Numerical Analysis on Vibration of Functionally Graded Carbon Nanotube Reinforced Composite Beams and Plates

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DECLARATION

I hereby declare that this submission is my own work and that, to the best of my knowledge and belief, it contains no material previously published or written by another person nor material which to a substantial extent has been accepted for the award of any other degree or diploma of the university or other institute of higher learning, except where due acknowledgment has been made in the text.
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ABSTRACT

This research investigates the free vibration of nanocomposite beams and plates reinforced by carbon nanotubes (CNTs). The distribution of the CNTs may functionally gradient (FG) through the thickness of beams and plates. The reinforcing CNTs are aligned along the axial direction for beams and at an angle to the length direction for plates. The energy equations of free vibration for the CNT composite beams and plates are obtained using the classic variational method of Hamilton’s principle and the geometric nonlinearity of von Kármán sense is considered for beam vibration. The eigenvalue equation for free vibration of the nanocomposite beams and plates are then derived by the p-Ritz method. Vibration frequency parameters for the uniformly distributed (UD) and functionally graded (FG) CNT beams and plates based on the first order and third order shear deformation theories are presented, and the differences in vibration frequencies between these two theories are highlighted. Parametric studies are carried out and the effects of CNT filler volume fraction, CNT distribution, CNT reinforcing angle, structure geometry, end/edge support conditions, nonlinear vibration amplitudes (for beams only) on the free vibration characteristics of the beams and plates are presented and discussed.

The present works may offer a better understanding of the vibration behaviours of FG-CNT composite beams and plates, and pave the way for their potential applications in actuators and resonators.
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CHAPTER 1  INTRODUCTION

Over the last several decades, fibre reinforced composites have found numerous applications in various industry sectors and have shown great superiority over conventional structural materials such as steel and alloys. Among many, glass fibre or carbon fibre reinforced polymers have been widely used on forming the fuselage of jumbo jets, blade of wind turbines, radar bonnets, cooling towers and racing cars etc. The merits are obvious in terms of high strength-weight ratio, high stiffness-weight ratio and economic benefits. However, the inadequacies are also looming. Firstly, this type of fibre reinforced composites often are produced as laminated composites with different fibre orientations at different layers, and the mismatching of the material properties may cause delamination. Secondly, because of its relatively high volume fraction of fillers, to some extent, debonding or microscopic defects between fibres and matrix are prevalent. As a consequence, the overall performance of structural elements is degraded (Ajayan, Schadler, & Braun, 2003).

It has been two decades since the discovery of carbon nanotubes (CNTs) (Iijima, 1991). Dubbed as “the ultimate fibre”, the theoretical prediction of the Young’s modulus of CNTs is even higher than the carbon fibre which is on the high end of the market. Besides, unlike its micron scale counterparts, the CNTs can be dispersed into the matrix within nanoscale, which increase the chance of perfect bonding and void bridging. Moreover, the miniaturization of devices can be readily achieved by using nanocomposites, which may provide the next generation material solutions used in the micro-electro-mechanical systems (MEMS).

With the rapid progress of advanced manufacturing technology, the functionally graded material (FGM) has emerged as a promising type of composites. By gradually varying the volume fraction of constitute materials, it not only combines the desired merits of several materials, such as the thermal resistance ability of ceramics and the strength of metals, but also eliminates the interlaminar stresses that usually exist in the traditional laminated composites (Kawamura, Matsuzaki, Hino, & Okazaki, 1997).
Furthermore, owing to the exotic physical properties of CNTs, such as high thermal conductivity and electrical conductivity for certain types of CNTs, it can be foreseen that the combined functionally graded carbon nanotube (FG-CNT) reinforced composites will offer lightweight, high strength, multifunctional and designable physical properties, which can be tailored for various industries.

As a novel concept of composites being brought up recently, the current research on the FG-CNT composites is scarce. Indeed, this is a multidisciplinary field which requires collaboration from scholars in different fields. The present research is focused on the computational method and numerical modeling of the vibration characteristics of the FG-CNT composites. It is hoped that this fundamental research work will provide a cost-effective way of evaluating the natural frequencies of common structural elements made of such FG-CNT composites, in this case being beams and plates, thus contributing to current knowledge in this emerging field.

Chapter 1 gives a brief introduction of the FG-CNT composites. Chapter 2 reviews the properties of CNTs as the reinforcing agent as well as the effort of incorporating the CNT into the composites, then introduces the concept of the aligned FG-CNT composite and discusses its merit and potential applications. A literature review of the computational methods and the recent studies on mechanics aspect of such composite are given. Chapter 3 details the mathematical modelling on vibration of aligned FG-CNT beams and plates based on the first and the third order shear deformation theories. The p-Ritz numerical procedure and algorithm are established and the governing eigenvalue equations are derived. Chapter 4 presents the comparison study on vibration frequency parameters of FG-CNT beams and plates based on the first and third order theories. Comprehensive parametric studies are listed and various vibration mode shapes are illustrated. Lastly, Chapter 5 summarizes the results and conclusions obtained from this study and list some suggestions on the future research directions in this area.
CHAPTER 2 REVIEW OF LITERATURE

2.1 Carbon Nanotube Composites

2.1.1 Carbon Nanotubes

Carbon Nanotubes (CNTs), since its discovery in 1990s (Iijima, 1991), is among the most spectacular materials of the 21st century. CNTs exist in the form of either single-walled or multi-walled CNTs (SWCNTs or MWCNTs). A SWCNT can be visualized as a nano-sized hollow cylinder formed by rolling a one atom thick graphene at certain direction. The MWCNT is comprised of two or more coaxially arranged SWCNTs with van der Waals forces holding them together. Depending on the synthesis techniques, the typical diameter of SWCNT is between 0.6 to 2.0 nm, while the length of it can be ranging from several hundred nanometers to one or two centimeters, which presents one dimensional rope-like shapes. The cylindrical nanotube usually has at least one end capped with a hemisphere of fullerene structure (Han, 2005).

