr>
<tr>
<td>Tutor:</td>
<td><em>Because I have to teach the concepts...</em></td>
<td>Student:</td>
</tr>
</tbody>
</table>

### In2.4 Difficulties

<table>
<thead>
<tr>
<th>Tutor:</th>
<th>What was the most difficult part of this tutoring?</th>
<th>Student:</th>
<th><em>Logarithms ... applying the formula into ... recognizing the question to apply the correct formula.</em></th>
</tr>
</thead>
</table>

### In2.5 Suggestion to current curriculum

<table>
<thead>
<tr>
<th>Tutor:</th>
<th>What is your suggestion about the current curriculum policy?</th>
<th>Student:</th>
<th>The current HSC program, there is not enough time for the senior students to prepare and fully understand into the understanding and learning the HSC syllabus, in my opinion there is not enough time for Year 12 students to understand that is needed for the exams in three terms time.</th>
</tr>
</thead>
</table>

### In2.6 Teaching style

<table>
<thead>
<tr>
<th>Tutor:</th>
<th>What happen if there is no time enough, does the teacher just simply explain the formula and then go?</th>
<th>Student:</th>
<th>There is not much time for the teacher to fully explain the concepts for students to <em>understand and question’s time mostly is usually short, in order to keep of the syllables.</em>*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tutor:</td>
<td><em>So what happen if you don’t understand the concept, you have to work out?</em></td>
<td>Student:</td>
<td>Work out by ourselves which sometimes ends up with student getting confused.</td>
</tr>
<tr>
<td>Tutor:</td>
<td>More comment and advice, the teaching style for example?</td>
<td>Student:</td>
<td><em>Uh, teacher’s altitude into the topic should be more enthusiastic to increase students ...</em></td>
</tr>
<tr>
<td>Tutor:</td>
<td>Thank you, I really thank you very much.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Notes and Ideas (In2Notel)
The conversation indicated that the student has actively engaged in the learning but he also suggested that there should be a larger time for so much material. (See In2.3, In2.2)
According to the student, the principle practiced by the tutoring process will be helpful to those students who are struggling to understand mathematical topics. It could be useful for teaching other science subject as well (In2.1).

The student also talked about the current curriculum. According to him, there is not enough time for the senior student to fully understand the mathematics and for the teacher to fully explain the concepts. As a result, the students have to work out on their own which might result in getting confused of the concepts.

**Data Collect from the Questionnaire**

After the tutoring process, the student is required to conduct self-assessment through a questionnaire. This is not only a source to measure the student’s engagement, but also provides opportunities that encourage the student to summarize what he has been doing in the learning (Munns, et al., 2006). Table 4.11 has presented all answers to the question by the student.

<table>
<thead>
<tr>
<th>Codes</th>
<th>Questions and Answers</th>
<th>Themes and ideas</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quest 1</td>
<td>The instruction presented during the tutoring process has attracted me to learn the material actively. “Actively” here means answering and asking questions enthusiastically.</td>
<td>Straightforward and explained fully</td>
</tr>
<tr>
<td>Instruction</td>
<td>☑Strongly Agree</td>
<td></td>
</tr>
<tr>
<td></td>
<td>□Agree</td>
<td></td>
</tr>
<tr>
<td></td>
<td>□Undecided</td>
<td></td>
</tr>
<tr>
<td></td>
<td>□Disagree</td>
<td></td>
</tr>
<tr>
<td></td>
<td>□Strongly Disagree</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Explain your Answer: The materials instructions provided is interesting and presented to me in an easy, straightforward way. There is room for me to ask questions when I’m unclear and is explained fully.</td>
<td></td>
</tr>
<tr>
<td>Quest 2</td>
<td>Summarising the concept during tutoring has allowed me to think more about the topic.</td>
<td>Summarising by the student help problem solving</td>
</tr>
<tr>
<td>Summarising</td>
<td>□Strongly Agree</td>
<td></td>
</tr>
<tr>
<td></td>
<td>☑Agree</td>
<td></td>
</tr>
<tr>
<td>Quest 3</td>
<td>Not sure</td>
<td>I want to apply my new knowledge to other problems outside of the tutoring class.</td>
</tr>
<tr>
<td>---------</td>
<td>----------</td>
<td>---------------------------------------------------------------------------------</td>
</tr>
<tr>
<td></td>
<td></td>
<td>☐ Strongly Agree</td>
</tr>
<tr>
<td></td>
<td></td>
<td>☐ Agree</td>
</tr>
<tr>
<td></td>
<td></td>
<td>☑ Undecided</td>
</tr>
<tr>
<td></td>
<td></td>
<td>☐ Disagree</td>
</tr>
<tr>
<td></td>
<td></td>
<td>☐ Strongly Disagree</td>
</tr>
<tr>
<td>Explain your Answer:</td>
<td></td>
<td>Allows me to review once again on areas and ask questions again on my unclear areas.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Quest 4</th>
<th>Deep understanding</th>
<th>The tutoring process has led me to a deeper understanding of the concept of logarithms.</th>
<th>Knowledge had increased by the time the test was taken.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>☑ Strongly Agree</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>☐ Agree</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>☐ Undecided</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>☐ Disagree</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>☐ Strongly Disagree</td>
<td></td>
</tr>
<tr>
<td>Explain your Answer:</td>
<td></td>
<td>Unsure where logarithm could be applied elsewhere other than math classes.</td>
<td>Allow me to understand where I have problems.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Quest 5</th>
<th>Do not agree</th>
<th>If I understand the mathematical concepts taught at School, I will participate in classroom activities more actively.</th>
<th>Participation in classroom activities depends on the teacher’s attitude</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>☐ Strongly Agree</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>☐ Agree</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>☐ Undecided</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>☑ Disagree</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>☐ Strongly Disagree</td>
<td></td>
</tr>
<tr>
<td>Explain your Answer:</td>
<td></td>
<td>Participation in classroom activities bases on the teacher’s attitude more than the understanding of the idea.</td>
<td>Allow me to understand where I have problems.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Quest 6</th>
<th>Concepts are important</th>
<th>A mathematical topic is not learned if its concept is not understood.</th>
<th>Understanding a concept enables unfamiliar types of questions to be</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>☑ Strongly Agree</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

96
<table>
<thead>
<tr>
<th>Quest 7</th>
<th>Made math fun</th>
<th>The teaching approach used in the tutoring encourages me to learn mathematics.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>☐ Strongly Agree</td>
</tr>
<tr>
<td></td>
<td></td>
<td>☑ Agree</td>
</tr>
<tr>
<td></td>
<td></td>
<td>☐ Undecided</td>
</tr>
<tr>
<td></td>
<td></td>
<td>☐ Disagree</td>
</tr>
<tr>
<td></td>
<td></td>
<td>☐ Strongly Disagree</td>
</tr>
<tr>
<td>Explain your Answer: Made maths fun.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Quest 8</th>
<th>Time management</th>
<th>I will try to do more mathematics after the tutoring process.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>☐ Strongly Agree</td>
</tr>
<tr>
<td></td>
<td></td>
<td>☑ Agree</td>
</tr>
<tr>
<td></td>
<td></td>
<td>☐ Undecided</td>
</tr>
<tr>
<td></td>
<td></td>
<td>☐ Disagree</td>
</tr>
<tr>
<td></td>
<td></td>
<td>☐ Strongly Disagree</td>
</tr>
<tr>
<td>Explain your Answer: Personal problems and the management to work on other subjects.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Quest 9</th>
<th>Concepts are important</th>
<th>I will pay more attention to understanding the mathematical concepts in later study.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>☑ Strongly Agree</td>
</tr>
<tr>
<td></td>
<td></td>
<td>☐ Agree</td>
</tr>
<tr>
<td></td>
<td></td>
<td>☐ Undecided</td>
</tr>
<tr>
<td></td>
<td></td>
<td>☐ Disagree</td>
</tr>
<tr>
<td></td>
<td></td>
<td>☐ Strongly Disagree</td>
</tr>
<tr>
<td>Explain your Answer: Concepts are more important than processes.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Quest 10</th>
<th>Depends on individual</th>
<th>Perseverance in learning is important.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>☐ Strongly Agree</td>
</tr>
<tr>
<td></td>
<td></td>
<td>☑ Agree</td>
</tr>
<tr>
<td></td>
<td></td>
<td>☐ Undecided</td>
</tr>
<tr>
<td></td>
<td></td>
<td>☐ Disagree</td>
</tr>
<tr>
<td></td>
<td></td>
<td>☐ Strongly Disagree</td>
</tr>
</tbody>
</table>

Just by working on exercises, if there is a slight twist in question; students can’t apply how something could change when the concept is unclear.
Notes and Ideas (Quest Note1)
The student’s self-assessment indicates that the instruction during the tutoring process had actively engaged him and led him to a deeper level of understanding of logarithmic concepts (see Quest 2, Quest 4). The material was interestingly presented and allowed the student to reflect and summarise what he had learned (Quest 1, Quest 2).

One thing that needs to be pointed out here is the test. According to the student, he realised that his knowledge had increased when attempting the test (Quest 4). This was unexpected. When I was writing the methodology Chapter (chapter 3), an idea sprang to mind: could I design a sequence of problems to help the student establish a link between conceptual and procedural knowledge? The result of this thinking has led to the idea of small problem solving during data analysis.
Chapter 5
Analysis of the Data

In this chapter further analysis of the data is carried out following the initial analysis presented in Chapter 4. As mentioned in section 3.7.1 and 4.1 there are five parts in the process of analysis. Chapter 4 completed the first two parts. The analysis presented in this chapter is the final three units of the process. A summary of the entire analysis will be presented in Chapter 6.

5.1 A Brief Description of Part 3, Part 4, and Part 5 of the Process of Data Analysis

Part 3 is called categorisation of themes. It is concerned with the categorisation of themes into groups. This process will be carried out separately for the different data sources. The method of thematic analysis introduced by Attride-Stirling (2001), Creswell (2008), and Meijer et al. (2002), will be used as a major tool for handling and encapsulating data as discussed below.

(a) Refining themes. Themes obtained from unit 1 and unit 2 of the analysis in chapter 4 will be refined whenever necessary.

This includes a close investigation of the text data, so that:

1. the themes obtained from the initial analysis (as presented in Chapter 4) indicate what was going on during the tutoring process, or will be refined and
2. the themes reflect significant events addressed by the research questions.

Attention will also be given to new patterns identified during analysis, thus allowing the newly emergent themes to be absorbed into the groups.

(b) Classifying themes into groups. The refined themes, which are called basic themes, will be classified into groups based on the research questions (See section 3.2). Each group will be labelled by an organising theme (Attride-Stirling, 2001).

A brief description for each group will be presented in the right column of tables. This process will include a brief discussion that outlines the emergent pattern and structure. Once identified, this emergent pattern and structure can be further analysed so that a conceptual link between different data can be established.
It is hoped that this presentation of basic themes and organising themes can provide not only a tool for systematically analysing the data, but also provide transparency for readers. Part 4 is called basic descriptions. It is principally concerned with describing and analysing what is going on depending on the information provided by the themes. The basic description of each group will be used as a guide for the researcher to go through each group of data again, looking for patterns or phenomena within its organising theme.

For all data in this study, part 3 and part 4 of the data analysis process will be presented together. Also, as the data are related to each other, they could sometimes be analysed simultaneously so that the contextual meaning of the tutoring process is not lost.

Part 5 is called triangulation. It will be used mainly to examine the effectiveness of Simple Steps and hence enhancing the internal validity (Yin, 2003). It is important to point out, however, that the term triangulation could have different meanings in the literature (Meijer et al., 2002; Moran-Ellis et al., 2006). As Sands and Roer-Strier (2006) commented that triangulation can be used to describe the “same story, same meaning” (p. 12); but it can also be used to describe different parts of a procedure. This is the complementary view of triangulation. This article will mainly take the complementary view, since data obtained here are from different periods of the process.

It should be mentioned that the results of data analysis from chapter 4 will be used if it is necessary. Finally, the summary will be given in chapter 6.

5.1.1 The Research Questions
Research questions were stated in section 3.1.2. They are:

1. How can the principle of direct instructional guidance, in the context of logarithms, be applied while keeping the student actively engaged?

2. How can the problem-solving approach, in the context of logarithms, proposed by the supporters of discovery learning co-operate in helping the student to understand a mathematical concept or a topic while keeping the misconception to a minimum?
5.2 Data Analysis (part 3 and part 4) for the Initial Interview

This section describes part 3 and part 4 of the analysis of the interview that was conducted before the tutoring process.

5.2.1 Categorisation of Themes for the Initial Interview (Part 3)

The categorisation of themes for the initial interview focuses on the student’s engagement in his classroom activities.

Nine basic themes obtained from the data (as presented in Chapter 4) are categorised into three groups (organising themes) presented in the second column and briefly described in the third column of Table 5.1.

The labels of the basic themes in the first column of the table refer to the basic theme number followed by the segment of the data it came from. For example, the label “1.In1.1” refers to basic theme 1 taken from interview 1, segment In1.1.

The labels of the organising themes in the second column of the table refer to the organising theme number followed by the data it came from. For example, the label Org1.In1 refers to organising theme 1 taken from interview 1.

Table 5.1
Categorisation of the Themes of Interview 1

<table>
<thead>
<tr>
<th>Basic Themes</th>
<th>Organising Themes</th>
<th>Descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. In1.1 Finds maths interesting</td>
<td>(Org1.In1) Needs help</td>
<td>The student seeks and needs help for he does not clearly understand the concept of logarithms</td>
</tr>
<tr>
<td>2. In1.2 Wants to know more about logarithms</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. In1.4 Seeks help from tutoring</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4. In1.3 Does not enjoy classroom learning environment</td>
<td>(Org2.In1) Passively taught</td>
<td>The student was not actively engaged in his classroom activities because of the lack of communication between teachers and the students.</td>
</tr>
<tr>
<td>5. In1.5 Passively taught</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6. In1.6 Does not realise the connection between logarithms and exponential</td>
<td>(Org3.In1) Lacks conceptual understanding</td>
<td>The concept of logarithms is not seen as a conceptual network with the related</td>
</tr>
</tbody>
</table>
functions  
fields and therefore may result in reasoning difficulties when performing problem solving.

7. In1.7 Can calculate logarithms correctly
8. In1.8 Feels confused

Notes
The categorisation of themes has divided the information into three groups based on aspects which relate to the research questions. The group labelled by the organising theme “Needs help” (Org1. In1) was supported by the themes that relate to the student’s purpose in coming for tutoring.

The group represented by the organising theme “Passively taught” (Org2. In1) shows that the student was not actively engaged in his classroom activities because of the lack of communication between him and the teacher.

The organising theme “Lacks conceptual understanding?” (Org3.In1) is related to misconceptions or reasoning difficulties. No new themes were generated in this analysis.

5.2.2 The Basic Descriptions of the Initial Interview
This section further describes the interview before the tutoring process using the organising themes listed in Table 5.1 as a guide.

Org1.In1 Needs help As noted in section 4.2.3, the student came to the tutoring session to gain a better understanding of logarithms. He is motivated to ask for help because he finds mathematics interesting, but the concept of logarithms does not make sense to him.

Org2.In1 passively taught. The student was not actively engaged in the classroom learning (See In1.5). Despite the fact that he had concentrated on the teacher’s explanation, he had no chance to ask questions until the end of the lessons. As a result, he was passively taught. This can be seen from the fact that he attempted the questions (given during the interview) by guessing rather than understanding the underpinning mathematical meaning (See In1.8).

Org3.In1 Lacks conceptual understanding This group of themes shows the reason why logarithms do not make sense to the student. He sees logarithms as a concept isolated from related fields such as exponential functions (See 6. In1.6). He could calculate specific
problems but failed to calculate others simply because the presentation of the problems changed.

5.3 Data Analysis (part 3 and part 4) of Field Notes for Simple Steps, Step 1

This section discusses the analysis of the field notes for Simple Step 1.

5.3.1 Categorisation of Field Notes for Simple Step 1

The 26 themes obtained in the initial analysis were classified into groups according to aspects of the tutoring.

Six new themes were added. These themes (marked as “New” in Table 5.2 below) were generated during the process of categorization. As a result, there are 32 basic themes in total and were classified into 5 groups.

The term “(Org 1 F1) Step by Step” means the first organising theme of Field note1 is “Step by Step”.

Table 5.2
Categorisation of Themes of Simple Step 1

<table>
<thead>
<tr>
<th>Basic Themes</th>
<th>Organising Themes</th>
<th>Descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. F1.1 Conception about logarithms and exponential functions unclear</td>
<td>(Org1. F1.) Step by Step</td>
<td>The step by step style of the revision that brought the learner’s knowledge to the prerequisite level.</td>
</tr>
<tr>
<td>2. F1.1 Not easy for a beginner</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. F1.1 Input a small amount of information</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4. F1.10 Introduction of abstract notion after grasping basic ideas</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5. F1.11 Chance of Automated Schema Construction</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6. F1.2 Replacing “a” with a number made question easier.</td>
<td>(Org2. F1.) Paradigm that Presents the Targeted Concepts.</td>
<td>Stimulated the student’s modes of thought during the tutoring process.</td>
</tr>
<tr>
<td>7. F1.3 Numerical Illustration helped reasoning</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8. F1.3 First step toward understanding</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td></td>
</tr>
<tr>
<td>9.</td>
<td>F1.6 Visualizes the correct meaning</td>
<td></td>
</tr>
<tr>
<td>10.</td>
<td>F1.6 Vehicle that targeted concept.</td>
<td></td>
</tr>
<tr>
<td>11.</td>
<td>F1.7 The key step carried out by the student (New)</td>
<td></td>
</tr>
<tr>
<td>12.</td>
<td>F1.3 Not sure (New)</td>
<td></td>
</tr>
<tr>
<td>13.</td>
<td>F1.6 Understanding the meaning of notions</td>
<td></td>
</tr>
<tr>
<td>14.</td>
<td>F1.6 Expressing thought using written language (New)</td>
<td></td>
</tr>
<tr>
<td>15.</td>
<td>F1.7 Expressed what he thought</td>
<td></td>
</tr>
<tr>
<td>16.</td>
<td>F1.3 Convert an abstract concept into a graph</td>
<td></td>
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<tr>
<td>17.</td>
<td>F1.6 Visualizes the correct meaning</td>
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</tr>
<tr>
<td>18.</td>
<td>F1.5 Provide opportunities to reflect on what has been done</td>
<td></td>
</tr>
<tr>
<td>19.</td>
<td>F1.4 Reasoning difficulties occurred</td>
<td></td>
</tr>
<tr>
<td>20.</td>
<td>F1.5 Uncertainties of the concepts</td>
<td></td>
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<tr>
<td>21.</td>
<td>F1.5 Ambiguous formulations of inverse function</td>
<td></td>
</tr>
<tr>
<td>22.</td>
<td>F1.8 Graphed a simple function and its inverse</td>
<td></td>
</tr>
<tr>
<td>23.</td>
<td>F1.4 An ambiguous description of inverse</td>
<td></td>
</tr>
<tr>
<td>24.</td>
<td>F1.6 Started to understand the concept</td>
<td></td>
</tr>
<tr>
<td>25.</td>
<td>F1.7 The meaning of inverse was so rich</td>
<td></td>
</tr>
<tr>
<td>26.</td>
<td>F1.6 Using written language (New)</td>
<td></td>
</tr>
<tr>
<td>27.</td>
<td>F1.7 Verbally expressing thought (New)</td>
<td></td>
</tr>
<tr>
<td>28.</td>
<td>F1.2 Sketch successfully</td>
<td></td>
</tr>
<tr>
<td>29.</td>
<td>F1.8 The first confident response</td>
<td></td>
</tr>
<tr>
<td>30.</td>
<td>F1.8 Solve the exercise quickly</td>
<td></td>
</tr>
<tr>
<td>31.</td>
<td>F1.8 Happiness from the tutor is the true reinforcement (New)</td>
<td></td>
</tr>
</tbody>
</table>

(Org3. F1.) Enhancing Reflections

(Org4. F1.) Problem Solving and Misconceptions

(Org5. F1.) Started to Understand and Reinforcement
Notes
The 32 basic themes were classified into 5 groups. The first organising theme “Step by Step” (Org1 F1), groups the basic themes that relate to the style of the revision that brought the learner’s knowledge to the prerequisite level.

The second organising theme, “Paradigm that Presents the Targeted Concepts” (Org2 F1), describes how the tutor stimulated the student’s modes of thought during the tutoring process. The third organising theme, “Enhancing Reflections” (Org3 F1), includes basic themes related to the student’s response and reflections on the subject matter.

The fourth organising theme, “Problem Solving and Misconceptions” (Org4 F1), describes the process of problem solving and the student’s misconceptions.

The final organising theme, “Started to Understand and Reinforcement” (Org5 F1), describes the beginning of the student’s deeper, conceptual understanding. As a result, the student made good progress and this evoked positive reinforcement from the tutor.

5.3.2 Basic Descriptions of Field Notes of Simple Step 1
One of the purposes of Simple Step 1 is to bring the learner’s knowledge to the prerequisite level. Otherwise, students may use their prior misconception to explain new information which could affect their level of understanding.

The step by step principle advocated by direct instruction was used to start the revision (See Org1 F1). At this stage, it was necessary for the tutor to use direct instruction to set the student on the right track towards the targeted concept. This was because time is limited and must therefore be carefully controlled, particularly in the context of tutoring.

But this does not imply passive learning. To actively engage the student in the process of integrating new information with his existing knowledge, Simple Steps uses various pedagogical methods to stimulate the learner’s mode of thinking (Bruner, 1964).
For example, the conversation about exponential functions (See theme 1 F1.1) was in fact the instruction that allowed the student to activate his knowledge via verbal thinking (Brumberger, 2007; Ormrod, 2008). This process also enabled the tutor to design an exercise involving graphic plotting in an attempt to push the student’s level of understanding, but within the learner’s zone of proximal development (See 7 F1.3).

This can be seen from the fact that the student did not know how to discuss the graph of $y = a^x$ for $0 < a < 1$ even though he had just finished plotting the graph of $y = a^x$ for $a > 1$ (See 12 F1.3).

I believe that this was the right place to assign the student a task that was “beyond their current capabilities” (Driscoll, 2005, p. 253). To provide scaffolding for the student, the “supportive tool for learners as they construct knowledge” (Driscoll, 2005, p. 257), I replaced the symbol “$a$” in the expression $y = a^x$ with $\frac{1}{2}$. This is, in fact, changing the mode of representation from symbolic representation to iconic representation.

Changing the representation enables the learner to approach a difficult concept from a different angle of perception. Different representations can stimulate different modes of thought, and thereby effectively engage the learner to reflect on what he still does not know. This view is also supported by the definition of understanding proposed by Heibert et al. (1992): “the mathematics is understood if its metal representation is part of a network of representations” (See section 3.1.1, p. 67).

The degree of understanding is determined by how well the information in representations that link ideas, facts, and procedures, is connected or recognised (Hiebert et al., 1992). When a concept is not fully understood, the student can encounter reasoning difficulties while he is working on a problem involving a different representation of the same mathematics concept. The theme 23 F1.4 describes such a situation where the student used an ambiguous description “opposite” to describe the concept of an inverse function. This provided the opportunity for the tutor to identify the misconception that blocked his understanding (See themes 25 F1.7).
On another occasion (See themes 21. F1.5), the student use the word “flip” to describe the geometrical relationship between \( y = f(x) \) and \( y = f^{-1}(x) \) without understanding the exact meaning of “flip” in this situation. This event enabled the tutor to design a problem for the student to explore concepts relating to functions, inverse functions and their graphs through the process of guided problem solving (See themes in Org4 F1).

Thus, a properly designed exercise or problem involving different representations of a concept will uncover the weakness of the learner in one or more of the representations and enable the tutor to detect the learner’s misconceptions or reasoning difficulties (See Org4. F1). The tutor is then able to provide scaffolding at an appropriate level to help the learner move from his “actual developmental level as determined by independent problem solving” to a higher level of “potential development as determined through problem solving under adult guidance” (Vygotsky, 1978, p. 86 as quoted by Driscoll, 2005). The group Org5 F1 describes how properly designed exercises will lead to understanding of the mathematics concepts being studied. The basic theme 31.F1.8 describes the joy that comes to both the student and the tutor at the moment when the student starts to progress.

5.4 Data Analysis (part 3 and part 4) of Field Notes for Simple Steps, Step 2

This section presents the categorisation and basic descriptions for field notes of step 2 of Simple Steps.

5.4.1 Categorisation of field notes of Simple Step 2

There were six themes from the initial analysis. Five new themes were generated during the categorisation as listed in Table 5.3.

Table 5.3
Categorisation of Themes of Simple Step 2

<table>
<thead>
<tr>
<th>Themes</th>
<th>Organising themes</th>
<th>Brief Descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. F2.2 Using simple expression to introduce concepts</td>
<td>(Org1. F2) Simple Illustrations</td>
<td>Describes the emergent idea of using simple illustrations during the process of tutoring.</td>
</tr>
<tr>
<td>2. F.2.1 Creating a numeric example before abstract concepts</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. F2.3 Connecting new definition to existing</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
knowledge
4. F2.2 Learning by doing
5. F2.3 Oral responses
6. F2.2 Verbal explanation
   (New)

7. F2.2 Write down the answer effortlessly (New)
8. F2.2 Got examples showing correct reasoning
9. F2.3 Understand the meaning of the inverse (New)
10. F2.3 Got the idea through one illustration (New)
11. F2.3 Ready for a higher level (New)

5.4.2 Basic Descriptions of Field Notes for Simple Step 2

Simple Step 2 is the step where the targeted topic is introduced at a level accessible to the learner. In this study, the concept of logarithms is considered to be almost new.

The challenge at this level was then how to present the basic mathematical structure clearly and at the same time, engage the student in learning actively.

Using Simple Illustrations to Present the Mathematical Ideas (Dis.5.4.2a)

One of purpose of this step is to help the learner select relevant incoming information. It corresponds to the first phase of the three cognitive processes in the SOI learning model as discussed in section 2.5.2.

But only directive instruction would not be enough. It should be seen as the stage of what Bruner (1964) called “constructing a model of knowledge through action and imagery”. As Bruner (1966) commented, the material or concept presented at this level (iconic representation) was not fully defined. The idea was to present the material in “a form simple enough so that any particular learner can understand in a recognizable form” (Bruner, 1966, p. 44).

The basic theme 1.F2.2 described the idea of using the simple expression
to refresh the student’s memory of the concept of an exponential function. Based on this relation, the concept of logarithms was introduced via the brief verbal explanation:

We say that 3 is the logarithm of 8 with base 2, (2)

and is written:

\[ 3 = \log_2 8 \quad (3) \]

(See basic theme 6. F2.2). Because of its simplicity, expression (1), which acted as visual information (Ormrod, 2008), could help the learner give meaning to the mental picture of an exponential function (Ormrod, 2008; Anderson, 2005). The retrieved information was then used to explain the meaning of logarithms via expression (3) with the help of the verbal explanation (2).

The theme “Simple Illustrations” (Org1. F2) describes how the student understands of the concept of inverse and logarithms were improved through just one simple illustration.

From now on, the term “simple illustration” will be used to mean either the process of an algebraic articulation or its resulted patterns.

**Direct Instruction and Discovery Learning (Dis.5.4.1b)**

It should be noted that the principle of direct instruction has been used in the sense that it was the tutor who directed the course of the learning toward the targeted concept and provided enough background information explicitly to the learner (Kirschner et al., 2006; Magliaro, 2005; Rosenshine, 1986). The direct instruction principle ensures that the tutor knows how well the student “is grasping the targeted content, as well as correcting any misconceptions” (Magliaro et al., 2004, p. 45). This was necessary because the result of the pre-test and the conversation in step 1 of Simple Steps showed that the student had some reasoning difficulties and misconceptions about exponential and logarithmic functions (See Org3. In1 of Table 5.1; and Org4. F1 of Table 5.2). If the ambiguities are not cleared in a timely manner in the context of tutoring, the chance of the learner “storing faulty schemata” will increase “as the size of the pool of schemata increases” (Bruning, et al., 2004), and as the level of abstraction increases.

But the flow of the discourse did not just pour the information into the learner’s head. The simple and obvious structures of expressions (1) and (3) were designed to focus the learner’s
attention when he arrived at his sensory register. Anderson (2004) and Matlin (2009) propose that at the sensory register, there are serial bottlenecks that limit the amount of information that can be processed. The bottleneck [as discussed in LE Chapter 2] in visual information processing is “even more apparent than the one in auditory information processing” (Anderson, 2004, p. 79).

The instruction strategy here, is to present information in a contextual manner, to provide opportunity for the learner to approach the mathematic concept using multiple mathematical representations, namely, that of visual imagery and verbal material (Gagatsis & Elia, 2004; Ormrod, 2008; Brumberger, 2007). For example, with the help of a verbal explanation, the concrete example of expression (1) might have helped the learner to visualise the exact meaning of the elements in expression (3) and thus establish a connection between the new information (expression (3)) and existing knowledge (expression (1)). This situation is convergent with the idea of spiral curriculum (Bruner, 1965).

But the understanding at this level might still be very weak. In order to improve the level of understanding, the knowledge network needs to be strengthened and tightened. The basic theme 5 F2.3 describes the event during which the student was probed to reflect on a deeper exploration of the algebraic relationship inherent in expressions (1) and (3). He was able to find out that the logarithmic function was the inverse of the exponential function (See 5. F2.3). This is one of the principles of discovery learning: that individual applies higher-level cognitive skill to make sense out of mathematics quantities as discussed in LE. (See Section 2.3.2).

This process is also consistent with the constructivist view of active learning “which posits that learning involves constructing connections between visual and verbal representations of a system” (Mcloughlin & Krakowski, 2010, p. 2).

In summary, direct instruction, cognitive learning theory, and principles of constructivism were all applied in this step of the tutoring process. Even though discovery learning was not directly applied, there was an element of discovery by the learner. The learner was not told everything, but rather guided towards making the required connection through the use of a simple illustration.
The tutor has been searching for the possibility of incorporating the advantages of both direct instruction and discovery learning. This approach has provided the opportunity for the student to progressively obtain a conceptual understanding of the mathematical topic.

The themes in the group Org2.F2 provide evidence for the effectiveness of this approach.

**Helping the Learner to Construct Schemata (Dis.5.4.1c)**

Another important aspect using a simple illustration is its value in guiding the learner through practice. Simple Steps considers this process as the process of constructing knowledge “through action, through imagery, and through language” (Bruner, 1964, p. 1).

Several things are involved in the design of the problems or exercises. They must be designed to test whether the student has understood the concept at the required level. If this is the case, a little bit harder exercise will be presented to provide opportunities for the student to construct schemas beyond the elementary level. As discussed in section 2.5.5, the teaching of mathematics should be designed to allow the learner to progressively construct schemata at higher levels of understanding. The acquisition of schema can be achieved through the use of worked examples (Sweller & Cooper, 1985). Basic theme 7 F2.2 showed that the student was able to understand the concept after studying just one simple illustration.

**5.5 Data Analysis (part 3 and part 4) for Field Notes of Step 3**

In this step, topics introduced in the previous steps will be presented at a higher level of abstraction. One of the important purposes of the tutoring process is for the student to learn how to manipulate symbolic expressions through developing an understanding of the mathematical concepts involved.

**5.5.1 Categorisation of Field Notes of Step 3**

The 18 themes obtained in the initial analysis were classified into three groups as listed in Table 5.4. Some themes that were refined are labelled as “changed”. Five new themes were generated and these are labelled as “new”. So the result of refinement of themes is the 23 themes.

<p>| Table 5.4 | Categorisation of themes of Step 3 | 111 |</p>
<table>
<thead>
<tr>
<th>Themes</th>
<th>Organising themes</th>
<th>Descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. F3.1 Replacing numbers with letters</td>
<td>Org1. F3 Understands the Roles of Elements in the Expression</td>
<td>Describes how Simple Steps enabled the student to make the transition from numerical expressions to a symbolic system.</td>
</tr>
<tr>
<td>2. F3.1 Not just replacing the numbers (New)</td>
<td></td>
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</tr>
<tr>
<td>3. F3.2 Engaging the student in deeper thinking (New)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4. F3.2 Replacing numbers with symbols gradually</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5. F3.2 Thinking actively on what was going on</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6. F3.2 Learning how to learn (Changed)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7. F3.3 Looking back to find connections (Changed)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8. F3.3 Seeing how a mathematical concept could be analysed</td>
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<tr>
<td>9. F3.2 Visualising the relationships between related functions</td>
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</tr>
<tr>
<td>10. F3.3 The structure was not told (New)</td>
<td>Org2.F3 Problem Solving</td>
<td>Describes the process of problem solving that resulted in the student’s discovery of new concepts.</td>
</tr>
<tr>
<td>11. F3.3 Integrating new knowledge with an existing knowledge base (New)</td>
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<tr>
<td>12. F3.3 Using an example to think about the properties of logarithms</td>
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<tr>
<td>13. F3.3 Probing the student to reflect</td>
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<tr>
<td>14. F3.3 Using an abstract notion to identify the curve (New)</td>
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<tr>
<td>15. F3.4 Amazing discovery (New)</td>
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<tr>
<td>16. F3.4 Finding a connection between different representations (Changed)</td>
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<tr>
<td>17. F3.3 The verbal form of information</td>
<td></td>
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</tr>
<tr>
<td>18. F3.4 Gradually changing in the representation</td>
<td>Org3.F3 Preventing Misconceptions by using a Simple Illustration</td>
<td>Describes how misconceptions were prevented during the process of problem solving.</td>
</tr>
<tr>
<td>19. F3.4 New knowledge is fragile and weak</td>
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<tr>
<td>20. F3.5 Recognises mistake by using an illustration</td>
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<tr>
<td>21. F3.5 Teaching discrimination</td>
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</tbody>
</table>

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by using illustrations

22. F3.6 Noticing and correcting a misconception
23. F3.6 A simple illustration helps the student notice a misconception

misconceptions were prevented during the process of problem solving.

Notes
The group Org 1 F3 describes how Simple Steps enabled the student to make the transition from numerical expressions to a symbolic system.

The group Org 2 F3 describes the process of problem solving that resulted in the student’s discovery of new concepts.

The group Org 3 F3 describes how misconceptions were prevented during the process of problem solving.

5.5.2 Basic Description of Field Notes of Step 3
In this step, the learner’s understanding of logarithms makes a transition from understanding numerical expressions to understanding the symbolic representation.

The following discussion will focus on the practical aspects of the tutoring process which had features of both direct instruction and discovery at the same time. Each feature will be related to the relevant theory.

Ideas Inspired by Discovery Learning and their Effect on Student Understanding (Dis.5.5.2a)
It would be very easy for the student to learn the notion of logarithms by first noticing the relationship introduced in step 2:

\[ 8 = 2^3 \quad \Rightarrow \quad 3 = \log_2 8 \quad (1) \]

then replacing 8, 2, and 3 with \( x \), \( a \), and \( y \) respectively, to get:

\[ x = a^y \quad \Rightarrow \quad y = \log_a x \quad (2) \]

This is what I had planned to do during the tutoring process. Here equations (1) and (2) could certainly be considered to be simple illustrations. But when I asked the student to talk about the idea of generalising numerical examples using the notation of a logarithmic function, he
seemed to have no idea (See basic theme 4. F3.2). Therefore I changed my plan. By analysing the mathematical concept, the student would be given the opportunity to reflect on his learning (See basic theme 6 F3.2 and 8 F3.3). Otherwise the student could have easily slipped back to having only a superficial understanding of logarithms.

This idea was inspired by the “instructional model” (Bicknell-Holmes, et al., 2000, p. 315) of discovery learning that encourages students to generalise knowledge through exploration and problem solving as discussed in section 2.3.2 (See Bicknell-Holmes et al., 2000).

I started the process by replacing 8 and 3 in the expression $8 = 2^3$ with $x$ and $y$ to get:

$$y = 2^x$$

The process of changing the representation enabled the student to associate the symbolic representation with a concrete meaning.