The bonding between the carbon atoms in the CNTs is essentially $sp^2$ and the in-plane $\sigma$ bond is the strongest in nature. Together with the delocalized $pi$ bond, it gives CNTs unique physical properties. Over the last two decades, numerous theoretical and numerical analyses as well as experiments have been carried out to probe the properties of CNTs (Srivastava, 2005). It’s well acknowledged that the CNTs have multiple attractive properties that may be tailored for future engineering applications. To name a few, the Young’s modulus of SWCNTs is several times larger than that of steel; certain types of CNTs, have much higher electrical conductivity than copper and better thermal conductivity than diamonds. Some properties such as optical, magnetic and chemical properties also show certain similarities if not superiorities when compared with the existing engineering materials (Han, 2005).

The aforementioned properties are largely depending on the particular structures of the CNTs and the synthesis methods. For example, the rolling direction of the graphite sheet, namely chirality, is one of the most important parameters in determining the
electrical conductivity of CNTs. For CNTs with different chiralities, the electrical conductivity can be like semiconductor or metallic. Other factors, such as defects, tube diameters also play a key role when considering the individual mechanical and thermal properties.

2.1.2 Dispersion and Functionalization of CNTs

In order to fully utilize the potential of CNTs as reinforcing agent in the composites, one of the major hurdles has to be overcome beforehand. At present, commercially supplied CNTs are mostly in the form of heavily entangle bundles. Due to the large surface areas of nanofillers, the CNTs tend to agglomerate under the influence of van der Waal forces if dissolved into the polymer matrix without any pretreatment. There is ample evidence showing that these bundles and agglomerates exert detrimental effect on the final product, either resulting in weak bonding between nanofillers and matrix or giving rise to heterogeneous or defects of the final product (Coleman, Khan, & Gun'ko, 2006; Fiedler, Gojny, Wichmann, Nolte, & Schulte, 2006). The imperfection of dispersion is often the source of much lower mechanical or electrical properties observed in the experiments in comparison with the results theoretically predicted based on the assumption of homogenous material properties.

There are several techniques to improve the CNTs dispersion in the matrix. Combined mechanical and chemical treatment of CNTs has been proved to be an effective way of nanofillers preparation. The mechanical exfoliation of the CNT bundles can be archived in different ways. Calendering, ball milling and stirring by shear mixing machines can separate heavily agglomerated CNTs into microscale CNTs particles. Ultrasonication, which often being used for machinery cleaning, is also one of the most frequently used methods for nanoparticle separation. In addition to the mechanical separation, because of the chemical stability of CNTs, chemical agents can be introduced to the surface of CNTs. These agents, called surfactants, can diminish the surface attraction between CNTs while lower the surface tension between CNTs and polymer molecule. There are abundant literatures covering the ionic and nonionic surfactants that can be used as wetting agent of CNTs (Nakashima, Tomonari, & Murakami, 2002; Tan & Resasco, 2005).
For fibre reinforced composites, the interface between the fibre and the matrix has long been an intensive research topic. For microscale fibre reinforced composite, owing to uncertainties and imperfections during manufacturing process, fibre discontinuity, debonding or microscopic defects between fibres and matrix may take place. Unlike their microscale counterparts, the physical properties of CNT composites can be altered at a very low weight percentage of nanofillers. For a given volume fraction, the interfacial regions between nanoparticles and matrix are significantly higher. In addition to that, just like the surfactant that can be used for increasing dispersion of the nanofiller, the surface properties of CNTs can also be altered by certain chemical modification, i.e. functionalization. The process of functionalization on one hand may increase bonding between CNTs and matrix and CNT dispersion in the matrix, and on the other hand will enable a wide variety of composite systems (He, Kuang, Chen, & Li, 2009).

Functionalization, however, is a double sword application. Certain functionalization procedures, on one hand, increase CNTs dispersion in the matrix and improve bonding between the CNTs and matrix molecules, on the other hand, decrease certain mechanical properties such as strength and ductility (Zhang et al., 2008). As most functionalization processes either damage the CNTs structure or introduce organic molecules that form new chemical group on the CNTs, the electrical and thermal properties of the composite as a whole will also be affected. There have been some theoretical studies on the impact of functionalization on the properties of CNTs; however the quantitative evaluation remains very difficult and is highly dependent on the functionalization process.

2.1.3 Aligned CNT Arrays

With the rapid progress of advanced manufacturing techniques, especially the technique of controllable growth of aligned CNT forest within nanoscale, vertically aligned CNTs arrays can now be fabricated in a large scale at a reasonably low cost. It has been reported that the super-aligned CNT arrays have been produced by continuous spinning of the normal aligned CNTs arrays (Di et al., 2012). Scanning electron microscopy (SEM) images indicate that the individual CNTs in the super-aligned CNT arrays are much straighter than the ones in normal aligned CNTs.
arrays, therefore forming better alignment. Depending on the usage, it can be manufactured either in the forms of ultra thin film or the super strong nanowires using special spinning techniques.