The Learning Atmosphere Supported by Discovery Learning (Dis.5.5.2a1)
The term “problem solving” or “doing exercise” in this study will be used interchangeably. According to Mayer et al. (2006)’s SOI model as discussed in section 2.4.2, problem solving can be thought as the cognitive process of thinking, reasoning, and creative thinking. This definition meets the need of this study.

For example, before introducing the inverse relationship between the logarithms and exponentials, the student was first encouraged to study the properties of a function and its inverses using a simpler case

$$y = 2x$$ and its inverse $$y = \frac{1}{2}x$$ \hspace{1cm} (4)

then the student was guided to find out the inverse of

$$y = 2^x$$ \hspace{1cm} (5)

By comparing system (4) and equation (5) the student was able to identify the similar position of $y = 2x$ and $y = 2^x$, which enabled him to find the inverse of $y = 2^x$. That is $x = 2^y$. These reasoning can be expressed as

$$y = 2x \rightarrow \text{Figure 4.3} \rightarrow y = \frac{1}{2}x \text{ (the inverse)} \hspace{1cm} (6)$$

$$y = 2^x \rightarrow \text{Figure 4.4a} \rightarrow x = 2^y \text{ (the inverse)} \hspace{1cm} (7)$$
(See F1.7 to F1.9 of Table 4.4 in chapter 4) where Figure 4.3 and Figure 4.4a were plotted by the student and the presentation will be called a reasoning matrix. Note that the notation used in the reasoning matrix can be found in books of linear algebra and abstract algebra (See, for example, Cullen, 1972; Bhattacharya, Jain & Nagpaul, 1995), but the meaning used here is different. This reasoning allowed the student to obtain the conceptual understanding of the logarithms \( y = \log_2 x \) through algebraic manipulation as indicated in the following process:

\[
8 = 2^3 \quad \Rightarrow \quad 3 = \log_2 8 \quad \Rightarrow \quad y = 2^x \quad \Rightarrow \quad x = 2^y \quad \Rightarrow \quad \text{Figure 4.4a} \quad \Rightarrow \quad y = \log_2 x \quad \text{(8)}
\]

The reasoning matrix formed by equation (6) and (7) show that the targeted solution \( x = 2^y \) is not just a surface copying of \( y = \frac{1}{2} x \). Rather, it catches the deep similarity related to the abstract meaning of an inverse function. So it is in agreement with Bruner’s idea that encourages a learner to discover the “regularities of previously unrecognized relations and similarities between ideas” (Bruner, 1965, p. 20). A more theoretical discussion will be given in section 5.7.5.

**Using the Principle of Direct Instruction to Control the Course of Concept Learning (Dis.5.5.2a2)**

One of the pedagogical methods of teaching harder concepts in this study is to present the material in different forms of representations in a gradual order (if the material allows doing so).

One important thing about sequence (8) is that it is not unique for different individuals. It was the tutor who was responsible to design hands-on simple illustrations according to the intellectual background of the student at different stages as the tutoring process was progressing. The tutor also needed to provide adequate underlying knowledge (Frederiksen, 1984) to the student to prevent working memory overloading and to guide the student on the right track to the targeted concept.

**A Brief Explanation (Dis.5.5.2a3)**

This progress may probably be explained by the following description. As the student went through the learning sequence (8), his different modes of thinking have been stimulated when working on different forms of representation through the process of articulating different
forms of simple illustrations. This enables the learner to approach the concepts from different angles in a spiral curriculum (Bruner, 1965). Moreover, the gradual changing of the presentation will allow the learner to reflect on what is going on by directly experiencing the calculation involving guessing, graphing, and comparing through the passageway of gradual change of representations of related mathematical contexts. This process will activate the student’s schemas “to either assimilate or accommodate new problem solution” (Steele, et al., 2004, p. 68) into the knowledge network.

In this way, the student established a conceptual connection between exponential function and logarithmic function (See basic theme 14. F3.3 & theme 16. F3.4), and acquired conceptual understanding.

**About the Analogical Reasoning (Dis5.5.2b)**

The theory of analogical reasoning can be used to support the idea of simple illustration. During the passage toward targeted formulas or concepts, the student was guided to work out almost each term in sequence (8). The result of these calculations was a set of algebraic patterns which will be called simple illustrations in this study. They provide concrete descriptions of more abstract concepts and allow the student to reason by analogy (Mayer, 1991) step by step during problem solving.

As Robins and Mayer (1993) pointed out that education psychology study supported the process of breaking down a cognitive task into component. And many researchers also agreed that:

“Direct instruction in component processes has been shown to improve student’s performance on inductive reasoning task such as analogical reasoning” (p. 529). Robins et al., (1993) also proposed that subsequent problems solving should help student form schemas. Many researchers agreed that analogical thinking and practice promoted understanding and schemata forming (Gick & Holyoak, 1983; Cooper & Sweller, 1987; Chen, 1999). Robins et al., (1993) stated that the processes of solving problem using analogy was active since it involved in the cognitive process of constructing individual knowledge.

A deeper and continual discussion will be given in section 5.7.4 and 5.7.5.
5.6 Data Analysis (Part 3 and Part 4) of Field Notes for Simple Step 4

The purpose of this step is to study the procedure used to calculate logarithms. The ideas that arose from the previous steps will be further investigated in relation to the domain of procedural knowledge.

5.6.1 Categorisation of Field Notes for Simple Step 4

In table 5.5, the themes obtained from the initial analysis have been classified into organising themes related to the topic of procedural learning. There are now 26 themes classified into three organising themes. One theme (F4.1) has been changed and five new themes have been added.

Table 5.5
Categorisation of themes of step 4

<table>
<thead>
<tr>
<th>Themes</th>
<th>Organising themes</th>
<th>Brief Descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. F4.1 Learning to learn (Changed)</td>
<td>Org1.F4. Searching for solutions</td>
<td>Describes how the student was encouraged to construct simple numeric illustrations and look for algebraic patterns.</td>
</tr>
<tr>
<td>2. F4.2 Solving a problem intuitively</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. F4.3 Using a concept to examine a pattern</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4. F4.5 Articulating simple illustrations to improve manipulation skills</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5. F4.6 Learning to construct a simple illustration</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6. F4.7 Pattern recognition leads to understanding</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7. F4.8 Exchanging between numeric and symbolic patterns (New)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8. F4.8 Triggering understanding (New)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9. F4.2 Experiencing cognitive conflict</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10. F4.2 Developing and acquiring schema</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11. F4.2 Recognising similar patterns</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12. F4.3 Being concerned about how to obtain results</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13. F4.4 Making knowledge automated</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14. F4.4 Approaching the formula via steps accessible to the learner</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15. F4.5 Simple illustrations allow learning to be monitored</td>
<td></td>
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<tr>
<td>16. F4.6 Learning how to use a</td>
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<td>17. F4.8 Analogical reasoning and schemata formation</td>
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<td>18. F4.7 Pattern matching helps reasoning</td>
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<td>19. F4.2 Reflecting on what mistake had been made</td>
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<td>20. F4.5 Using numeric illustrations to articulate the meaning of algebraic expressions</td>
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<tr>
<td>21. F4.5 Expressing thinking using a simple illustration (New)</td>
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<tr>
<td>22. F4.5 Using a numeric illustration to visualise abstract expressions</td>
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<tr>
<td>23. F4.6 Identifying a pattern can correct mistakes</td>
<td></td>
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<tr>
<td>24. F4.7 Simple numeric illustrations carry logical clues</td>
<td></td>
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<tr>
<td>25. F4.2 Ambiguous notation (New)</td>
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<tr>
<td>26. F4.3 Cautiousness could reduce the chance of errors (New)</td>
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</table>

**Note**

Org1.F4 describes how the student was encouraged to construct simple numeric illustrations and look for algebraic patterns.

Org2.F4 describes how the instructions were given to engage the student in the exploration and construction of a general formula using algebra.

Org3.F4 describes how the tutor and the student worked together to minimise student’s errors or misconception.

### 5.6.2 Basic Description of Field Notes for Simple Step 4

As discussed in section 4.3.2, research on procedural knowledge has not received enough attention. According to Hostos, Hostos, and Bronx (2004) “the current characterization of conceptual and procedural knowledge reflects limiting assumptions about how procedures are known” (p. 404). Most classroom studies focus on the computational model of procedural knowledge rather than studying the depth of meaning inherent in this knowledge. The current study explores a way of teaching procedural knowledge that emphasises the underpinning concepts (Star, 2005).
Exploration of Procedural Knowledge with Concepts (Dis.5.6.2a)

One of the ways of acquiring deep understanding of a formula, according to the tutor in this study, would be to learn how to prove or derive each formula encountered. The first formula needed to be proved was:

\[ \log_a xy = \log_a x + \log_a y \quad (1) \]

The student had no idea how to do this. According to Mayer (2004), in this case, the student would fail to select the relevant material. He mentioned that direct instruction could be used to promote the effectiveness of this step. Thus this was the type of situation where the principle of direct instruction was applied. The method used was to change the presentation of equation (1) into a numeric example:

\[ \log_3(9 \times 27) = \log_3 9 + \log_3 27 \quad (2) \]

The purpose of this instruction was a little different from the direct instruction used in step 3 of Simple Steps. As discussed in Dis5.6.2a, now that the student had learnt the concepts needed, it was time for him to experience how these concepts could be used to solve problems in a new situation. This process is similar to discovery learning in that the emphasis is on the analysis and interpretation of information in order to gain a deeper level of understanding (Castronova, 2001). The purpose of learning a formula is not just to enable calculation. The process of learning a formula also enables the student to reflect on his or her learning.

For example, the purpose of giving equation (2) was to help the student visualise the abstract meaning of equation (1). It provided an opportunity for the student to realise that one could tackle a problem from a different perspective. This active effect can be seen from the fact that the student was immediately able to construct the equation

\[ x = \log_2 243 \quad (3) \]

when he was required to prove equation (2). But then he stopped. So the student would not move too far away from the right track, I suggested that he concentrate on smaller numbers such as:

\[ 8 = 2^3 \quad \rightarrow \quad \log_2 8 = 3 \quad (4) \]

This equation had enabled him to recognise that the \((9 \times 27)\) in equation (2) could be treated in the same way as 8, so by analogy he got

\[ (9 \times 27) = 3^5 \quad \rightarrow \quad \log_3(9 \times 27) = \log_3(3^2 \times 3^3) = \log_3 3^5 = 5 \quad (5) \]
Previously, the student had “cancelled” within the equation:

\[ \log_3 3^5 = 5 \quad (6) \]

but without knowing the reason (See theme 2. F4.2). The interaction between the tutor’s instruction and the student’s response is diagrammatically described in Figure 5.1a.

![Diagram of interaction between tutor's instruction and student's response](image)

Figure 5.1a

In Figure 5.1a, “Equ1” represents the equation (1). “Ins” represents the tutor’s instructions. “Res1” represents the response 1 from the student. The arrow from Equ1 to Equ5 represents the main direction of the learning guided by the tutor. The arrow from Equ2 to Equ3 represents the student’s response to the instruction represented by Equ2. Similarly, Equ6 was the result or influence of Equ5. The arrow that points in a different direction represents the student’s different way of reasoning. When I asked him to explain the reason why \( \log_3 3^5 = 5 \), he answered:

S: Confused, I’m not sure how to take this (\( \log_3 3^5 \)) to that (5).

I considered it critically important that the student used the word “confused”. I realised there was a need to explore the formula:

\[ \log_a a^x = x \quad (7) \]

But I did not directly present this formula. Instead, I gave the student an opportunity to deepen his level of understanding by reflecting on his confusion (See theme 25. F4.2). This is where the principle of discovery learning came into play, with a slight difference. I first showed him that because \( 8 = 2^3 \), one could directly obtain the 3 using the definition:

\[ \log_2 2^3 = \log_2 8 = 3 \quad (8) \]

This was neither a proof nor a solution to the puzzle hidden in equation (6). It was a fact that the student was already familiar with. I expected he would answer the question using a
similar argument but he used another approach as indicated in the following. From what he got before, he wrote

\[ x = \log_3 243 = \log_3 3^5 \]  \hspace{1cm} (9)

and using the definition of logarithms he got:

\[ 243 = 3^5 = 3^x \]  \hspace{1cm} (10)

So

\[ x = 5 \]

Thus the simple illustration of equation (7) had triggered a connection between new knowledge and existing information. Perhaps this event illustrates what Bruner (1965) said about discovery – “discovery of regularities of previously unrecognized relations and similarities between ideas” (p. 20).

As discussed in sections F4.3 and F4.4 of Table 4.7, after several practices the student obtained the generalisation

\[ \log_a a^x = x \]  \hspace{1cm} (11)

With the help of equation (11), the student was able to show:

\[ \log_3 (9 \times 27) = \log_3 9 + \log_3 27 = 2 + 3 = 5 \]  \hspace{1cm} (2)

When he replaced 9 and 27 with \( x \) and \( y \), he got the identity

\[ \log_a xy = \log_a x + \log_a y \]  \hspace{1cm} (1)

For the proving of equation 1, it was important to allow the learning to progress in steps that were accessible to the learner. Since in stage 6 mathematics many topics usually involve several basic formulae, teachers need to treat the topic flexibly (See section F4.4 of Table 4.7). Figure 5.1b describes the interaction between the tutor and the student in this process. It is a continuation of the diagram in Figure 5.1.a:

![Figure 5.1.b](image-url)
In Figure 5.1.b, Ins1 represents the tutor’s instruction, using equation (8). “Verbal” represents the tutor’s verbal instructions that directed the student’s reasoning from equation (6) to the main stream. Equation (7) is not in the diagram because this equation was not actually given. The dotted line represents the tutor’s intended direction of the learning. Res1 and Res2 indicate that the student was trying to use his own way of reasoning via equations (9) and (10) to equation (2) then reasoned further (Res3) from equation (2) to equation (1). The path joining Equ9 and Equ10 is parallel to the main stream indicating that the student was reasoning (or calculating) as the tutor intended. This brought him back to the targeted identity (Equ1). A summary of this interaction is given in the next section.

After obtaining this identity, the student was guided to prove a generalised form of the equation (7) \( \log_a a^x = x \):

\[
\log_a x^n = n \log_a x \quad (12)
\]

in just three steps, namely:

Step1: \( x = a^y \) (so \( y = \log_a x \)) \quad (13)

Step2: \( x^n = a^{yn} \) \quad (14)

Step3: \( \log_a x^n = yn = n \log_a x \) (by definition) \quad (15)

Done! What a beautiful proof! (See section 4.5 of Table 4.7). Careful analysis of the pattern something enabled the student to prove some quite complicated. For example the student proved the identity:

\[
\log_b x = \frac{\log_a x}{\log_a b} \quad (16)
\]

For details, see section F4.6 of Table 4.7. During such a learning environment, the student’s understanding of logarithms was gradually improved.

Summary of the Process of Learning a Basic Procedure (Dis. 5.6.2a1)

In step 4 of Simple Steps, the student was required to learn how to prove or derive all the formulae studied. This was quite different from step 3, where the numeric illustration was used to teach the concept being introduced. Here in step 4, the emphasis is on “discovery”, in the sense that the student must finally work out the formula under the guidance of the tutor. The process of constructing simple illustration was used as a major tool to develop the skill of algebraic manipulation. Moreover, by observing how the student articulated the illustration,
the tutor could easily identify student errors or misconceptions as they occurred (See Org3.4F).

Figure 5.1.b reveals some feature resulted from the incorporation of direct instruction and the process of simple illustration manipulation. Firstly, one can see that the direction of the learning is guided by the tutor. This allows Simple Steps to utilise the advantage advocated by direct instruction that “provide organizing schemes for novices in a domain that help coordinate information in working memory” (Rittle-Johnson, 2006, p. 1).

On the other hand, as the student is constructing an illustration, he has to think clearly about the role of each element in the equation he is manipulating. He is also assimilating information from the illustration given by the tutor. This process allowed the student to select the relevant information effectively through the process of constructing algebraic patterns under guidance, to see how the individual formula was blended coherently into a knowledge network allowing him to built connections between familiar examples and more abstract concepts (Mayer et al., 2006).

5.7 Data Analysis (Part 3 and Part 4) of Field Notes for Simple Step 5

Steps 1, 2, and 3 of Simple Steps focus on conceptual understanding. Step 4 focuses on procedural knowledge. In Step 5 the aim is to establish a connection between conceptual understanding and procedural knowledge.

In Step 5, the student was introduced to specific types of problems involving logarithms that he was required to do in stage 6 Mathematics. The analysis of this step explores the possibility that a student could obtain a deeper understanding of a topic via the learning and practice of procedural skills.

5.7.1 Categorisation of Field Notes for Simple Step 5

The themes obtained from the initial analysis will now be classified into groups under an organising theme. As some of the individual themes were found to overlap, they needed to be refined. They also needed refining to focus attention on the salient issues that arose from the analysis (Attride-Stirling, 2001; Creswell, 2008).
The process of data reduction used here is a modification of the process described by Creswell (2008). Initially the text data segments were examined using the themes as a guide. Themes with a similar meaning were then combined, provided that an organisng theme did not become so broad that it failed to adequately describe the specific situation (Attride-Stirling, 2001). Table 5.6 shows the result of this refinement.