It has been proven that using aligned CNT arrays, thin films or wires rather than random orientated CNTs can remarkably improve the dispersion and alignment of the CNTs in polymer matrix. Liquid polymer matrix can be infiltrated to the super-aligned CNTs arrays, by using the technique of resin transfer molding (RTM) or the Latex technology. As a result, super aligned CNTs polymer composite can be manufactured (Ma & Kim, 2011). The super-aligned CNTs arrays, thin films or wires, once incorporated into the composites, lead to anisotropic or orientation dependent properties of the composites.

From another perspective, in order to fully exploit the exceptional mechanical, electrical or thermal properties of CNTs in certain direction, the CNTs in the matrix should be aligned to the designated direction in order to attain the maximum improvement for the composites.

2.1.4 CNT Reinforced Composites

Just as their micro-scaled glass or carbon fibre counterparts, the nano-scaled CNT fibres have been added to the matrix in the hope of exploiting its outstanding properties. There have been tremendous research efforts on the potential use of CNT reinforced composites, and progress has been made in various fields. New multifunctional materials or “smart materials” have emerged from research labs as a result of the overall development of CNT composites.

In the field of electronics, non-conductive polymers can be turned into partial electrically conductive composites by adding certain amount of CNTs (Cochet et al., 2001; Xiao & Zhou, 2003). The CNT fillers can be uniformly dispersed into polymer matrix within nanoscale, as a result, optically transparent, mechanically strong as well as electrically conductive film can be manufactured, which may find potential applications in a wide range of consumer goods such as touch screens, displays etc. (Ma, Menamparambath, Nikolaev, & Baik, 2013).
In the automotive, aviation and energy industry sectors, where light weight is crucial for the purpose of cost saving and performance boosting, CNT based polymers or metals provide especially lightweight feature, in combination of high levels of stability and strength. Besides, from functional point of view, structural components made of CNT reinforced polymers can improve antistatic properties, thermal and electrical conductivity as well as thermo-mechanical properties (Howe, Tibbetts, Kwag, & Lake, 2006; Njuguna & Pielichowski, 2003). Because of that, CNT based structural components may no longer need major surface treatment or electrostatic dissipative coating, which is a significant factor of cost saving. Research also shows that the CNTs are highly ductile, and CNT reinforced foams are able to absorb a significant amount of deformation energy, which makes them ideal for impact insulation or vibration isolation.

In the civil engineering practices, there have been mounting interests on developing the new high strength CNT based concrete, in which CNTs have been added and dispersed evenly into the traditional concrete. Scanning electron microscope images show the nanofillers forming networks between crystallized cement particles, which can bridge gaps and prevent the development of microcracks. When under seismic loadings, the ductility and the ability of energy dissipation of the concrete can also be improved by the pull-out of the CNTs fillers and frictions between fillers and matrix. Furthermore, there have been researches on evaluating the possibility of utilizing the electrical conductivity of CNT fillers to form the sensor network for structural health monitoring (Zhao et al., 2010). Research has shown that, when the CNT reinforced structural members are subject to cracks or other discontinuities, the structure will experience slight difference of electrical conductivity, thus provide an opportunity of monitoring the stress in these novel composite structural members (de Villoria, Yamamoto, Miravete, & Wardle, 2011).

2.2 Functionally Graded Materials

2.2.1 Introduction to Functionally Graded Materials

The idea of functionally graded materials (FGM) was originated from the space
programme in Japan. Instead of applying the thermal resistant coating on the structural elements, by gradually varying the material content of ceramic and metal along the thickness of the structure, the structure itself possesses both merits of thermal resistant from ceramic and mechanical toughness from metal. Besides, because of the existence of transition zone of mixture between pure ceramic and metal, it alleviates the mismatching of two distinct material properties, thus eliminates the interlaminar stresses that usually exist in the laminated plates or shells. This type of functionally graded materials later found many applications in extreme temperature environment, such as on forming thermal barrier in the nuclear reactors or turbine blades in the jet engines (Reynolds, 2012).

Many research works have focused on the functionally graded material made of metals and ceramics. Recently, research on functionally graded polymer, also called polymer graded materials (PGM), is emerging. For example, the research on the functionally graded polyvinyl chloride (PVC) and polyalkyl methacrylate (PMA) materials has shown that, the tensile strength and the ability of thermal shock resistance of the polymer graded composite are superior to the ones formed by pure constituent materials (Agari, 2008).

### 2.2.2 Manufacture Techniques

As there are a large number of different techniques to produce functionally graded materials, the variation of the material constitution of the functionally graded materials is highly dependent on the manufacture techniques.

The most commonly used functionally graded materials may consist of two or more of the following materials: ceramics, metals and polymers. For FGMs which consist of ceramics and/or metals, self-propagating high-temperature synthesis (SHS) method has the advantage of producing high purity products in short period of time with low energy consumption. Other methods include chemical vapour deposition (CVD), plasma spraying, powder metallurgy, gravity sedimentation, laser sintering with the additive manufacture technique or other solid freeform fabrication techniques etc. Graded polymer blends can be prepared by the following methods: surface inclination in the melt state; surface inclination in the solution method; dissolution-diffusion
method; and centrifugal sedimentation. Given the same preparation method, the variation of the material volume fractions is also controllable, for example, to have narrow graded interface, wide graded interface, or full graded interface between different constitutive materials (Agari, Anan, Nomura, & Kawasaki, 2007).

2.2.3 Advantages and Applications

Comparing to the engineering components made of homogenous materials, functionally graded materials can be regarded as multifunctional composites that incorporate the advantages of each material constituents, while suppress the disadvantages. One typical application is thermal barrier coatings of high temperature components in gas turbines. In addition to the application of thermal barriers, the ceramic metal functionally graded materials may also find applications in corrosive barriers and abrasion resistance, surface hardening for tribological protection and graded interlayers used in multilayered microelectronic and optoelectronic components. Recently, the graded polymer materials have been applied in the medical field. For example, artificial bones or prostheses, dental implant and biochemical coatings.

Functionally graded materials also have the unique properties which can be tailored for multifunctional or “smart” responses, such as different thermal or hydroelasticity expansion rates at different position within one body. The laminated piezoelectric ceramic layers, once incorporated into the FGM structures, can act as sensors and actuators for vibration control. Functionally graded polymer optical material may find applications in the telecommunication and automobile industry for its graded optical property. Owing to the inherent properties, polymer graded materials can also act as the damping materials which may be used for vibration alleviation or acoustic screen.

2.3 Functionally Graded Aligned CNTs Reinforced Composites

Shen (Shen, 2009b) in 2009 incorporated the concept of functionally graded materials into the CNTs reinforced composites. Under this concept, the CNT volume fraction along the cross section of the structure is no longer constant but can follow a certain distribution pattern. From the nanotube chemistry point of view, the aligned CNT array,
thinflm or nanowires are still acting as both reinforcing agent and property modifying agent. However unlike the aforementioned uniformly distributed aligned CNT (UD-CNT) reinforced composites, the functionally graded aligned CNT (FG-CNT) reinforced composites have heterogeneous properties along the cross sections of structure. As a result, different types of composites can be conceived and fabricated just by altering the volume fraction distribution of the CNTs along the cross section of structures.

Generally speaking, the matrix of the composite can also be functionally graded materials, such as those mentioned in the previous sections. However due to the complexity of manufacturing, only pure matrix material reinforced by CNTs are considered in the current study. It is anticipated by this means that the final structures can offer unique structural and functional merits that can be designed to meet various industry requirements.

2.3.1 Structural Merits

CNT reinforced composites have better structural functionality than conventional structural elements in terms of high strength-weight ratio, high stiffness-weight ratio, ductility and damping mechanisms. Research confirms that the existence of CNTs in the matrix can induce the frictional sliding damping mechanism, which increases the energy dissipation and suppress the vibration of the composite (Rajoria & Jalili, 2005; Zhou, Shin, Wang, & Bakis, 2004), meanwhile, introduction of CNTs into some thermoplastics can enhance stiffness without sacrificing ductility (Ma & Kim, 2011).

From the cost effective point of view, the CNT reinforcement can be maximum utilized when using the functionally graded CNT distribution. It has been reported that the FG-CNTs reinforced aluminum (Al) matrix composites have been successfully fabricated by a powder metallurgy route. The manufacture technique can be applied to other nanoreinforced composites, such as ceramics or complex matrix materials. The results show that the final structure has highly strengthened surfaces and highly enhanced ductility inside (Kwon, Bradbury, & Leparoux, 2011; Kwon, Cho, Leparoux, & Kawasaki, 2012; Kwon, Kurita, Leparoux, & Kawasaki, 2011; Kwon & Leparoux, 2012).
The structural parts used in various industries often under dynamic/impact mechanical loadings, frictions and wears, environmental corrosive attacks etc. The microcrack usually starts from surface defect, which would penetrate through the thickness direction, and then open an oxygen/chemistry path to the internal portion of the structure. As a result, degradation of structure occurs and cause premature failure. It is anticipated that by using the FG-CNT distribution, the surface of the structure can be heavily armored by a relatively high fraction of CNTs which can be used to prevent the microcracks from taken place, and a medium to low fraction of CNTs concentration in the internal portion of the structure can still provide certain stiffness and strength enhancement. Comparing to the UD-CNT and the traditional non CNT reinforced FGMs, the durability and fatigue strength of the FG-CNT composites can be improved, meanwhile the CNT agents are effectively used. In addition to that, the flexure rigidity and the vibration or damping characteristic of the structure can be designed and improved.

2.3.2 Functional Considerations

The thermal conductivity of FG-CNT composites generally depends on two factors. Firstly, CNTs are insulating in the transverse direction and a high thermal conductor along the tube length. Therefore the thermal conductivity of the composite is dependent on the CNT alignment. Secondly, dispersion and distribution of the CNTs in various matrices have a significant impact on the thermal conductivity of the composites and the general rule of mixture applies (Biercuk et al., 2002). Based on the above two factors, it is conclude that the thermal conductivity of the aligned FG-CNT composite is anisotropic and heterogeneous, it can be designed to cater for many applications, such as thermal conducting pads, insulation panel etc.