Table 5.6
Categorisation of Themes of Step 5

<table>
<thead>
<tr>
<th>Themes</th>
<th>Organising Themes</th>
<th>Descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. F5.4 Teaching explicitly with modified ideas</td>
<td>Org1. F5 TS examples and gradual change</td>
<td>Describes how TS examples illustrate a concept, and stimulate the student to look for algebraic patterns.</td>
</tr>
<tr>
<td>2. F5.6 TS examples (New)</td>
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<tr>
<td>3. F5.6 Small problem and TS example</td>
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<td></td>
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<tr>
<td>4. F5.1 Learning mathematics through problem solving</td>
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<tr>
<td>5. F5.3 Avoid complicated theoretical reasoning when introducing a concept</td>
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<tr>
<td>6. F5.5 Building up from a small problem</td>
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<tr>
<td>7. F5.2 Requiring a high level of abstraction</td>
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<tr>
<td>8. F5.5 Procedural skill is being improved (Changed)</td>
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<tr>
<td>9. F5.9 Discussion of a misconception and mistake</td>
<td>Org2. F5 Small problems and misconceptions</td>
<td>Describes how instructions were given to engage the student in the exploration and construction of a general formula using algebra.</td>
</tr>
<tr>
<td>10. F5.6 Small problem could be given any time</td>
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<tr>
<td>11. F5.9 Using concepts to analyse a problem</td>
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<tr>
<td>12. F5.9 Positions of numbers might cause confusion</td>
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<tr>
<td>13. F5.6 Using a small problem to detect the student’s errors</td>
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<tr>
<td>14. F5.10 Took a long time to realise the fact</td>
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<tr>
<td>15. F5.12 Critical factors related to reasoning difficulty</td>
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<tr>
<td>16. F5.12 Lack of discrimination between algebraic structures (changed)</td>
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<tr>
<td>17. F5.13 Eliminate possible obstacles to the individual</td>
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</table>
18. F5.13 Teaching is needed to enable discrimination
19. F5.8 Probing for the correct answers (New)
20. F5.6 A little confusion was expected (New)
21. F5.7 Identifying reasoning difficulty
22. F5.7 Could not denote the point (Change)
23. F5.6 Small problem and gradual change
24. F5.10 Notice when x=1, then y=0
25. F5.5 Using worked examples

26. F5.11 Using a simple illustration as a benchmark
27. F5.6 Simple illustration as a reminder
28. F5.6 Making mistakes (New)
29. F5.8 Improving procedural skill by doing harder problems.
30. F5.6 Small problems act as simple illustrations
31. F5.7 Express thought using written language
32. F5.8 Following what the student thought
33. F5.8 Learning how to express mathematical relations
34. F5.14 The linkage between existing knowledge and the procedure (New)
35. F5.15 Create mathematical representations according to own understanding
36. F5.14 Keep track of the calculations
37. F5.8 Raise the level of abstraction
38. F5.7 Reminder to help retrieve knowledge
39. Giving insinuation to guide reasoning

Org3. F5 Representations and simple illustrations

Describes how the tutor and the student worked together to minimise the student’s errors or misconceptions
Note
There are 35 themes obtained from the initial analysis, three of them were changed, and five new themes had been added. The result was 40 themes in total. They were classified into 3 groups.

Group Org1.F5 describes how a TS example (See following discussions) illustrating a concept, stimulated the student to look for algebraic patterns.

Group Org2.F5 describes how instructions were given to engage the student in the exploration and construction of a general formula using algebra.

Group Org3.F5 describes how the tutor and the student worked together to minimise the student’s errors or misconceptions.

5.7.2 Basic Description of Field Notes of step 5
This unit of the analysis will discuss the data obtained from step 5 of the tutoring process. As mentioned previously, this step aims to establish a connection between conceptual understanding and procedural knowledge. The discussion is about the same events described in Chapter 4, so there will inevitably be some repetition of the information in Table 4.8 and its initial analysis.

Several types of problems were studied during the tutoring process (see table 4.9 of chapter 4). The discussion here mainly focuses on problems related to the concept of increasing and decreasing functions. This topic was chosen because it requires a level of abstraction that is well beyond the accessibility of most students at the senior high school level. As a result, the computations involved are usually passively taught. The challenge was therefore to avoid deep theoretical reasoning and teach difficult concepts using a more intuitive approach that lay within the zone of proximal development of the learner at the introductory level of a topic (See basic theme Org1).
Step 5 is different from the previous steps in that it unavoidably introduces into the problem solving process, some knowledge and concepts beyond those of logarithms. This is after all the nature of the problem-solving.

Problems involving logarithms can be quite complicated for a beginner. In order to engage the student in active learning and not impose cognitive overload on his working memory, it was suitable to teach problem solving explicitly using a worked example as suggested by Sweller et al. (1985) (Sweller, 2004; Robins et al., 1993) but the methods used here are somewhat different to the use of worked examples. This difference is discussed in the following sections.

**The TS Example (Dis.5.7.2a1)**

The name TS example appeared in F5.5 of Table 4.8 where T stands for “Teacher” and S for “Student”, meaning teacher and the student worked together to get the solutions (See basic theme 2.F5.6).

One aspect of understanding (discussed in section 3.1.1) is schema acquisition. Problem solving using worked examples is recommended by many researchers (Star, 2001; Sweller, 2003) as a means of enhancing schema acquisition. This process occurs in Simple Steps through using the TS example, but with a modified perspective.

The effectiveness of using worked examples as a substitute for problem solving has been proposed and studied by Sweller et al. (1985), Cooper et al. (1987), Sweller (2004), and Robins et al. (1993). The basic assumption is that analogical reasoning enhances schema acquisition during the process of problem solving (Chen, 1999; Cooper et al., 1987 Gick et al., 1983; 1987; Sweller et al., 1985).

Simple Steps is quite different from other pedagogical approaches in the way that worked examples are used. It tries not to allow the students to just sit silently watching the tutor’s illustration. Instead, the tutor tries to work together with the student to find solutions, even when a new topic or a formula is being introduced (see F1.9 in Table 4.4 of chapter 4). The tutor creates a “small problem” that directly relates to the content being studied. (See, for example, the discussion of Dis.5.6.2a and F4.1 in Table 4.7). This process results in a set of cues and paradigms that the tutor can then use to illustrate different aspects of the concept.
The basic Theme 25. F5.5 gives an example of how a TS example was presented. When the student was asked to work out where the graph of \( y = \log_a x \) would cut the \( x \)-axis. He had no ideas of how to start. Perhaps the presentation is too abstract at the first glance. Instead of directly telling him how to solve the problem, I remind him of using the relation:

\[
8 = 2^3 \quad \rightarrow \quad 3 = \log_2 8 \quad (20)
\]

This is not a direct example that solved the problem but only a pattern constructed earlier by the student. It could therefore serve as a cognitive cue that directs the student to relate new information to the prior knowledge stored in the long term memory (Goldstein, 2011). To accomplish a TS-example, however, the student must take part in the process of constructing the example, so he was guided to draw the graph that could inform the solution of the problem as shown in Figure 5.2

![Figure 5.2](image)

During the process of sketching, he recognised that the role of \( a \) was similar to 2 while that of \( x \) was similar to 8. So he carried on

\[
y = 0 \quad \rightarrow \quad 0 = \log_a x \quad (21)
\]

and

\[
a^0 = x \quad \rightarrow \quad x = 1 \quad (22)
\]
and the TS-example was accomplished (See A). The student’s procedural skill was improving through the process of working out solutions to examples in co-operation with the tutor. This example was later used as a guide to explore a deeper problem in the next section.

Note that a TS example uses some features of direct instruction such as the provision of step by step instructions and adequate information and assistance to the student. But it is different from direct instruction in that it provides more space for the student to reflect on abstract concepts and gradually blend them with numerical understandings through the process of “small problem solving”.

**Small Problem Solving (Dis.5.7.2a2)**
Immediately after the TS example given above, the student was asked to investigate the properties of a decreasing function using $y = \log_a x$ (for $0 < a < 1$). This problem was the same as the one just given but with the condition of “$a$” changed. I call this a “small problem” (see theme 23. F5.6).

An earlier example of a small problem was when, after plotting the graph of the inverse function of $y = 2x$ (see theme F1.7 in Table 4.4 of chapter 4), the student was asked to sketch the graph of the inverse function of $y = 2^x$ (see theme 5, F1.11 in Table 5.2). In doing so he had to understand the analogy. This is not a surface similarity. It “involves understanding a principle to solve a problem in one domain, and creating an analogous solution for a new problem in another domain” (Murphy et al., 1999, p.664).

Small problems could be given at any time (see theme 10. F5.6). Whenever the tutor detects an error or a sign of a misconception, or wants to clarify an ambiguity, or raise the level of the understanding, he could give a small problem. The content given in the small problem must be similar to what is being discussed and the small problem should be easy for the student to solve if he pays attention to the similarity.

Theme 10.F5.6 illustrates such a situation. The student was asked to plot the graph of $y = \log_a x$ but the condition changed from

$$y = \log_a x \text{ (for } a > 1) \text{ to } y = \log_a x \text{ (for } 0 < a < 1)$$

(23)
The student needed to find the $x$ co-ordinate when $y = 1$. As discussed, the student made a mistake by letting

$$x = a^0 \quad (24)$$

I believed that he confused $y = a^x$ with $x = a^y$. But by comparing the pattern in equation (20) and Figure 5.2, he recognised that when

$$y = 1, \ x = a^1, \text{ not } x = a^0$$

The student wrote:

"$y$" $x = a^1 \quad (25)$

so

$$x = a \quad (26)$$

The student wrote “$y$” in equation (25) to remind himself that “1” corresponded to “$y$” and not “$x$” (See Figure 5.3 A). So the small problem guided his reasoning.

Equation (25) could be considered a simple illustration constructed by the student. It indicated that the student had identified his misconception so the tutor could now design another TS example.

Simple illustrations allow the student to think by analogy (Murphy & Panchanadam, 1999). Recognising a pattern makes schema recognition possible (Milligan, 1979, p. 198).
Identifying Misconceptions with the Help of Simple Illustrations (Dis. 5.7.2a3)

After the brief introduction, the student was required to express his thinking using both oral and written language. He mistook the role of an independent variable with that of a dependent variable due to a misconception.

Figure 5.4 shows a brief summary given by the student under the guidance of the tutor:

\[
\text{If } y = f(x), \text{ if } x_2 > x_1, \text{ and } f(x_2) > f(x_1), \text{ then } f(x) \text{ is increasing in that domain} \quad (27)
\]

The student wanted to draw a graph for an arbitrary increasing function. He drew the x variable correctly but could not denote the point \((x, f(x))\) correctly and was not sure how to draw the corresponding graph (See Figure 5.4 A). I believe this was because the discussion was too abstract. A more concrete illustration was needed for him to be able to visualise the mathematics.

This step was different from the previous ones in that it required knowledge of concepts other than logarithms to be introduced, thus allowing the possibility of additional misconceptions being revealed during the learning process. It also provided opportunities for the tutor to help the student by referring back to simple illustrations constructed previously.
Another TS example (Dis. 5.7.2a4)
The fact that the student received both verbal and visual information could have stimulated his thinking (Bruner, 1985; Bruner, 1964) and allowed the tutor to follow his train of thought from a different angle.

The student’s algebraic and graphic representations indicated a confusion that could become an obstacle to learning. The tutor was able to help by explicitly pointing out the similarities and differences (see F5.11, F5.12 and F5.13 of Table 4.8 in chapter 4).

Another TS example (See theme 29 F5.8) was presented to:
1. allow the student to reflect further on new concepts
2. increase the student’s level of understanding within his zone of proximal development
3. make the tutor aware any additional misconceptions.

What follows, is a segment of the conversation surrounding the TS example chosen from F5.8 in Table 4.8. The conversation is about the increasing and decreasing properties of logarithms, beginning with the graph of the increasing function \( y = \log_a x \) (for \( a > 1 \)).

T: Since \( y_2 > y_1 \), the function is increasing. \( (28) \)

While I was speaking, the student was required to write down what he thought. He wrote:

Since \( y_2 > y_1 \), then \( \log_a x \) is increasing \( (29) \)

Furthermore, the student was required to draw the graph. The result is shown in Figure 5.5.

![Figure 5.5](image)

A comparison of Figure 5.5 with Figure 5.4 shows the student’s progress. For example:
a. The value of the function $y = f(x)$ has been correctly chosen in Figure 5.5 while in Figure 5.4 the student was unable to plot the general form of the curve $y = f(x)$ for an increasing function.

b. The $x$ variables in Figure 5.4 were specifically chosen so that one of them was negative and the other was positive, while the variables in Figure 5.5 were randomly chosen, reflecting increased confidence (theme 19 F5.8).

The student’s graphical interpretation shows that his misconception presented in Figure 5.4 has been replaced by a correct supplementary concept.

Next, we talked about the condition when $0 < a < 1$.

T: Now what happens if $0 < a < 1$? (30)

S: Decreasing...

Then he added hesitantly:

S: ...In the case of an exponential function we have $y = a^x$.

His hesitation showed that he was not sure about the case of $y = \log_a x$ ($0 < a < 1$). Supplementary concepts were therefore needed for clarification, so I moved to the topic of exponential functions (See F5.8 of Table 4.8).

T: Now, if $a > 1$, and $x_2 > x_1$, what can you say? (31)

S: $a^{x_2} > a^{x_1}$ (32)

T: and if $0 < a < 1$, what can you say (for $x_2 > x_1$)? (33)

S: $a^{x_1} < a^{x_2}$ (34)

The student was incorrect. After I insinuated he was wrong, he thought for a while then corrected his answer to

$a^{x_1} > a^{x_2}$ (35)

T: Good.

Next, the relationship between the exponential function and the logarithms was revised, and the student was asked to draw the graph of a decreasing function using $y = \log_a x$ (for $a < 1$) as an example. The result is shown in Figure 5.6.
He summarised this case as:

since $y_2 < y_1$, then $\log_a x$ is decreasing. \hfill (36)

After this, the student was ready to tackle harder problems.

**Tackling Harder Problems (Dis.5.7.2a5)**

The student was presented two harder problems:

3. Compare the value of $\log_5 6$ and $\log_6 5$. \hfill (37)

4. A harder question (See Appendix B). \hfill (38)

The student’s solution to the second problem is presented in Appendix B: a harder question.

The most significant event while solving problem (37) is briefly summarized in the following:

At first, the student found it hard to distinguish the roles of “5” and “6” (See Theme 12 F5.9). But after the discussions with the tutor, he was able to interpret the meaning of the algebraic expression by comparing the graph he drew (See Figure 5.7)
He also noticed the connection between

\[ \log_a x^n = n \log_a x \]  

(39)

and

\[ \log_5 5 = 1 \]  

(40).

Using this relation, together with the properties of an increasing function, he got:

\[ 1 = \log_5 5 < \log_5 6 \]

but

\[ \log_6 5 < \log_6 6 = 1 \]

So

\[ \log_6 5 < \log_6 6 = 1 = \log_5 5 < \log_5 6 \]  

(41)

and

\[ \log_5 5 < \log_6 6. \]

An important part of this process was to distinguish between the roles of “5” and “6”. Also which allowed the student to discover the critical benchmark (See Theme 26 F5.11)

\[ \log_5 5 = 1 \]

The establishment of this benchmark triggered the correct reasoning. The critical factors (theme 15 F5.12) are:

1) If \( x \leq a \) then \( \log_a x \leq 1 \) (for \( a > 1 \))  

(42)

2) If \( x \leq 1 \) then \( \log_a x \leq 0 \) (for \( a > 1 \))  

(43)
These critical factors are associated with the point (0, 1) which divides the graph into two parts (see A in Figure 5.7). In the process of problem solving, these two factors might have been entangled with each other resulting in an obstacle to understanding.

The above discussion illustrates the necessity of direct teaching to enable the learner to discriminate between “the unique aspects of seemingly similar situations” (Snowman, Dobozy, Scevak, Bryer, Bartlett & Biegler, 2009, p. 233).

### 5.7.3 A further Discussion on the Data of Step 5

To explain what was going on during this step, it has been divided into a sequence of events. These events resulted from the interactions between the tutor and the student as the tutor attempted to improve the student’s understanding through teaching procedural skills. The events show the students’ conceptual progression. The data discussed in this step is summarised in Table 5.7.

#### Table 5.7

**Brief Summary of the Data of Step 5**

<table>
<thead>
<tr>
<th>Name and Location</th>
<th>Representations</th>
<th>Content</th>
</tr>
</thead>
<tbody>
<tr>
<td>TS example 1 (p. 126)</td>
<td>Equations, Figures</td>
<td>$8 = 2^3 \rightarrow 3 = \log_2 8$ , (20)</td>
</tr>
<tr>
<td></td>
<td></td>
<td><strong>Figure 5.2</strong></td>
</tr>
<tr>
<td>Small problem solving 1</td>
<td>Equations, Figures, Written reflection</td>
<td>$x = a^0$ (24), “y”, $x = a^1$ (25),</td>
</tr>
<tr>
<td>(p. 128)</td>
<td></td>
<td><strong>Figure 5.3</strong></td>
</tr>
<tr>
<td></td>
<td></td>
<td><strong>Verbals expressions (27),</strong> Figure 5.4</td>
</tr>
<tr>
<td>Small problem solving 2</td>
<td>Verbal expressions, Figures</td>
<td>Verbal expression (29), Inequality (31),(32),(33),(34), (35),</td>
</tr>
<tr>
<td>(p. 129)</td>
<td></td>
<td><strong>Figure 5.5,</strong> Figure 5.6</td>
</tr>
<tr>
<td>TS example 2 (p.130)</td>
<td>Inequalities, verbal expressions, Figures</td>
<td><strong>Problem (37), Problem (38)</strong>,</td>
</tr>
<tr>
<td></td>
<td></td>
<td><strong>Figure 5.7</strong></td>
</tr>
</tbody>
</table>
The Diagrammatic Representation of the Data

Figure 5.8 was developed during the process of data analysis. It shows the interaction between the student and the tutor that resulted in the student’s conceptual progression. The arrows going straight across the figure represent the intended track of instruction by the tutor. The other line of arrows represents the student’s track. If the student’s work is correct, it is placed above the intended track, and if it is incorrect, it is placed below the intended track.

These abbreviations have been used in the diagram:

- “Sp” means the small problem solving carried out by the student.
- “Res” means the response of the student to the tutor’s instruction.
- “Equ” means equation.
- “Ine” means inequation.
- "Verbal 29” represents the verbal statement (29).
- “Suppl” means supplementary idea 1.
- “Fig” means Figure.
- “Ref” means the student’s reflection
Figure 5.8: The Conceptual Progression
The first small problem solving began after the first TS example. The student made a mistake in Equation 24. Following reflection, he corrected the mistake and got Equation 25, then Figure 5.3. After a second small problem solving, the student arrived at Figure 5.4, which was incorrect. By doing a second TS example in which a supplementary idea was provided, the student produced a much improved graph (Figure 5.5). In response to further questioning he made another mistake (Ine 34) so the tutor gave him another supplementary idea, resulting in Ine 35. From this, a third small problem solving was carried out resulting in Figure 5.6.