In the research of conductive polymers, by varying the CNT contents along the thickness of the structures, one can achieve the top layer of the structure to be metallic while the bottom layer of the structures to be semi-conductive or electrical insulated. This nanocomposite may find special applications in the electrostatics protection, electro-magnetic screen or forming radar wave absorption materials, micro electro mechanical systems (MEMS) and physical or chemical sensors.
Many of the structural elements are under rigorous geometrical restrains, in other words, in certain engineering systems, the dimensions of the engineering components are predetermined. Under such circumstances, the FG-CNTs based materials not only offer a possible way of producing reliable multifunctional materials, but also offer a way of designing controllable component properties only by altering the section properties (Cho, Boccaccini, & Shaffer, 2009; Estili, Takagi, & Kawasaki, 2008; Otieno et al., 2010). For example, turbine blades used in power plants usually have to meet multiple design requirements such as thermal shock resistance, corrosion protection, strong mechanical properties, light weight as well as vibration being within certain limitations to avoid premature blades fatigues. Carefully designed functionally graded carbon nanotube reinforced composites may be capable of fulfilling all these design criteria and offer exceptional engineering performances.

2.4 Vibration of Composite Beams and Plates

There are two main reasons for performing vibration analysis, one is to eliminate or suppress the undesirable vibrations, and the other is to generate useful vibrations.

2.4.1 Structures at Macroscales

For structural elements made of composite materials, just like their metal counterparts, excessive vibration is inevitably accompanied by microcracks, which will cause material fatigue. This will deteriorate the physical properties of the composites and further significantly shorten the life expectation of the structural elements (Silva, 2000). For a given engineering system under free vibration, there is one or more “natural frequencies”. It’s well known that resonance occurs and vibration amplitude reaches the maximum if the frequency of the external excitation is the same as one of the natural frequencies of the system. For the aforementioned reason, resonance should be avoided in engineering design. By varying the CNT contents along the section thickness, it can be foreseen that the natural frequencies of the functionally graded CNTs based composites will be altered, which offers a way for mechanical property optimization via section design and vibration modal analysis.
There has been research on the nanocomposite with actuator layers embedded in the structural element. The actuators, either known as the piezoelectric or magnetostrictive materials, can induce mechanical stress and strain when under the influence of an external electric or magnetic field. Therefore this type of composites will be able to translate input electrical signals into applied mechanical actuations. Similarly, CNT based electromechanical actuators will also be able to generate high stresses and strains. Research shows that low-operating voltages of a few volts can generate larger actuator strains than high-modulus ferroelectrics and higher stress than natural muscle. This may become an alternative piezoelectric material in the future (Aliev et al., 2009). These “smart” structures may find applications in the active vibration suppression.

Useful vibrations include vibratory parts feeders, ultrasonic devices used in machinery cleaning and dentistry etc. An ultrasonic motor is another example of the usefulness of vibration.

2.4.2 Vibration of Microbeams and Microplates - MEMS

With the rapid progress in micro machining techniques, there has been an increasing research and development of Micro-Electro-Mechanical Systems (MEMS). This type of mechatronic devices is often in the “Micro” scale, ranging from micrometers to several millimeters. Because of the nature of miniaturization, MEMS are highly sensitive to environmental variation, which makes it ideal for sensors and actuators. MEMS can be designed to sense almost every physical quantity such as acceleration, pressure, gas and mass, temperature, force and humidity. Examples of actuators are micromirrors, RF switches and microrelays, microgrippers and generic force and displacement actuators (Borboni, 2006).

One of the typical applications of MEMS is the scanning probe microscopy (SPM). By sensoring the surface of the specimen using microscale probe, the SPM has become an important scientific research tool for surface characterization. In common commercialized SPMs the most crucial part is the probe, which is essentially a cantilever microbeam with a small protuberance attached at its free end. When working under the non-contact or tapping mode, the probe will oscillate at its resonant
frequency, and by measuring the change in amplitude, phase or frequency of the vibrating microbeam, the topography of the surface can be obtained (Magrab, 2012).

Green technology and alternative energy harvesting have become a hot topic recently. In this field, micro-resonator energy harvesting devices is the active research subject and many innovations exist (Hajati, Bathurst, Lee, & Kim, 2011). The basic principle of such device is that the mechanical vibratory energy can be converted and stored as electricity by using piezoelectric materials. Other sensors devices have been reported by various researchers. To name a few, by sensing the change of resonance frequency, amplitude or other variables of the micro structures, the pressure, chemical substance or the biomagnetic field of the surrounding environment can be measured (Keskar, Elliott, Gaillard, Skove, & Rao, 2008; Marauska et al., 2012; Voiculescu, Zaghloul, McGill, Houser, & Fedder, 2005).

Most of the microstructures are in the form of microbeams or microplates because they can be easily fabricated. However, due to the manufacturing defect, there are still uncertainties such as the boundary conditions or the residual stresses after the structure being fabricated. For those devices working under vibration conditions such as resonators, filters, and resonant sensors, predicting accurately the natural frequencies of the structure is essential for calibration of the devices and charting the external voltage excitation.

2.4.3 Challenges in Vibration Analysis

There are several challenges in modeling the mechanical responses of the aligned FG-CNT reinforced composites. They can be generally categorized into the following groups:

- Anisotropic and heterogeneous material properties
- Geometric, material or boundary nonlinearities
- Stochastic uncertainties, such as the boundary uncertainty or residual stress
- Damping mechanisms and nonconservative force
- Multiphysics coupling (thermal, electrical, fluids fields etc.)
- Scaling effects, such as those exist in the MEMS or NEMS
Each of the above challenges deserves a monograph. In the present study of the FG-CNT composites, the main focus is on modeling the vibration characteristics of beams and plates using high order theory with anisotropic and heterogeneous material properties, geometric nonlinearities.

### 2.5 Recent Research on Mechanics of CNT Composites

Wuite and Adali (Wuite & Adali, 2005) analyzed isotropic beams composed of randomly oriented CNTs dispersed in a polymer matrix. They found the stiffness of CNTs reinforced beams can be improved significantly by homogeneous dispersion of a small percentage of CNTs. A type of novel actuator comprised of a piezoceramic matrix reinforced by SWCNTs was proposed for the application of active controls of smart structures by Ray and Batra (Ray & Batra, 2007). The effective properties of the resulting piezoelectric composites were determined by a micromechanical analysis and the active control system exhibited better performance in the controllability of vibrations.

Formica et al. (Formica, Lacarbonara, & Alessi, 2010) presented the vibration behaviour of CNT reinforced composite plates by employing an equivalent continuum model based on the Mori-Tanaka approach. An eigenvalue analysis was conducted varying the CNTs alignment in different matrices. They found that the improvement achieves a maximum when the carbon nanotubes are uniformly aligned with the loading direction. The fundamental frequency of CNT-reinforced plates can increase by nearly 500 percent, without practically altering the mass density of the material.

Inspired by the development of FGMs, Shen (Shen, 2009b) first studied the nonlinear bending behaviour of CNT reinforced composite plates and found that the load-bending moments curves of the plates can be considerable improved through the use of the functionally graded distribution of CNTs in the matrix. Ke et al. (Ke, Yang, & Kitipornchai, 2010) studied the nonlinear free vibrations of FG-CNT Timoshenko beams with the Ritz method and direct iterative technique. They found both linear and nonlinear frequencies of FG-CNT reinforced composite beams with symmetrical distribution of CNTs are higher than those of beams with uniform or unsymmetrical
distribution of CNTs. Recently, the same author studied the dynamic stability of CNT reinforced beams using the differential quadrature method (Ke, Yang, & Kitipornchai, 2013). Yas and Heshmati (Yas & Heshmati, 2012) analysed the dynamic behaviour of a functionally graded nanocomposite beams reinforced by randomly oriented carbon nanotube traversed by a moving load. Wattanasakulpong and Ungbhakorn considered the beams resting on the Pasternak elastic foundation which included a shear layer and Winkler spring based on various shear deformation theories and studied the bending, buckling and vibration behaviours of FG-CNT reinforced composite beams (Wattanasakulpong & Ungbhakorn, 2013). The behaviours of large amplitude vibration, nonlinear bending and thermal postbuckling of FG-CNT beams resting on an elastic foundation in thermal environments were studied by Shen and Xiang (Shen & Xiang, 2013). They found that a CNT reinforced beam with intermediate CNT volume fraction does not necessarily have intermediate nonlinear frequencies, buckling temperatures and thermal postbuckling strengths, and also the thermal postbuckling path of unsymmetric FG-CNT beams is no longer the bifurcation type. Wang and Shen (Wang & Shen, 2011, 2012) studied the nonlinear vibration of FG-CNT sandwich plates and plates rest on elastic foundation in thermal environments using the two step perturbation technique. The results confirmed that a functionally graded reinforcement has a significant effect on the nonlinear vibration characteristics of CNT reinforced composite plates. Zhu et al. (Zhu, Lei, & Liew, 2012) studied the static and free vibration of FG-CNT plates using the finite element method. They concluded that for the free vibration analysis, both the CNT volume fraction and the width-to-thickness ratio have pronounced effect on the natural frequencies and vibration mode shapes of the CNT reinforced composite plate. Shen and Xiang (Shen & Xiang, 2012) investigated the nonlinear vibration of FG-CNT shells in the thermal environments. Aragh and Hedayati (Aragh, Barati, & Hedayati, 2012) studied the vibration behaviour of FG-CNT shells with different shell angle using the 2-D generalized differential quadrature method.

Similarly, the bending, buckling and postbuckling behaviours were also investigated recently by several researchers. Mehrabadi et al. (Mehrabadi, Aragh, Khoshkhahesh, & Taherpour, 2012) studied the mechanical buckling of FG-CNT plates. Shen and Zhu (Shen & Zhu, 2010, 2012) studied the buckling and postbuckling behaviour of
FG-CNT plates in thermal environment and sandwich plates with FG-CNT face sheets resting on elastic foundations. In a series of papers, Shen (Shen, 2011a, 2011b) presented the postbuckling in thermal environment for axially and pressure loaded cylindrical shells. The results revealed that the CNT volume fraction has a significant effect on the buckling load and postbuckling behaviour. Shen (Shen, 2012) also presented the bucking and postbuckling of FG-CNT cylindrical shells induced by uniform temperature rise and found that in most cases the CNT reinforced composite shells with intermediate nanotube volume fraction does not have intermediate buckling temperature and initial thermal postbuckling strength.