Critical factors were found in response to solving two more problems (Ine37 and Ine38). For the problem solving process used for the problem in Ine37 see Dis.5.7.2a5; and for the problem in Ine 38, see Appendix B

**The Description of Conceptual Progression**

Several practical elements of Simple Steps have been developed during this study. These elements are the use of simple illustrations, gradual change, supplementary ideas, TS examples, and small problem solving. In this section, the integration of these elements (Figure 5.8) and their relationship with the principles of direct instruction and discovery learning will be discussed.

The development of the practical principle of Simple Steps has been guided by several pedagogical approaches. Some of these principles (such as persistently providing adequate information to the student at any time in a step by step manner) are features of direct instruction but the instruction has been modified in Simple Steps (by introducing the idea of TS examples and small problem solving) to actively engage the student in the learning process. Each box of Figure 5.8 is associated with a certain kind of mathematical representation obtained during the process of articulating or explaining a concept based on a simple illustration. It is therefore the product of the cognitive interaction between the tutor and the student.
Before preparing the diagrammatic presentation of the conceptual progression (Figure 5.8), I did not notice the oscillation that was occurring between correct and incorrect reasoning. This oscillation is related to the idea of gradual change, which is consistent with Piaget’s view of learning. The oscillation shows the cognitive changes an individual must perform in order to adapt to a changing environment.

When the individual cannot interpret new information with his existing schema, disequilibrium is created (Sandwell, 1995; Slavin, 2000; Piaget, 1967). In order to re-establish equilibrium, the individual needs to ‘accommodate’ the information by changing his cognitive structure (Piaget, 1967). As Piaget (1964) pointed out: “there is needed when something either outside ourselves or within us (physically or mentally) is changed and behaviour has to be adjusted as a function of this change” (p. 6). In the next section, TS example 2 is used as a basis for discussion about some of the important issues relating to this change.

The Process of Gradual Change (Dis.5.7.3b1)
The idea of gradual change arose from step 1 (as discussed in section 5.3.2 and sequence (8) of Dis. 5.2a1). It refers to the process of presenting targeted materials gradually in order of increasing difficulty or abstraction. The order of presentation is not unique; it varies according to the student’s level of understanding i.e. the knowledge the student possesses that would enable him to tackle the problem. Figure 5.8 illustrates one such order.

In Figure 5.8, the box Fig 5.4 is the point where the student could not construct a correct picture corresponding to the abstract statement (27):

\[
\text{If } y = f(x), \text{ if } x_2 > x_1, \text{ and } f(x_2) > f(x_1), \quad \text{then } f(x) \text{ is increasing in that domain.}
\]

At this point, a “deliberate arrangement of events to facilitate a learner’s acquisition of some goal” (Driscoll, 2005, p 23) was needed, so direct instruction came into play. De Jong (2005) calls “support that steers the learner in a certain direction” as directive support (p. 217). Directive support was carried out by a TS example, which presented the material gradually in a form that would stimulate the student’s thinking. For example, TS example 2 started with Fig
5.4. In response to TS example2, the student made a correct verbal statement (29) and plotted Fig 5.5, which was much better than Fig 5.4. After comparing

\[ a^{x_1} \quad \text{and} \quad a^{x_2} \quad (\text{for} \quad 1 < a) \]  

the question was changed to comparing:

\[ a^{x_1} \quad \text{and} \quad a^{x_2} \quad (\text{for} \quad 0 < a < 1) \]  

The change from (44) to (45) is gradual. The student made a mistake as a result of this change (See Ine 34 in Figure 5.8). Supplementary ideas were immediately provided using TS example 2. After discussion, the student corrected the mistake resulting in Ine 35. He then made great progress as already described in F5.11 and F5.14 of Table 4. This process can be represented as the sequence:

\[ \text{Fig5.4} \rightarrow \text{TS example2} \rightarrow \text{Verbal 29} \rightarrow \text{Fig5.5} \rightarrow \text{Ine 34} \rightarrow \text{Ine 35} \rightarrow \text{Sp 3} \rightarrow \text{Fig5.6} \rightarrow \text{Targeted problems} \]  

(46)

Sequence (46) began with the act of TS example2. The act of TS example1 resulted in another sequence:

\[ \text{TS example1} \rightarrow \text{Sp1} \rightarrow \text{Equ 24} \rightarrow \text{Equ 25} \rightarrow \text{Fig 5.3} \rightarrow \text{Sp2} \rightarrow \text{Fig 5.4} \rightarrow \text{TS example2} \]  

(47)

Sequence (46) and (47) showed that each TS example is responsible for the gradual change within different periods of the process toward the targeted problems. For this reason, small problem solving can be considered as the components of a TS example.

In what follows, the box representing targeted problems (Ine37 and Ine38) in Figure 5.8 will be generalised to include any mathematics materials such as formulas or concepts. Because the only things that need to be changed when the targeted material has changed are the structure of TS examples and the contents in each box. The structure of the diagram will basically remain the same. In fact, the idea of gradual change, TS example and small problem solving were emergent at the beginning of the tutoring process as can be seen from field notes of step 3 and step 4. The brief theoretical descriptions of these ideas will be given in the following sections.

5.7.4 TS Example and Direct Instruction

As discussed in Dis.5.7.2a1, TS examples possess some features of direct instruction, but differ in that the tutor always works together with the student to find the solutions. In a TS
example, the teacher does more than just allow the student to ask questions about a worked example. The student is required to carry out the calculation and answer questions posed by the tutor whenever the tutor considers that the student is ready. (See, for example, Dis.5.6.2b.)

Almost every item in sequences (46) and (47), or in Figure 5.8, is associated with a different simple illustration, constructed by the student or the tutor during learning. Each of these simple illustrations serves as a cue or worked-out examples. They acted as reminders and carry much more specific information for performing the task (Ross, 1984).

As Robins et al. (1993) pointed out that this type of learning would “be especially helpful when students are confronted with new problems that require the use of the relations they have just learnt” (p. 529).

Moreover, as can be seen from Figure 5.8, the process of incorporation of TS example and small problem solving can provide opportunities for the learner to directly observe the gradual change of the concrete structure of the algebraic patterns. This allows the student to build conceptual connections between the individual formulae and link all information gradually into the existing cognitive network to reach the state of understanding.

It could also be said therefore that the process of presenting a TS example engages the student in learning actively.

A further explanation will be given in section 6.3.1.

5.7.5 TS Example and an Atmosphere of Discovery Learning

Small problem solving was introduced in Simple Steps to provide a platform for the student to “perform actions and reflect on them to develop a schema” (Steele et al., 2004, p. 68). In the small problem solving and also in the harder problem solving phases of Simple Steps, the student was in a learning atmosphere typical of one created by discovery learning: the student was encouraged to discover the “regularities of previously unrecognized relations and similarities between ideas” (Bruner, 1965, p. 20). The recognition of relations and similarities
is an important element which Simple Steps has in common with Bruner’s description of discovery learning. Finding relations between pieces of information to form a knowledge network connection is also advocated by Hiebert et al. (1986), who said that understanding is “the term used most often to describe the state of knowledge when new mathematical information is connected appropriately to existing knowledge” (p. 4) the following sections discuss the terminologies to describe a small problem solving.

**The Reasoning Matrix (Dis.5.7.5a)**

In Simple Steps, many but not all materials were presented gradually and sequentially. Materials not presented in a sequence, required the student to recognize the connection between different pieces of information. For example, after discussion surrounding the functions

\[ y = 2x \rightarrow \text{Figure 4.2} \rightarrow y = \frac{1}{2}x \]  \hspace{1cm} (48)

together with the verbal statements about inverse functions constructed by the student:

The graph of \( y = f(x) \) and the inverse \( y = f^{-1}(x) \) is symmetry about the straight line \( y = x \) \hspace{1cm} (49)

the student was asked to find out the inverse of

\[ y = 2^x \]  \hspace{1cm} (50).

This is a small problem solving.

The process can be explained using the reasoning matrix as introduced in section Dis.5.5.2a. Firstly, the prior knowledge as represented in equation (48) can be put on the first row of the matrix (See sequence 51), and the new information or corresponding unknown is placed in the second row (See sequence 52):

\[ y = 2x \rightarrow \text{Figure 4.2} \rightarrow y = \frac{1}{2}x \]  \hspace{1cm} (51)

\[ y = 2^x \rightarrow \text{Figure 4.3a} \rightarrow ? \]  \hspace{1cm} (52)

The question mark in the diagram represents the desired answer.

It was found, up to this stage, that pure discovery alone is not sufficient to describe the nature of learning for harder questions. Instruction was needed to help the student to select correct
information. TS example was therefore introduced and the process can be described using the SOI model (See section 2.25): the first process is selecting, during which the tutor may help the student to realise the new information such as \( y = 2^x \), this was the place where the TS example came into play. When the preparation is enough, the learner is then encouraged to carry out “the organising and integrating process, yielding a meaningful learning outcome” (Mayer et al., 2006) as discussed in the following. This process is similar to the situation when SOI model was used to explain the process of guided discovery methods.

**Relational Data and Sequences (Dis.5.7.5b)**

If the verbal statement (49) is added to sequence (51), then the new sequence (53) is obtained which represents the knowledge stored in the student’s long-term memory related to the concept of inverse:

\[
y = 2x \quad \rightarrow \quad \text{Verbal statement (49)} \quad \rightarrow \quad \text{Figure 4.2} \quad \rightarrow \quad y = \frac{1}{2} \quad (53).
\]

Since most of the elements in sequence (53) were previously constructed by the student or the tutor, the student was able to apply his understanding to the new situation sequence (52) to get the answer \( y = log_2(x) \) as illustrated below.

\[
y = 2^x \quad \rightarrow \quad \text{Figure 4.3a} \quad \rightarrow \quad x = 2^y \quad \rightarrow \quad y = log_2(x) \quad (54).
\]

(See Table 4.6 of chapter 4). In this analysis, sequences such as sequence (8) in section 5.5 and sequences (53) and (54) will be called relational data or relational sequences. These sequences show the student’s train of thought as he coherently integrated new information into his existing network. This is the cognitive processes corresponding to organizing and integrating in the SOI model.

Note that the similarities between sequences (53) and (54) are not superficial. The student used knowledge from different domains, namely, the domain of linear functions and the domain of exponential functions and logarithmic functions. What the student has caught was the “process likeness”. As Harrison et al. (2006) mentioned: “process likenesses between domains are strongly relational and are found in analogies that build concept-process knowledge in new contexts” (p. 16). This illustrates what Mayer (1991) meant by analogy, in that it “involves understanding a principle to solve a problem in one domain, and creating an analogous solution for a new problem in another domain” (Murphy et al., 1999, p. 664). In this regard, the learning
environment is similar to that advocated by the principle of discovery learning. In this environment, the student had been able “to recognize the connections within what they have learned—the kind of internal discovery that is probably of highest value” (Bruner, 1966 p. 96).

### 5.7.6 Reducing Misconceptions

This section summarises the method used to handle the student’s errors and misconceptions. Chi, Slotta and de Leeuw (1994) broadly define conceptual change as “learning that changes some existing conception” (p. 27). Miller et al. (2009) considers the process of reducing misconceptions, or repairing misconceptions to be a special case of learning.

According to Thompson, and Logue (2006), “misconceptions” are “ideas that provide an incorrect understanding of such ideas, objects or events that are constructed based on a person’s experience” (p. 553). Misconceptions are generated as the learner incorrectly combines their prior knowledge with new information (Murphy et al., 2006; Gijlers & de Jong, 2005). Although many researchers identify students’ misconceptions, they seldom show how those misconceptions could be removed (Kowalski & Taylor, 2009; Chi, 2005).

**Reducing Misconceptions via Small Problem Solving (Dis.5.7.6a)**

Many researchers suggest that student’s misconceptions are related to difficult concepts particularly in the field of physics and mathematics (Miller, Streveler & Yang, 2009; Enderle, Smith & Southerlan, 2006). Chi (2005) states that: “Many science concepts are extremely hard for middle and high school students to learn with deep understanding. This lack of deep understanding is displayed in their naïve explanations for various concepts and phenomena” (p. 162).

Making sure that the student has really learned the concepts taught is fundamental. Accordingly, communication between the tutor and the student is the first aspect to be considered.

After achieving the understanding of the expression

\[ y = \log_2 x \]
the next goal was

\[ y = \log_a x \]

But when the student tried to convert

\[ x = a^y \quad (55) \]

into a logarithm, he made a mistake:

\[ x = \log_a y \quad (56) \]

The correct answer was:

\[ y = \log_a x \quad (57) \]

Equation (56) shows that even if a concept was correctly learned, it could still be very fragile and weak (Heibert et al., 1992) (See Theme 19 F 3.4 in Table 5.4).

This section discusses the method of removing (or repairing) misconceptions using equation (55), (56), and (57) as examples. The method has two major aspects: (a) raising the level of understanding gradually; and (b) restructuring knowledge to remove the misconception.

The theoretical support of this study comes from more than one source because no single theory could be used as a unified tool to direct the investigation. As Harrison and Treagust (1999) argue, “conceptual change is best understood when multiple perspectives are used” (p. 45). Three main sources used to support this study are Rumelhart and Norman (1976), Chi (1994) and the theory of analogy and representation.

Rumelhart and Norman (1976) proposed three types of learning, namely: (a) accretion, a process that assimilates information using the existing schemata; (b) tuning, a process involves “slow modification and refinement of a schema as a result of using it in different situations” (Shuell, 1986, p. 421); (c) restructuring, or a process of schema creation (Shuell, 1986).

According to Chi (1994), misconceptions are repaired if a student’s conception is shifted from an incorrect category to a correct category (Mayer, 2004). The theories of Rumelhart and Norman (1976) and Chi (1994) are both related to the Piagetian mechanism of assimilation and accommodation (Ozdemir et al., 2007; Shuell, 1986) and both view the process of repairing a misconception as a special case of learning. In this study, the process involves a series of “small problem solving”.

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Reminding and Raising the Level of Understanding (Dis. 5.7.6a1)

To help the student interpret new information correctly, the student was asked to review existing knowledge. From the expressions $8 = 2^3$ and $3 = \log_2 8$, the result of this process can be diagrammatically expressed as

$$8 = 2^3 \rightarrow 3 = \log_2 8 \rightarrow y = 2^x \rightarrow x = 2^y \rightarrow \text{Figure 4.3a} \rightarrow y = \log_2 x \quad (58)$$

New knowledge was related to the student’s prior knowledge. All elements were derived through the process of TS examples and small problem solving. The patterns seen now served as reminders of that knowledge. According to Ross (1984), early cognitive skills including earlier episodes or “a set of well-learned cognitive procedures for accomplishing a constrained set of goals” (p. 371) play “the role of reminding in learning” (p. 372). This is the function of the simple illustrations (For example, see theme 12 F3.3, 13 F3.3 in Table 5.4 and theme 11 F4.2, in Table 5.5). But this was not a normal reviewing. The student knew that when he transferred the system

$$x = 2^y \rightarrow y = \log_2 x \quad (59)$$

to a system with more abstract notion, he made a mistake:

$$x = a^y \rightarrow x = \log_a y \quad (60)$$

even though he did not know what was wrong. Therefore his reasoning by examples was directed by a goal (Mason, 1994; Gick et al., 1983). This goal directed his search for the mistake he had made. A different goal could have led him to a different selection of knowledge from the same data (Mason, 1994; Ross, 1984). During the process of reviewing sequence (58), the student might discover something which he was not sure about and decide to deepen his understanding.

However, there is a problem: will the student interpret the example correctly? Harrison et al. (2006) pointed out that when a concept was difficult to understand, alternative conceptions could be generated by students using their prior experiences to encode the analogy. To help the student to interpret the meaning of the example correctly, Simple Steps uses the idea of presenting examples via the process of small problem solving. This enables the tutor to determine whether the student has really understood the subject. If misunderstanding is
detected, then a TS example could intervene. In Dis. 5.62b, for example, where the student used the formula

$$\log_3 3^5 = 5$$

without knowing its exact meaning, this led to a series of small problem solving (See F 4.3 of Table 4.7). The sequence corresponding to this small problem (presented in Figure F 5.1b in Dis. 5.6.2b) was:

$$\log_3 3^5 = 5 \rightarrow x = \log_3 3^5 \rightarrow 3^5 = 3^x \rightarrow x = 5 \rightarrow \log_a a^x = x \quad (61)$$

The student was trying to clear the ambiguity through assimilation of knowledge. As a result, the student’s naïve explanation about $$\log_3 3^5 = 5$$ was replaced by a deeper understanding of the mathematical content, $$\log_a a^x = x$$.

This process is similar to learning by accretion as suggested by Rumelhart and Norman (1976). It describes the situation where new information can be interpreted with the existing schemata (Shuell, 1986). According to Murphy et al. (2006), conceptual change by accretion corresponds to Piaget’s mechanism of assimilation. During the process, the learner “used the instantiated schemata to reconstruct the original experience, thereby ‘remembering’ that experience” (Rumelhart et al., 1967, p. 13). In this way, the student was able to assimilate information with the existing schema, strengthening and tightening the network structure (Hiebert et al., 1992). His level of understanding gradually rose while he was reflecting on the relational data and simultaneously removed his misconception. This is a constructivist view of conceptual change: “because constructivism sees students’ existing ideas as a primary source for learning, erasing misconceptions” (Özdemir et al., 2007).

**Restructuring Knowledge to Remove the Misconception (Dis. 5.7.6a2)**

Giving a reminder alone could not remove the misconception. A process was needed. The process described in this section is not clearly distinguishable from that described in the previous section. As Rumelhart et al. (1976) pointed out: “learning takes place whenever people modify their knowledge base, and no single theoretical description will account for the multitude of ways by which learning might occur” (p. 24).

Repairing misconceptions with the help of relational data

The student’s error was detected when the algebraic expression was generalised from
The misconception could be related to the student’s lack of familiarity with algebraic symbols. In order to provide an opportunity for the student to visualise the concept through algebraic manipulation, the student was guided to replace the “2” in \( y = 2^x \) with “a” obtaining \( y = a^x \). Next he was asked to find the inverse of \( y = a^x \) and was referred to Figure 4.3a of sequence (58), so he could visualise the meaning of “inverse”. Then he may try to sketch a similar picture in his head. If he could not sketch the picture, he might move back to \( y = 2^y \) in sequence (58), from which he might get a deeper understanding of inverse functions and so on. This example of gradual change could be expressed diagrammatically as:

\[
\begin{align*}
y &= 2^x & \rightarrow \text{Figure 4.3a} & \rightarrow \ x &= 2^y \\
y &= a^x & \rightarrow \text{imaginary picture} & \rightarrow ? \quad (62)
\end{align*}
\]

Each vertical arrow is a small problem solving and the question mark indicates the answer needed. Suppose he found the answer (not necessarily correct) and carried out the next calculation. In this way he might obtain another sequence:

\[
\begin{align*}
y &= a^x & \rightarrow ? \text{imaginary picture} & \rightarrow ? \text{inverse of } y = a^x & \rightarrow ? \ x &= \log_a y \quad (63)
\end{align*}
\]

This sequence represents the new knowledge the student had just learned. The question marks indicate the answer needed to be found, and “imaginary picture” is the picture the student draws mentally, which was similar to Figure 4.3a. The last term indicates the mistake the student made. Since sequence (58) is mainly numeric, while sequence (63) has more abstract symbols, the misconception might have arisen during the transformation from a numeric representation to a symbolic one. It was hoped therefore that the process of transforming numeric representations to symbolic representations through the process of small problem solving would allow the student to observe directly what was going on and thus identify his misconceptions. The tutor wanted to know if the student was mentally sketching the graph correctly and if he was, could he then find the inverse of \( y = a^x \).
Chi (1998) pointed out that one effective way of repairing a misconception is to give the correct potential explanation once the error is detected, but it can be difficult for the teacher to detect the error. She suggested that the teacher obtain this information by asking the student to give his own explanations and “to probe more deeply in order to understand what the student was explaining” (p. 43). The active interaction between the tutor and the student through the incorporation of TS example and small problem solving is one of the most important characteristics of Simple Steps. This situation was described in “The TS Example (Dis.5.7.2a1)” and “Small Problem Solving (Dis.5.7.2a2)”. The student was required to show his idea in almost all steps using different forms of representation involving verbal and visual information, as well as algebraic calculations. The result of these activities was a set of relational data (See Dis.5.7.5b) that enabled the tutor to understand the student very well and identify the misconceptions when they occurred.

Chi (1994, 1998, 2005) distinguishes between two types of misconceptions: the ordinary misconception and the misconceptions generated by a miscategorised concept. Repairing the second type of misconception requires a radical change. It “involves recategorizing a concept from an incorrect category to a correct category” (Mayer, 2002). Özdemir et al. (2007) points out that radical change do not “take place suddenly. Rather, they involve a gradual and time-consuming process” (p. 354). Chi agrees that under some circumstances, conceptual change involves adding new pieces of knowledge and revision. It is a process that consists of changing many small pieces of knowledge “rather than a process of sudden accommodation” (Mayer, 2002, p. 5; Chi, 1994, 1998). This view is consistent with the idea of gradual change.

The reasoning matrix (62) shows firstly that the change of the elements from the first row to the second row was very “small” in the sense that the form change from $y = 2^x$ to $y = a^x$ is easy to understand. This allowed the student to examine carefully the role of an element in the algebraic expressions as well as the relationship between different formulas in the same sequence. Secondly, the similarity was not superficial. If the similarity had been on the surface it would have been possible for a student to give a correct answer without understanding the concept (National Academy of Sciences, 1997). Even if the student had attempted the question correctly before, the small problem solving was time for him to reflect on what he had learned.
The process is similar to “thinking by example” as described by Mayer (1991). According to Mayer, “thinking by examples involves understanding the solution procedure of a worked-out example in one domain to create a solution procedure for a related problem within the same domain” (Murphy et al., 1999, p. 664). As Robins and Mayer (1993) point out, it is helpful “when students are confronted with new problems that require the use of the relations they have just learned”.

In Simple Steps, a TS example would intervene if difficulties arose for the student. This situation was described in section 5.7.3. The tutor directed the student so he was able to detect his own errors (See F3.4 to F3.5 of Table 4.6 or Theme 20 F3.5 of Table 5.4 in section 5.5.1). The whole process can be summarised below:

The sequence corresponding to prior knowledge is:

\[ 8 = 2^3 \rightarrow 3 = \log_2 8 \rightarrow y = 2^x \rightarrow \text{Figure 4.3a} \rightarrow x = 2^y \rightarrow y = \log_2 x \quad (58) \]

The sequence corresponding to new knowledge is:

\[ y = a^x \rightarrow \text{? imaginary picture} \rightarrow \text{? inverse of } y = a^x \rightarrow \text{? } x = \log_a y \quad (63) \]

Note that the expression \( x = \log_a y \) is the targeted misconception.

The small problem-solving involved to remove the targeted misconception are:

\[ y = 2^x \rightarrow \text{Figure 4.3a} \rightarrow x = 2^y \rightarrow \text{? imaginary picture} \rightarrow \text{? inverse of } y = a^x \rightarrow \text{? } x = \log_a y \quad (64) \]

It is important to notice that almost each question mark “?” corresponds to a small problem solving similar to the reasoning matrix (52) in Dis.5.7.5a and the atmosphere is “discovery”.

The results of these small problem-solving can be represented as:

\[ y = 2^x \rightarrow \text{Figure 4.3a} \rightarrow x = 2^y \rightarrow y = \log_2 x \quad (65) \]

\[ y = a^x \rightarrow \text{imaginary picture} \rightarrow x = a^y \rightarrow y = \log_a x \quad (66) \]

Since the answer corresponding to the last “?” was correct and the misconception was removed.
This gives the answer to the second research question from a practical point of view, more theoretical discussion will be given in section 6.3.2.

5.8 Data Analysis (Part 3 and Part 4) of Test 2
This section discusses data collected from the test given after the tutoring process.

5.8.1 Categorisation of Themes
The test given after the tutoring process (test 2) was designed to measure the student’s conceptual understanding of logarithmic functions. It contained ten questions, six of which had three sub questions (see Appendix A). In Table 5.8, themes from this test are categorised. “1.test2” means the first theme obtained from the test2 data.

Table 5.8
Categorisation the of themes from written test

<table>
<thead>
<tr>
<th>Themes</th>
<th>Organising Themes</th>
<th>Descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Test2 Mistake due to careless reading of the question</td>
<td>(Org1. Test) Minor mistake</td>
<td></td>
</tr>
<tr>
<td>2. Test did not express clearly</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. Test2. Attempted most questions correctly</td>
<td>(Org2. Test) Good performance</td>
<td></td>
</tr>
<tr>
<td>4. Test2. Did not simply immerse himself in calculation</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5. Test2. Used a simple illustration as a guide</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Discussion
The themes collected here are the same as those from the initial data analysis:
Org1. Test2 describes the minor mistake.
Org2. Test2 summarises the major events during the test.

5.8.2 Basic Description of the Test
The questions in this test were the same as those given to the student before the tutoring process, but at that time the student could only attempt the first few questions and he was
uncertain about his answers. He could calculate one question but could not calculate the same question presented in a different form. I concluded that he was guessing answers rather than calculating.

After the tutoring process, he could attempt most of the questions successfully and only made three mistakes. These mistakes were in sub questions. One was a careless error and the other two were due to unfamiliarity with the notation. The student corrected these errors immediately after they were detected by the tutor when he faced a difficult question in the test. The student used a simple illustration to help him understand the meaning of the letters in an algebraic expression (See Test aft 2).

5.9 Data Analysis (part 3 and part 4) of the Feedback Interview

This section discusses data collected from the feedback interview immediately after the second test. It was originally designed to help the student to overcome difficulties encountered in the test. Because the student did well in the test, there was time for the tutor and the student to exchange their ideas about the tutoring process and about the study of Mathematics in general.

5.9.1 Categorisation of Themes

Themes from the interview segments are categorised in Table 5.9. “1.In2.1” means the first theme obtained from interview 2, section 1.

Table 5.9
Categorisation of themes from the feedback interview

<table>
<thead>
<tr>
<th>Themes</th>
<th>Organising Themes</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. In2.1 Difficulties</td>
<td>(Org1. In2) Student’s opinion of Simple Steps</td>
<td></td>
</tr>
<tr>
<td>2. In2.1 Helps student struggling to understanding</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. In2.1 Sketching may be useful for science and economics</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4. In2.2 Procedure can be worked out from the concepts</td>
<td>(Org2. In2) Student engagement</td>
<td></td>
</tr>
<tr>
<td>5. In2.3 Actively engaged in the learning</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
6. In2.3 Both actively and passively taught
7. In2.4 Choosing the correct formula is difficult

8. In2.6 Student getting confused
9. In2.2 Should have a longer time to learn so much
10. In2.5 Not enough time for students to digest

(Org 3 In2) Student’s opinion of the Mathematics curriculum

Discussion
10 themes were classified into 3 groups. No new themes were generated.

The Group Org1.In2 describes how the student thought about the idea of Simple Steps.
The group Org2.In2 indicated that the student’s had actively engaged in the process of tutoring.
The Group Org3.F4 expresses student’s ideas about the educational curriculum.

5.9.2 Basic Description of the Feedback Interview
The conversation indicated that the student was actively engaged in the learning (See theme 5In2.3). According to him, Simple Steps is a useful process for helping students who are struggling to understand mathematical topics. He also thought it could be used in the teaching of science subjects and economics (See theme 2In2.1 and 3In2.1).

In relation to the current senior Mathematics curriculum, the student said there is not enough time for students to fully understand the mathematics and for the teacher to fully explain the concepts (See theme 9In2.2). As a result, students have to work out concepts on their own and they might become confused (See theme 8In2.6).

5.10 Data Analysis (Part 3 and Part 4) of the Questionnaire
After the tutoring process, the student was required to conduct self-assessment by answering a questionnaire. This questionnaire was not only to measure the student’s engagement, but also
to provide an opportunity for the student to summarise what he had been learning (Munns, et al., 2006).

5.10.1 Categorisation of Themes

There are 10 themes corresponding to 10 questions in the questionnaire. These 10 themes were classified into three groups according to three aspects of a student’s engagement as proposed by Munns et al. (2006) and discussed in section 3.4.2, namely:

- Reflectively involved in deep understanding and expertise (high cognition).
- Genuinely valuing what they are doing (high emotion).
- Actively participating in school and classroom actives (high behaviour).

Here “Quest1” “Quest” referrers to “questionnaire” and “1” means the first theme obtained from the student’s answer to question 1 in the questionnaire.

Table 5.10
Categorisation of themes from the Questionnaire

<table>
<thead>
<tr>
<th>Themes</th>
<th>Organising Themes</th>
<th>Descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Quest1. Straightforward and explained fully</td>
<td>Org1. Quest Actively participating</td>
<td>Gives the student’s self-assessment about his engagement in the tutoring process.</td>
</tr>
<tr>
<td>2. Quest 5. Participation in classroom activities depends on the teacher’s attitude</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. Quest 8 Personal time constraints affect learning</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4. Quest 10 Perseverance varies between individuals</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5. Quest 7 The approach made maths fun</td>
<td>Org2. Quest Genuinely valuing</td>
<td>These answers are related to the student’s internal feeling (Munns et al., 2006).</td>
</tr>
<tr>
<td>6. Quest 6 Understanding a concept enables unfamiliar types of questions to be answered</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7. Quest 3 Unsure where to apply</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8. Quest 4 Knowledge had increased by the time the test</td>
<td>Org3. Quest Deep understanding</td>
<td>Allows the student to express how well he</td>
</tr>
</tbody>
</table>
Discussion

The Group Org1.Quest gives the student’s self-assessment about his engagement in the tutoring process.

The group Org2.Quest2 is related to the student’s internal feeling (Munns et al., 2006).

The Group Org3.Quest allows the student to express how well he was doing during learning.

5.10.2 Basic Description of the Questionnaire

The student’s self-assessment showed that the tutoring process had actively engaged him and led him to a deeper level of understanding of logarithmic concepts (see 1.Quest1 and 8.Quest4). The learning environment provided by Simple Steps had not only enabled him to gain a deeper understanding of the topic, but also enabled him to think reflectively, resulting in him genuinely valuing what he was doing during the tutoring process.

5.11 Triangulation of the Data (Unit 5 of the Data Analysis)

As mentioned in section 5.1, the main purpose of triangulation is to examine the effectiveness of Simple Steps and determine the internal validity of the qualitative research in the light of the research questions. This is accomplished by comparing data chosen from different sources so that they are

(a) a contrast to each other, or

(b) in agreement with each other.

All data including data obtained in Chapter 4 will be used when necessary and the process will be largely descriptive. The analytic methods are mainly modified from Greene et al. (1985), Oliver-Hoyo, and Allen (2006), and Meijer et al. (2002).
5.11.1 Comparative Group 1: Engagement

The comparative data in Table 5.11 have been taken from the first interview (In1.5 of Table 4.2) and the interview after the tutoring process (In2.2 of Table 4.11).

Table 5.11
Comparison of segments from two interviews

<table>
<thead>
<tr>
<th>First Interview</th>
<th>Second Interview</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tutor: Do you just sit there?</td>
<td>Tutor: Do you think you are actively engaged in this learning?</td>
</tr>
<tr>
<td>Student: Sit there.</td>
<td>Student: Yes.</td>
</tr>
<tr>
<td>Tutor: Why?</td>
<td>Tutor: Do you enjoy the learning?</td>
</tr>
<tr>
<td>Student: <em>Concentration on teacher’s explanation of concepts and no questions asked.</em></td>
<td>Student: Yes.</td>
</tr>
<tr>
<td>Tutor: Do the teachers ask you some questions during the study?</td>
<td>Tutor: Were you pressed to learn?</td>
</tr>
<tr>
<td>Student: No, until the end when he gives us examples.</td>
<td>Student: No, not pressed.</td>
</tr>
<tr>
<td>Tutor: Ah, he never, doing introduction of a new topic, or knowledge, he never tried to, never asks your questions?</td>
<td>Tutor: Do you think you have actively learned or passively taught?</td>
</tr>
<tr>
<td>Student: No.</td>
<td>Student: Uh, both.</td>
</tr>
<tr>
<td>Tutor: Until the lesson is finished?</td>
<td>Tutor: Because I have to teach the <em>concepts</em>...</td>
</tr>
<tr>
<td>Student: Until the end</td>
<td>Student: Yes, the concepts (Org2.In2).</td>
</tr>
</tbody>
</table>

The interview segment in the first column shows that the student was not actively engaged in learning Mathematics at school. Despite the fact that he had concentrated on the teacher’s explanation, he was not questioned until the end of the lessons. As a result, he was passively taught. The situation was quite different during the tutoring process as can be seen from the interview segment in the second column.

5.11.2 Comparative Group 2: Understanding

Table 5.12 compares extracts of the results from two tests in which the questions were the same. The first test was given orally before the tutoring process and the second test was a written test given after tutoring. In the first test, the student did not attempt all the questions.

Table 5.12
Comparison of the results from different tests
Interview segment from first test

Tutor: Do you remember how to do it? Try.
Student: I can try.
Tutor: *So we start at* ... *9 = 3²*, can you calculate the logarithm of 9 to base 3?
Student: Er, yes.
(The student calculated it correctly).
\[ \log_3 9 = 2 \]

But in the next question:
Within the field of logarithms, give an equation to express the relationship between 2, 10 and 100, identify the names for each quantity.
The student incorrectly wrote:
\[ \log_2 100 = 10 \]
(In 1.7, In 1.8 in Table 4.3).

Some result of the second test

Firstly, the student calculated the answer effortlessly
\[ \log_{10} 100 = 2. \]
The other question was to simplify the expression:
\[ y = 10^{\left(\frac{\log_{10} b^a}{a}\right)} \]
His answer was “b”, which is correct; but his notation was not quite right as shown
\[ y = 10^{\left(\frac{\log_{10} b^a}{a}\right)} \]
\[ y = 10^{\left(\frac{\log_{10} b^a}{a}\right)} \]
\[ y = b^a \]
\[ y = b \]
The incorrect writing might be due to unfamiliarity with the formula
\[ x = a^{\log_a x} \]
or carelessness.
Most answers were correct, such as the following questions:
Simplify the expression
\[ (\log_9 20)(\log_{20} 81) \]
Show that: \[ \log_{10} 5 = \log_{10} 5 - \log_{10} 8. \]
These answers are given in Appendix A (See Test aft 2 in Table 4.10)

By comparing the materials provided in Table 5.12, we see that the student has made great progress. More importantly, this progress was not the result of rote memorising but an improved conceptual understanding as can be seen in the next comparative group.

5.11.3 Comparative Group 3: Misconception and Discovery

The data in this group are obtained from the second test and the final questionnaire. They relate to problem solving.

Table 5.13
Comparison of test 2 and the questionnaire

<table>
<thead>
<tr>
<th>Data from test 2</th>
<th>Data from questionnaire</th>
</tr>
</thead>
</table>

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It is worth pointing out that during the test, the student did not simply immerse himself in calculations. He also paid attention to the conceptual meaning of each problem. He used simple illustrations to help obtain correct answers. For example, when he solved the following question
\[3^{-\log_2 x} = \frac{1}{\sqrt{27}}\]
he encountered the following expression
\[-\log_2 x = -\frac{3}{2} \quad (9)\]
This confused him, so he used the simple illustration:
\[
\begin{align*}
\log_2 8 &= 3 \\
\log_2 x &= \frac{3}{2}
\end{align*}
\]
obtaining
\[\frac{3}{2} = x,\]
giving
\[\sqrt{8} = x, \text{ so } x = 2\sqrt{2}\]
(Section Testaft 2 of Table 4.10 in chapter 4)

The question in the questionnaire was:
The tutoring process has led me to a deeper understanding of the concept of logarithms.
The options were:
  - Strongly Agree;
  - Agree;
  - Undecided;
  - Undecided; and
  - Strongly Disagree.
The student chose: Strongly Agree
He explained:
  through clear and straight forward explanations, my limited knowledge has increased when I attempt the test (Quest4 in Table4.12).

The data in first column shows that when the student encountered difficulties, he would use a simple illustration to help him to remember the conceptual meaning and clear possible misconceptions.