2.6 Computational Method on Aligned FG-CNT Composites

2.6.1 Mathematical Modeling

2.6.1.1 Engineering Constants of the Composites

The load transferring mechanism at the interfacial regions between the CNTs and the associated matrix to a large extent dictates the mechanical properties of the composites. However, due to the small size of the nanofiller, not only the measurement of the mechanical properties of the CNT itself is extremely difficult to obtain from experiment, but also there are substantially difficulties in performing conventional mechanical tests such as the fibre pullout test. Computer simulation has therefore become an alternative way of predicting the engineering constants of the nanofillers and the nanocomposites. One of the most effective computational approaches for obtaining the mechanical properties of nanocomposites is the molecular dynamics (MD) simulation. However under the current computer power, simulating the life-sized material by MD is not possible. Even using the high performance computing or parallel computer clusters, the achievable size of modeling is limited to about 0.1 micrometers (Wu, 2005). Most of the engineering components used in macroscale are in the unit of meters. Even the components in the MEMS have the scale of micrometers or millimeters. Therefore, various theoretical modeling, called the micromechanics approach, is used to predict the overall material properties based on properties of representative micro unit (Shen & Hu, 2006; Zeng, Yu, & Lu, 2008).

Wang and Zhang (Wang & Zhang, 2008) assessed the elastic properties and the
effective wall thickness of the SWCNT and concluded that the wall thickness of the SWCNT should be less that 0.142 nm. Zhang and Shen (Zhang & Shen, 2006) reported the mechanical properties of SWCNTs using MD simulation. The results suggested that for a (10, 10) SWCNT, the wall thickness should be taken as 0.067 nm. Therefore the modulus of the nanotube should be even higher than previously reported with the wall thickness of 0.34 nm. The mechanical property simulation of the SWCNT using finite element method with intermolecular energy can also be found in the research work by Lu et al. (Lu & Hu, 2012) and Giannopoulos et al. (Giannopoulos, Kakavas, & Anifantis, 2008).

Using the MD simulation method, Griebel and Hamaekers (Griebel & Hamaekers, 2004) investigated the SWCNT-Poly (ethylene) composite, and Han and Elliott (Han & Elliott, 2007) modelled the SWCNT-PMMA and SWCNT-PmPV system. Both of the researches indicated that CNT could be used to reinforce an appropriate matrix and the CNT should be aligned along the loading direction in order to achieve the maximum improvement. They also suggested that the simple rule of mixture may not be accurate and an extended rule-of-mixtures should be used. Bohlen and Bolton (Bohlen & Bolton, 2013) simulated the pullout of a SWCNT from the PVDF matrix using MD simulation. The simulation showed that the Young’s modulus in the CNT alignment direction can be significantly enhanced. On the other hand, there are negligible improvement with respect to the bulk or shear modulus of the composite system.

2.6.1.2 Classification of Beam and Plate Theories

From microstructure point of view, common structural elements such as beams, plates, shells are essentially 3-D solids, therefore 3-D solid elasticity theory applies. However, for the thin-walled structural elements, the transverse thickness are relatively thin compared to the other two dimensions, therefore the displacement fields were developed based on different 2-D hypothesis, known as the Kirchhoff-Love (classical) theory and the Mindlin-Reissner (first-order shear deformation) theory.

The classical theory and the first order shear deformation theory were derived under certain assumptions. In both theories, the transverse normals are assumed to be
remaining straight under deformation. In the first order shear deformation theory, this assumption leads to a constant transverse shear strain distribution along the thickness of the structure which is required to be amended using the shear correction factors (Mindlin, 1951; Timoshenko & Gere, 1961; Timoshenko & Woinowsky-Krieger, 1959). For homogenous and isotropic beams, the value of shear correction factors may take the values of 5/6 and $\pi^2/12$ or be a function of the Poisson ratio. However, these values may not be appropriate for the FGM as the material properties vary through the thickness of structures, which is extremely difficult to assess for composite materials (Reddy, 2004b).

As most of the envisaged functionally graded CNT reinforced composites are in the scale of millimeters or nanometers and are working under sensitive manner, better theory should be used. The third order beam/plate theory relaxes the straightness and normality of transverse normal after deformation. The unknown displacements field can be expand as cubic functions of thickness coordinate which allows parabolic transverse normal deformation, thus refined solution can be obtained kinematically at the expense of computational cost. Besides, no shear correction factor is needed in the third order theory. Other theories, such as higher order theory that expands the displacement components into power series of the thickness coordinates, may attain similar accuracy as the third order theory, but with substantially more computational cost (Matsunaga, 2008). Based on the aforementioned reason, the third order shear deformation theory is being adopted in the present research.

2.6.1.3 Constitutive Equation

Functionally graded aligned CNT composites are essentially made of heterogeneous orthotropic materials. Unlike the isotropic homogenous counterparts, the coupling effects have become a major issue. The bending – extension – twisting couplings are reflected by the global stiffness matrix assembled by coordinates-wise integration along the thickness direction. As mentioned before, most structural elements made of such type of material are thin beams, plates or shells. Therefore plane state stress assumption is valid for such structural elements, and the order of the stiffness matrix can be reduced accordingly.
According to the law of conservation of energy, thermal strains can be introduced to the constitutive equations which connect the temperature field with the stress and strain. Similarly, the hydroelasticity, electroelasticity and magnetostrictive phenomenon connect other physical field to the mechanics.