The data in the second column showed the student was genuinely valuing what he was doing. He said that his limited knowledge increased as he did the test. This was unexpected. I interpreted this as a small discovery.

5.11.4. Comparative Group 4: Skill of Algebraic Manipulation
The data in Table 5.14 shows that the student’s skill of algebraic manipulation has been improved through the process of TS examples. The first column is the material extracted from the first test (during the first interview) and the second column shows how the student’s mathematics skill was improved during the tutoring process.
Table 5.14
Comparison of manipulation skills

<table>
<thead>
<tr>
<th>Data from the first test</th>
<th>Data from field notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Here the tutor was asking the student to show the following identity:</td>
<td>In order to prove the formula:</td>
</tr>
<tr>
<td>$\log_3(9 \times 27) = \log_39 + \log_327$</td>
<td>$\log_a xy = \log_a x + \log_a y$</td>
</tr>
<tr>
<td>Tutor: Do you have any idea to simplify this question?</td>
<td>the student was asked to prove a more concrete form:</td>
</tr>
<tr>
<td>Student: Em, no. Not like that.</td>
<td>$\log_3(9 \times 27) = \log_39 + \log_327$</td>
</tr>
<tr>
<td>Tutor: OK, we stop here.</td>
<td>from which he got a series of formulae such as</td>
</tr>
<tr>
<td>(See section In1.8 of Table 4.2)</td>
<td>$\log_33^5 = 5 takeaway = x ….. then obtain:</td>
</tr>
<tr>
<td></td>
<td>$\log_3(9 \times 27) = \log_39 + \log_327 = 2 + 3$</td>
</tr>
<tr>
<td></td>
<td>$= 5$</td>
</tr>
<tr>
<td></td>
<td>and finally proved: $\log_a xy = \log_a x + \log_a y$</td>
</tr>
<tr>
<td></td>
<td>(See also Section Dis. 5.6.2b).</td>
</tr>
</tbody>
</table>

The information in the first column shows that the student had no idea how to prove the formula. Perhaps this is because the student saw the identity as a piece of isolated information. In order to help the student to establish a conceptual link between his existing knowledge and new information, Simple Steps did not just teach the student how to prove a mathematical equation, instead, it tried to guide the student through all relevant formulae enabling the student to recognise the formula to be proved as a part of a conceptual algebraic system.

According to Star (2005), research on procedural knowledge has not received enough attention since the “current characterization of conceptual and procedural knowledge reflects limiting assumptions about how procedures are known” (p. 404). Most classroom presentations focus on the computational model of procedural knowledge rather than on the study of deep meaning inherent in it (Hostos, Hostos & Bronx, 2004; Star, 2005).

Table 5.14 illustrates how Simple Steps tries to help the student establish the conceptual connection between a set of related formulae, during which the misconceptions are cleared until the targeted formula is achieved.

5.11.5. Comparative Group 5: The learning environment
This section discusses the effect of the pedagogical approach practiced by Simple Steps. The interview segment in the first column is drawn from the first interview (See In1.2, In1.3 of
Table 4.2) while the data in the second column is from the questionnaire (Quest.1 of Table 4.12).

Table 5.15
Information about the learning environment

<table>
<thead>
<tr>
<th>Interview segment from the first interview</th>
<th>Data from the questionnaire</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tutor: In your environment, does your teacher emphasise concept or procedure?</td>
<td>The question was: The instruction presented during the tutoring process has attracted me to learn the material actively.</td>
</tr>
<tr>
<td>Student: Ah, no, not quite. (Here the student didn’t answer the question clearly; I should have asked him to further clarify this question.)</td>
<td>Questionnaire noted: “Actively” here means answering and asking questions enthusiastically.</td>
</tr>
<tr>
<td>Tutor: Do you enjoy the classroom learning environment in school?</td>
<td><strong>The options are:</strong> Strongly Agree; Agree; Undecided; Disagree; Strongly Disagree</td>
</tr>
<tr>
<td>Student: Ah, not, not quite. Can you tell me why?</td>
<td>The student chose: Strongly Agree.</td>
</tr>
<tr>
<td>Student: It’s too small, lack of equipment, like the blackboard whiteboard is not big enough for a big example ... and poorly renovated</td>
<td>He explained: The materials instructions provided is interesting and presented to me in an easy, straight forward way. There is room for me to ask questions when I’m unclear and is explained fully.</td>
</tr>
</tbody>
</table>

As Appleton et al. (2008) point out, a student could be bored and unmotivated but one could not be sure whether the student was just simply being on task as their teacher wished (Munns et al., 2006). Self-assessment expresses internal feeling and can provide both theoretical and practical information to educators (Munns et al., 2006).

According to the student’s self-assessment, he had actively engaged in the learning. The fact that he used expressions such as “straight forward” and “explained fully” implied that the principle of direct instruction had been used. On the other hand, the expression “there is room for me to ask question” is associated with a learning environment that supports learning by reflection. These expressions indicate incorporation of principles of both direct instruction and discovery learning in the tutoring process.
Chapter 6
Summary and Discussion

This chapter is a summary or interpretive discussion of the results of the data analysis. The first part of this chapter gives a brief description of the results of the triangulation and a summary of Simple Steps. The second part discusses other aspects of the findings of this study.

6.1 The Triangulation and the Process of Simple Steps

The original process of Simple Steps was mentioned in 3.11, but the properties of this process were not made clear until the data was compiled and analysed in Chapter 5. These properties are summarised and discussed below.

6.1.1 The Triangulation

The main purpose of the triangulation process was to examine the effectiveness of Simple Steps and to enhance the internal validity of the qualitative research (Meijer et al., 2002). The triangulation mainly involved comparisons between the tests before and after the tutoring process; the interviews; and the student’s self-assessment. These data provide important information for evaluating the process of Simple Steps in terms of student engagement as defined by Munns and Woodward (2006). Munns and Woodward defined engaged students as those that are:

- Reflectively involved in deep understanding and expertise (high cognition).
- Genuinely valuing what they are doing (high emotion).
- Actively participating in school and classroom actives (high behaviour).

The student’s response to Simple Steps will be evaluated according to these three aspects of student engagement, namely cognition, emotion and behaviour. According to Munns et al. (2006) “this definition captures student engagement as a substantive sense of satisfaction with, and a psychological investment in, the classroom work being undertaken” (p. 194).

Cognition

The process of Simple Steps improved the student’s level of understanding of logarithms. In the test prior to tutoring, he could only attempt the first few questions and he was uncertain
about his answers. He could calculate one question but could not calculate the same question presented in a different form. After the tutoring process, he could attempt most of the questions successfully and only made three mistakes: one careless error and two errors in notation.

More importantly, a comparison of the pre and post-test results showed that the student’s progress was not the result of rote memorizing but an improved conceptual understanding. The student did not simply immerse himself in calculations (see Table 5.13). He also paid attention to the conceptual meaning of each problem. He used simple illustrations to help him understand the algebraic expressions and thereby find the correct answers. For example, in the pre-test, when he was asked to prove the identity \(\log_3(9 \times 27) = \log_3 9 + \log_3 27\), his answer was: “Em, no. \textit{Not like that}”. By using simple illustrations he learnt to select the relevant formulae even though he made some mistakes during the process (see Table 5.14).

**Emotion**

The information obtained from the questionnaire after the tutoring process showed that the student genuinely valued what he had done.

He described what happened to him during the test as: “my limited knowledge has increased when I attempt the test” (See Table 5.13). This answer was unexpected. The student was continuing to value what he was doing to improve his understanding of logarithms.

**Behaviour**

The data in Tables 5.11 and 5.15 show that the student enjoyed the tutoring process and had been actively engaged. On the contrary, in the formal classroom at school he said he just sat there because he was not questioned or given the opportunity to question the teacher.

The student “strongly agreed” that he had been actively engaged in the learning. Interestingly, he used expressions such as “straight forward” and “explained fully” to describe the process of learning, indicating that he had received direct instruction. At the same time, he said that “there is room for me to ask question”, indicating that the learning environment supported discovery learning. These comments indicate that his active engagement in learning resulted from the
integration of principles of both direct instruction and discovery learning in the tutoring process.

The Process of Simple Steps is given is the following sections.

6.1.2 Step 1 Review Concepts and Notation
The purpose of this step is to bring the learner to the prerequisite level and at the same time, clear possible misconceptions that could be blocking the student’s understanding. Because time is limited in the context of tutoring, the teaching is directive.

To actively engage the student in the process of integrating new information with his existing knowledge, Simple Steps uses various pedagogical methods to stimulate his mode of thinking (Bruner, 1964). This includes working with the student on carefully designed exercises or small problems involving different representations of a concept. The student’s mistakes provide opportunities for the tutor to recognise the misconception and help the student overcome it (see F1.4, F1.6, and F1.7 of Table 4.4).

Note that this step is not pure revision. Rather, the teacher tries “something a bit harder” but not in excess of the learner’s zone of Proximal Development i.e. something within “the range of tasks that children cannot yet perform independently but can perform with the help and guidance of other” (Omrod, 2008, p. 332).

6.1.3 Step2 Introduce the New Topic
The purpose of this step is to present the new concept in a stimulating and simple manner by actively engaging the learner in selecting relevant incoming information. The method of simple illustration is used to help the student visualise the new concept. The student is required to take part in the construction of the patterns related to the concept being learned.

Examples of properly designed simple illustrations that could help the learner select new information meaningfully are:

\[ 8 = 2^3 \]  \hspace{1cm} (1)
and
\[ 3 = \log_2 8 \quad (2) \]

These simple illustrations remind the student of the structure of the algebraic expressions. According to Ross (1984), early cognitive skills including earlier episodes or “a set of well-learned cognitive procedures for accomplishing a constrained set of goals” (p. 371) play “the role of reminding in learning” (p. 372).

The problems given must be designed to test whether the student has understood the basic concept at the required level. If the student is successful at an elementary level, a slightly harder exercise is presented to provide opportunities for the student to construct a more advanced schema (See section 2.5.5).

### 6.1.4 Step 3 Presenting the Topic in a More Generalised Form

The concepts introduced earlier in basic form will be generalised to a more abstract and advanced level. The concrete numerical patterns constructed earlier will now act as a set of relational data or a sequence of related formulae on which new information is built.

What important is, that the new knowledge is not directly taught, rather, if the structure of the formulae allows, the student is guided to change the elements from a concrete to an abstract representation through the gradual process of small problem solving.

For example, the concrete patterns related to the concept of logarithms constructed earlier are:

\[ y = 2^x \quad \rightarrow \quad x = 2^y \quad \rightarrow \quad \text{Figure 4.3a} \rightarrow \quad y = \log_2 x \quad (3) \]

as discussed in section Dis.5.5.2a1. Then the student is supposed to be guided to work out the following corresponding patterns:

\[ y = a^x \quad \rightarrow \quad x = a^y \quad \rightarrow \text{some Figure} \rightarrow \quad y = \log_a x \quad (4) \]

where \( y = \log_a x \) is the targeted expression in this example. The actual structure of the sequence will depend on the content of the material and the intellectual level of the student.

This approach will allow the learner to reflect on what is going on by directly experiencing the calculation involving guessing, graphing, and comparing through the passageway of gradual change of representations of related mathematical contexts. It will be hoped that this process
will activate the student’s schemas “to either assimilate or accommodate new problem solution” (Steele, et al., 2004, p. 68) into the knowledge network (See basic themes 18 F3.4 & themes 16 F3.4).

Moreover with conceptual progression, another set of relational patterns could be generated. The generation of these patterns allows the tutor to catch the student’s train of thought and so repair any misconceptions as they occur.

6.1.5 Step 4 Obtain Useful Formulae and Information Based on a Conceptual Understanding to Gain Procedural Knowledge.

The purpose of this step is to learn basic procedures. The student is required to learn how to prove or derive all the formulae studied. This is quite different from step 3, where concepts and the notations were new to the student. Here the emphasis is “discovery” in the sense that the student must work out the formula under the guidance of the tutor.

The formulae being introduced are mostly taught using a sequence. In doing so, a concrete form of the formula is first presented, for example:

\[ \log_3(9 \times 27) = \log_3 9 + \log_3 27 \quad (5) \]

Then the student is asked to prove it under guidance. The tutor knows that the student is probably unable to work it out independently, so the tutor gives the student clues which encourage her/him to carry on until the problem is solved. The aim is to stimulate the student’s curiosity and encourage a positive attitude toward a challenging situation. For example, the result of proving of equation (5) is the sequence

\[ \log_3 3^5 = 5 \rightarrow \log_2 2^3 = 3 \rightarrow x = \log_3 3^5 \rightarrow 3^5 = 3^x \rightarrow x = 5 \rightarrow \log_a a^x = x \quad (6) \]

and

\[ \log_3(9 \times 27) = \log_3 9 + \log_3 27 \rightarrow \log_a xy = \log_a x + \log_a y \quad (7) \]

Almost all the terms in sequence (6) were constructed by the student or by the tutor. In the process of constructing these algebraic expressions, the student repeatedly applies basic concepts previously learned. This allows the student to see how concepts are blended into related formulae and to finally discover the connection between his/her prior knowledge and the targeted formula. The process is diagrammatically presented in Figure 5.1b. If the process
of TS example is included, then the resulting picture will be similar to that of conceptual progression as shown in Figure 5.8.

6.1.6 Step 5 - Solve Basic Problems and Clear any Misconceptions
This step has two major purposes. The first purpose is to introduce some problems of specific types that are compulsory for stage 6 mathematics. The second is to clear any ambiguity or repair any misconceptions induced during learning. Step 1, 2, and 3 mainly involve concept learning, step 4 mainly involves procedural learning, and step 5 is the logical combination of the two.

The tutor always starts from what he thinks is the student’s level of understanding. In presenting the material, the student works together with the tutor. This is a process of gradual change during which the content of the material moves, if possible, to higher levels until the targeted problem is reached. The process was described in section 5.7.2. An example is sequence (4) taken from section Dis. 5.7.3b:

TS example1 ➔ Sp1 ➔ Equ 24 ➔ Equ 25 ➔ Fig 5.3 ➔ Sp2 ➔ Fig 5.4 ➔ TS example2
Verbal 29 ➔ Fig5.5 ➔ Ine 34 ➔ Ine 35 ➔ Sp 3 ➔ Fig5.6 ➔ Targeted problems (8)

A detailed analysis of sequence (8) led to the concept of conceptual progression as indicated in Figure 5.8 (See Dis.5.7.3a). It is important to note that the sequence is not universal. It depends on the student’s level of understanding. For example in sequence (4) there are some misconceptions (See Figure 5.8).

6.2 Answering the Research Questions
The two research questions are:

1) How can the principle of direct instruction, in the context of logarithms, be applied while keeping the learner actively engaged?

2) How can the problem-solving approach, in the context of logarithms, proposed by the supporters of discovery learning, help the learner to understand a mathematical concept or topic while keeping the possibility of a misconception to a minimum?
Detailed analyses in relation to these questions are in sections 5.7.3 and 5.7.4 (for the first question) and in Dis. 5.7.6a2 (for the second question). Here these analyses will be discussed in relation to the process of Simple Steps.

6.2.1 The First Research Question

How can the principle of direct instruction be applied while keeping the learner actively engaged?

According to Resnick (1984), instruction that “had a clear instructional goal in mind and deliberately constructed exercises intended to promote it” is “in the spirit of direct instruction” (p. 16) so from this viewpoint, TS examples possess features of direct instruction. Worked examples are constructed to enable the acquisition of schemata in preparation for more complicated topics (Sweller et al., 1985; Cooper et al., 1987; Sweller, 2004; Robins et al., 1993).

However, the way Simple Steps approaches examples is quite different from the conventional method that tends to present a complete example; even some teachers allow questioning during the process, but it is mainly a passive process. During a TS example in Simple Steps, the tutor always works together with the student to find the solutions. The student is required to carry out the calculation and answer questions posed by the tutor whenever the tutor considers that the student is ready (See Dis. 5.6.2b).

To present a new topic, a TS example always tries to decompose an example into several “small examples” (if the material allows this). Figure 6.1 below illustrates the action of a TS example diagrammatically where “TS example 1” means the first step of the example. The diagram shows the student being asked to work out the “small problem 1”(Sp1) and making a mistake (“Incorrect answer1”). “Res1” is the student’s response to the correction. This enables him to solve “small problem 2” (Sp2) to get “correct answer2”. After obtaining “Correct Answer 2”, the student is given another slightly changed problem (Sp3) and so on.
This example shows how the principle of direct instruction is applied via the use of a TS example when the student does not know how to start a problem. According to Mayer (2004), this could be because the student “fails to promote the first cognitive process, namely, selecting relevant incoming information” (p. 17) as discussed in the SOI model. Mayer (2004) suggested that “In some cases, direct instruction can promote constructivist learning” (p. 17). Simple Steps uses this idea but only within each small problem distributed over the whole process, thus providing more freedom for the learner to actively take part in the process of selecting incoming relevant information, organising it and integrating it coherently with prior knowledge in the student’s long term memory, as described in the SOI model.

Furthermore, In order to integrate deeper understanding of the material with existing knowledge, additional small problems which are slightly changed from the previous ones, are introduced to help the student further digest the material, a cognitive process called “refreshed by rehearsal” (Baddeley, 2002, p. 2). So TS examples are now acting as a series of examples. Cooper and Sweller (1987) demonstrated the value of worked examples on schema acquisition and problem-solving transfer. They explained that when a novice is faced with an unfamiliar
problem, random components will inevitably result in their working memory overloading (Sweller, 2003). However, when a novice is presented with a large number of worked examples, learning occurs because they worked examples act as “a substitute for schemas that are unavailable to novices” (p. 39). However, TS examples are not ordinary worked examples: they are designed by the tutor who is gradually changing the small problems according to how well the student is doing.

As the illustrations are repeatedly used or modified, the individual is constantly engaged in mental practice (Cooper, Tindall-Ford, Chandler & Sweller, 2001), resulting in the acquisition of schemas that “allow patterns of information to be recognized as belonging to a previously learned category” (Hiebert et al., 1992, p. 69). With increased activation, the schemas become automated and the state of understanding is increased (Marcus et al., 1996; Robins et al., 1993; Cooper et al., 1987). The learner is the sense maker “who actively strives to build a coherent and meaningful representation of personal material” (Mayer, 1996, p. 364). Furthermore, as indicated in the analysis of data by triangulation (Section 5.11.3, Table 5.13), the student stated that his knowledge had increased while he attempted the test. This is in agreement with the engagement framework of Munns and Woodward (2006) that student “genuinely valuing what they are doing (high emotion)” (Munns et al., 2006, p. 194). Thus the incorporation of TS example and small problem solving, shows that the principle of direct instruction can be applied while keeping the learner actively engaged.

### 6.2.2 The Second Research Question

**How can the problem-solving approach, in the context of logarithms, purposed by the supporters of discovery learning, help the learner to understand a mathematical concept or topic while keeping the possibility of a misconception to a minimum?**

The practical aspect of removing misconceptions was discussed in Dis. 5.7.6a2. Further discussion is presented here to give a more complete answer.

In this study, repairing or reducing misconceptions is considered as a case of conceptual change which is broadly defined by Chi et al. (1994) as “learning that changes some existing conception” (p. 27) and “misconceptions” are “ideas that provide an incorrect understanding of
such ideas, objects or events that are constructed based on a person’s experience” (Thompson et al., 2006, p. 553). This discussion will concentrate on the particular circumstance where the student’s misconceptions are detected during the process of tutoring.

One of the most important characteristics of Simple Steps is that almost all the materials are presented through the incorporation of TS examples and small problem solving. This process usually results in different sets of simple illustrations (called relational data) depending on the student’s misconceptions. These data help to correct the student’s misconceptions later. For example, when the student makes a mistake in an expression, the tutor guides him to step back to some point related to the misconception, and reminds him of some algebraic concept or pattern. In Sequence (9) this point is represented as A, from which the student is guided to remember the related expressions B, then C and so on.

\[
\text{A} \rightarrow \text{B} \rightarrow \text{C} \rightarrow \text{D} \rightarrow \text{E} \rightarrow \text{F} \quad (9)
\]

The numeric expressions in Sequence (9) are conceptually related and thus give a concrete example of how things are linked. This process of reminding and deriving formulae sequentially through small problem solving is similar to learning by accretion as described by Rumelhart and Norman (1976) where new information is interpreted using existing schemata (Shuell, 1986). According to Murphy et al. (2006), conceptual change by accretion corresponds to Piaget’s mechanism of assimilation. It is a constructivist view of conceptual change “because constructivism sees students’ existing ideas as a primary source for learning, erasing misconceptions” (Özdemir et al., 2007).

If reminding or learning by accretion alone cannot remove a misconception, then a second process is used. This process begins by introducing a more basic sequence (See sequence 10 below) from the point at which the misconception occurs. The concepts in this sequence are related to those in the original sequence but may be less abstract. This is the place where a TS example comes into play. For example, suppose the expression “a” is chosen to correspond to “A”. The student is then required to work out the related expression, say “b” so that the relation between “a” and “b” is similar to that between A and B. Step by step through the process of small problem solving, the student is guided towards the targeted concept. The reasoning matrix is:
The sign “?c” represents the expression to be found that corresponds to “C” in sequence (9).

An example of the process was already analysed in Dis. 5.7.6a2 (See page 151) as illustrated in the following sequences:

\[
\begin{align*}
\text{Figure 4.3a} & \quad \rightarrow \quad y = 2^x & \rightarrow & \quad x = 2^y & \rightarrow & \quad y = \log_2 x \\
y = a^x & \quad \rightarrow & \quad \text{? imaginary picture} & \rightarrow & \quad \text{? inverse of } y = a^x & \rightarrow & \quad ? x = \log_a y
\end{align*}
\]

It was this process that helped the student to identify the potential misconception as described on page 151.

Note that the form of the sequence (10) is not universal. It depends on the student’s level of understanding and the nature of the structure of the materials. The theoretical description of this process is consistent with the SOI model described in Dis.5.7.5a, where the reasoning matrix was discussed.

According to Presmeg (1997), all types of imagery “have some role and function in mathematical problem solving” (as cited in Gardern, 2006, p. 497). They are stored in long term memory as the internal abstraction of mathematical ideas or schemata (Pape, et al., 2001). Sequence (9) can be seen as concrete representation of the conceptual connection between the formulae in sequence (10) (Hamilton & Rajaram, 2001). For example, to work out the form of “?c” from “b”, he could use the relationship between B and C in sequence (9). If there were some incorrect conceptions in the student’s mind, then there would be a small violation in the individual’s mental model (Chi, 1998). More importantly, the student is not told that his idea is wrong. The student knows that the relation between B and C in sequence (9) is correct because the pattern was constructed by him (or the tutor) earlier! So according to Chi (1998), self-repair of student’s misconception takes place. This is an example of learning by restructuring: when existing knowledge, which may be incorrectly constructed, is not adequate to explain new
information, then new schemata are created to solve the encountered trouble, or existing schema are modified (Rumelhart et al., 1976).

All the time, the tutor is monitoring the process. Since the patterns in sequence (9) and (10) are changing slowly (See Sequences 58 and 63 in Dis. 5.7.6a2), the student has to show his understandings in all aspects related to the sequence. This enables the tutor to understand the student very well and identify misconceptions when they occur.

In this way, the student could abandon all emergent incorrect conceptions gradually as the result of recognizing the conceptual connection between the algebraic patterns in sequence (9), as well as the abstract relation inherent in sequence (10). This process coincides with what Bruner (1966) pointed out, “to recognize the connections within what they have learned---the kind of internal discovery that is probably of highest value” (P. 96).

Therefore it could be said that the problem-solving approach of discovery learning can help the learner to understand a mathematical concept or topic while keeping the possibility of a misconception to a minimum.

6.3 Discussion

6.3.1 Implications of the Finding

Debates about the educational effectiveness of direct instruction compared to discovery learning have been ongoing (Klahr et al., 2004; Dean JR et al., 2006; Mayer, 2004; Kirschner et al., 2006). Literature from both sides of the debate has informed the development of the establishment of Simple Steps. In fact many researchers have felt the need to combine the advantages of both approaches for various educational purposes (Cohen, 2008; Dean JR, et al., 2006; Kuhn, 2007). As Dean JR commented: “a single instruction session is insufficient to produce the desired mastery” (p. 396).

The necessity and significance of this research can be found in the official reports of National Council of Teachers of Mathematics cited by Steen (1999):
“According to supporters, constructivism focuses education on the learner (what happens in students’ minds), on inquiry (seeking the right questions, not just the right answers), on relevance (questions of natural interest to children), and on activity (learning with both hands) … yet critics … contend that constructivist methods too easily slight the importance both of didactics (systematic instruction) and drill (systematic practice)” (p. 279).

This study provides an example of the advantages of using principles of both approaches in the teaching of mathematics in a tutoring context. It highlights the importance of perceiving instructions and individual thinking as mutually related elements in the process of teaching and learning. One of the most important findings of this study is that direct instruction and discovery learning reflect two important aspects of successful teaching and learning, particularly in the context of tutoring.

Figure 6.1 and Figure 5.8 may shed light on this view. The oscillating feature of the graph has provided information about the activity of the mind of the learner tied to the experience of interacting with the changing world (Thompson, 1994, as cited in Steele et al., 2004). In some aspect, this agrees with Piaget’s (1964) version of change: “there is needed when something either outside us or within us (physically or mentally) is changed and behaviour has to be adjusted as a function of this change” (p. 6).

When the individual cannot interpret new information with his existing schema or encounters reasoning difficulty, directive help is needed such as direct instruction through the process of worked example. This idea is convergent with Sewler et al. (1998), Tuovinen et al. (1999); Sweller et al. (1985), Cooper et al. (1987), Sweller (2004), and Robins et al. (1993)’s finding as mostly discussed in section Dis. 5.7.2a1. These literatures provide theoretical descriptions for the effectiveness of using worked examples as a substitute for schema acquisition. Based on this finding, Simple Steps modified the idea of worked example into TS example and introduced the concept of small problem solving, which is, on the other hand, developed on behalf of the learner.
Its idea arose from the fact that almost every time when the student was facing a new problem such as converting $x = 2^y$ to the form of logarithm, he seemed to have difficulty in stating the first step. It seemed to be even more important to help the student to develop some skills that enhance the connection between prior knowledge and new information than to just learn the specific formulae themselves. This thought led to the idea of small problem solving. In other words, one of the important functions of small problem solving is that it can be used as a tool for the individual to discover the relationship between the prior knowledge and the incoming information to form a knowledge network connection.

6.3.2 Limitations and Suggestions for Future Research

This study has some limitations. Firstly, it is qualitative research and therefore provides no statistically generalisable results. However, most past research in this field was usually carried out by dividing the participant students into two groups under opposite conditions. These methods might have artificially separated the two mutually related elements in the process of learning, namely, the instruction of the teacher and the individual thinking of the student and therefore could have ignored the relational aspect of the two. While the present study explored the possibility of incorporating the advantages from both approaches into one practical scheme in the case of one-on-one tutoring, which also restricted the boundary of its implication. Thus, a more general method will be needed to improve the investigation. As Hodkinson and Hodkinson (2008) state that “case studies bring it closer to the experience of teachers and trainers than is possible with some other forms of research” (P. 3).

Secondly, it is a single case study. Its results might have just reflected some specific phenomenon bounded to individual differences. It should also be acknowledged that the dual roles taken by the researcher also as the tutor could have influenced the student’s response and thus have placed a potential limitation on the study. Further case studies using the same material but involving students with different intellectual backgrounds would enable a more complete view of the relationship between instruction and individual thinking.
Another limitation of this study is that it only focuses on a few specific properties of direct instruction and discovery learning. Further investigations could focus on other components of these two pedagogical approaches.


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Appendix A Test and Answers

The exercises are designed as means for learning material given in the tutoring process. They consist of questions involved in basic calculations and the understanding of concepts.

1)  
   a) Given three numbers 2, 10 and 100, Within the field of integer, use the operator ‘+’ ‘-’ ‘×’ or ‘÷’ to give a relationship between 10 and 100 (e.g. 10+90=100), and 2 and 10 (Answers may vary).
   
   b) Within the field of exponential function, use equation of the form: \( y = a^x \) to give a relationship between 2, 10 and 100, (e.g. \( 3^2=9 \)).
   
   c) Within the field of logarithms, give an equation to express the relationship between 2, 10 and 100, identify the names for each quantity.

2)  Complete the questions following the suggested steps
   a) Express 9 x 27 in the form of \( a^{n+m} \) using 3 as the base, show your working,
   
   b) Using the result from part a), find a possible relationship between 243, 3 and 5 within the field of logarithms.
   
   c) Using the result from a) and b), to show \( \log_3(9 \times 27) = \log_3 9 + \log_3 27 \). Show your working.

3) Let \( 10^x = 5, 10^y = 8 \)
   
   a) Express 5 ÷ 8 as a power of 10, show your working
   
   b) Within the field of logarithms, give a equation to express the relation between \( \frac{5}{8}, 10 \) and \( x - y \)
   
   c) Hence, show that \( \log_{10} \frac{5}{8} = \log_{10} 5 - \log_{10} 8 \)

4) Let \( 3^x = 9, \) and \( 3^y = 10 \)
   
   a) Express 90 as a power of 3, show your working
   
   b) What is the logarithm of 90 to base 3?
   
   c) Hence show \( \log_3(9 \times 10) = \log_3 9 + \log_3 10 \)
5)  
   a) Given and identify the names for each quantity $125 = 5^3$, express 3 in terms of logarithms to base 5  
   b) Hence show $125 = 5^{\log_5{125}}$  
   c) Given $y = a^x$ express y in terms of logarithm in a structure similar to b).  
To complete the following questions, you may need the following formulae  
\[ \log_a x^n = n \log_a x, \log_a b = \frac{\log_x b}{\log_x a} \] 

6) Simplify the following expressions  
   a) $3^{2 \log_3{5}}$  
   b) $5^{\log_5{8} + \log_5{9}}$  
   c) $10^{n \log n}$  

7) Simplify the expression $y = 10^{\frac{\log b^a}{a}}$  

8) Simplify the following expression  
   $(\log_9{20})(\log_{20}{81})$  

9) Simplify the expression  
   $\log_3{(\log_3{6}) + \log_3{\left[3 - \frac{\log_3{2.4}}{\log_3{6}}\right]}}$  

10) Given the equation  
    $3^{\log_2{x}} = \frac{1}{\sqrt{27}}$  
   a) Express $\frac{1}{\sqrt{27}}$ as a power of 3  
   b) Using the result of a) to solve the equation
Answers

1) 
a) 10 X 10=100, 10-8=2.
b) 100=10^2
c) 2=log_{10}100. Where 2 is the logarithm, 10 is the base, and 100 is the exponential.

2) 
a) 9 \times 27 = 3^2 \times 3^3 = 3^{2+3} \text{ where } n = 2, m = 3
b) \log_3 243 = 5
c) Since \log_3 9 \times 27 = 5, \text{ but } \log_3 9 = 2, \log_3 27 = 3
   \therefore \log_3 9 \times 27 = 5 = 2 + 3 = \log_3 9 + \log_3 27

3) 
a) 5 \div 8 = \frac{5}{8} = \frac{10^x}{10^y} = 10^{x-y}
b) \log_{10} \frac{5}{8} = x - y
c) We have x = \log_{10} 5, y = \log_{10} 8 \text{ from b) } \log_{10} \frac{5}{8} = x - y = \log_{10} 5 - \log_{10} 8

4) 
a) 90 = 9 \times 10 = 3^x \times 3^y = 3^{x+y}
b) \log_3 90 = x + y
c) Since x = \log_3 9, y = \log_3 10, \therefore \log_3 90 = x + y = \log_3 9 + \log_3 10

5) 
a) 3 = \log_3 125
b) Using 3 = \log_3 125 then 125 = 5^3 = 5^{\log_5 125}
c) y = a^{\log_3 y}

6) 
a) 25
b) 72
c) n^n

7) y = 10^{\frac{1}{a} \log b} = 10^{\log (b^{\frac{1}{a}})} = 10^{\log b} = b

8) (\log_9 20)\left(\log_{20} 81\right) = \log_9 20 \left(\frac{\log_9 81}{\log_9 20}\right) = \log_9 81 = 2

9) \log_3 (3 \log_3 6 - \log_3 24) = \log_3 (\log_3 216 - \log_3 24) = \log_3 (\log_3 9) = \log_3 2

10) 
a) \frac{1}{\sqrt{27}} = 3^{-\frac{3}{2}}
b) 3^{-\log_2 x} = 3^{-\frac{3}{2}}
   \log_2 x = \frac{3}{2}
Thus, $x = 2^{\left(\frac{3}{2}\right)} = 2\sqrt{2}$. 
Appendix B The harder problem

The harder problem mentioned in Dis.5.7.2a5

Find the relationship between $x_1$ and $x_2$ for

(2) $\log_a x_1 > \log_a x_2 \quad (a > 1)$
(3) $\log_a x_1 > \log_a x_2 \quad (0 < a < 1)$

Answer: the student use $a = 2$ as an example to discuss case (1) and use $a = \frac{1}{2}$ to discuss case (2). With the help of the graph plotting, he found that $y = \log_{\frac{1}{2}} x$ ($a > 1$) is decreasing in the region $x > 0$; and $y = \log_{\frac{1}{2}} x$ ($0 < a < 1$) is increasing in the region $x > 0$.

Therefore, for:

$$\log_{\frac{1}{a}} x_1 > \log_{\frac{1}{a}} x_2 \quad (a > 1)$$

$x_1 < x_2$

And for:

$$\log_{\frac{1}{a}} x_1 > \log_{\frac{1}{a}} x_2 \quad (0 < a < 1)$$

$x_1 > x_2$. 
Appendix C Participant Information Sheet

Project Title

A Case analysis of Stage Six Mathematics Students in a Tutoring Intervention

Who is carrying out the study?

Xia Miao Zeng - Postgraduate Research Student

Under the supervision of A/Professor Allan White and Dr Dacheng Zhao

You are invited to participate in a study conducted by Xia Miao Zeng, School of Education, University of Western Sydney.

What is the study about?

The study will investigate the learning process and effectiveness of a practical teaching scheme in a tutoring context.

What does the study involve?

The study will involve one student and a tutor who will be conducting the tutoring process. All documents required will be analysed and reported along with the data collected by the researcher in the thesis. The identity of the participants will be protected and will not appear in any document.

How much time will the study take?

There will be one participant and the teacher who will participate in the study. The study will run over approximately 2 weeks and will involve two face-to-face interviews, a tutoring process, a test after the tutoring process, and the completion of one questionnaire. The interviews and the tutoring process will be recorded by digital voice recording.

Will the study benefit me?

The study will provide you with the opportunity to discuss aspects of and reflect upon your mathematics learning experiences that result from the program.

Will the study involve any discomfort for me?

No.

How is this study being paid for?

There is no funding required for this study.
Will anyone else know the results? How will the results be disseminated?

All aspects of the study, including results, will be confidential and only the researcher will have access to information on participants. A report of the findings will be submitted as a thesis.

Can I withdraw from the study?

Participation is entirely voluntary: you are not obliged to be involved and, if you do participate, you can withdraw at any time without giving any reason and without any consequences.

Can I tell other people about the study?

Yes, you can tell other people about the study by providing them with the chief investigator's contact details.

They can contact the chief investigator to discuss their participation in the research project and obtain an information sheet.

What if I require further information?

You can contact the principal researcher: walterz3@optusnet.com.au, or call at (02) 9804 1943. You can also contact the supervisors of the study: A/Professor Allan White at al.white@uws.edu.au or Dr Dacheng Zhao at d.zhao@uws.edu.au

What if I have a complaint?

This study has been approved by the University of Western Sydney Human Research Ethics Committee. The Approval number is XXXX

If you have any complaints or reservations about the ethical conduct of this research, you may contact the Ethics Committee through the Office of Research Services on Tel 02-4736 0083 Fax 02-4736 0013 or email humanethics@uws.edu.au. Any issues you raise will be treated in confidence and investigated fully, and you will be informed of the outcome.

If you agree to participate in this study, you will be asked to sign the Participant Consent Form.