2.6.1.4 Strain – Displacement Relations

Many of the structural elements are designed to be compliant and may undergo relatively large rotations and deflections under working conditions. Midplane stretching, rotary inertia effect and curvature nonlinearities can contribute to the response of a beam which undergoes large displacement and rotation (Mohammad, 2011). Such beams and plates can usually be encountered in the MEMS devices, aviation or marine industries. If the rotations of the transverse normals are moderate (for example, between 10° and 15°), the linear strain-displacement relations are no longer valid. Under the framework of classical theory, the von Kármán strains take into account geometric nonlinearities which involve large deflections and rotations in the elastic vibration state (small strain with modest rotations and deflections). Similarly, in the first order and third order shear deformation theory, the geometric nonlinear strain of the von Kármán sense can be used to reflect the geometric nonlinearity.

2.6.1.5 Equations of Motion

The principle of conservation of linear momentum and Newton’s second law states the governing equations. Often it will result in the form of partial differential equations (PDEs). Certain mathematical techniques can be used to cast the equivalent “weak form” of the PDEs which can be assembled to algebraic equations.

Variational method or energy based method can also be adopted to obtain weak forms of the governing equations. In the present study, the Hamilton’s principle has been used to obtain the variational energy functional.

2.6.2 Analytical and Numerical Solutions

Navier expanded the displacement fields and loads into double trigonometric series for simply supported rectangular plates and gained the closed-form solution. Lévy’s
single Fourier series solution can be applied to rectangular plates with two edges simply supported and the other two either be free, simply supported or fixed. Much effort has been devoted by researchers on obtaining exact solution for buckling and vibration of plates and shells (Wang, Wang, & Reddy, 2005). The analytical results have provided important benchmarks for engineers and researchers.

For complicated problems such as irregular geometries or governing equations involving nonlinearity, exact analytical solutions are often not available. Approximate numerical solutions can be deduced asymptotically, provided that the numerical solution will converge to the real solution.

The Ritz method is a variational approach that can be used to obtain natural frequencies. It is restricted to systems of equations of motion that can be casted in energy or weak forms. Galerkin's method, a special case of weighted residuals method, addresses the differential equation form of a variational problem. By minimizing the residual of the differential equation integrated over the domain with a weight function, algebraic equations which equivalent to the weak forms of the governing differential equations can be derived (Reddy, 2002). One of the merits is its capability of dealing with nonconservative and nonlinear systems.

Perturbation method is an approximate analytical method for solving nonlinear systems, by decomposing the small perturbation parameter into power series. The nonlinear solution can be obtained. Shen et al. extended the classic perturbation method into a two-step perturbation technique and applied it to analyse postbuckling, nonlinear bending and vibration of structures (Shen, 2009a).

Differential quadrature (DQ) method approximates the derivatives of a smooth function at a point as a weighted sum of function values at selected Chebyshev nodes. As a high-order numerical method, it can provide a cost-effective tool for solving nonlinear partial differential equations (Zong & Zhang, 2009).

Global methods such as the Ritz, DQ or Galerkin methods are highly accurate. But local methods are more flexible in handling complex geometries, boundary conditions
and internal supports by discretizing the problem region into local domains. The finite element methods can be computationally very expensive, especially for multiphysics, nonlinear, and time-dependent problems, such as those commonly encountered in MEMS.

In the aforementioned methods, selecting appropriate interpolation function becomes the linchpin. Instead of refining the size of the mesh, p-FEM and spectral element method with the higher degree piecewise polynomial trial functions have been developed, to exploit the so-called speed of exponential convergence. Wei et al. developed the Discrete Singular Convolution (DSC) algorithm and constructed several singular kernels and applied the algorithm to various types of PDEs (Wei, Zhao, & Xiang, 2002; Xiang, Zhao, & Wei, 2002). Xiang et al. incorporate DSC delta type wavelet kernel as a trial function with Ritz method for vibration analysis of rectangular Mindlin plates with mixed edge supports, and demonstrating great numerical accuracy and stability (Xiang, Lai, Zhou, & Lim, 2010), and may be suitable for handling complex geometries, boundary and support conditions and high-frequency vibrations problems.

As a mature computational method, the p-Ritz method is a high order polynomial numerical approach. It is versatile, and accurate, and possesses rapid numerical convergence rate and reasonable computational cost. The energy variational approach together with the generalized p-Ritz method will be used in the current study.

2.7 Research Methods and Significance

From the above discussions, it is concluded that the FG-CNT composite have many advantages and may find its niche position in the next generation multifunctional engineering applications. However, it is noted that the vibration behavior of structures made of this novel type of FG-CNT composites haven’t been thoroughly assessed. It is crucial to understand the vibration characteristics of certain engineering product made of FG-CNT composites before bringing such devices to market. Therefore there is an urgent need to develop an accurate and efficient computational model to evaluate such novel composites.
To the best of the author’s knowledge, the computational model using the third order shear deformation theory, variational formulation and p-Ritz method have not been attempted previously on evaluating the natural frequency of FG-CNT beams and plates. Besides, there is no comparison study on the results of FG-CNT beams and plates by using the third order shear deformation theory and the first order shear deformation theory. The current work is aiming to fill this gap by using this new model and provide some new results. By doing this, it is hoped to shed some light on the usability of such composite structures.

In the present study, the mathematical modeling of free vibration will be established for the aligned FG-CNT reinforced beams and plates with certain essential boundary conditions. Energy functional will then be derived and solved numerically by the generalized p-Ritz method. Geometric nonlinearity will be considered for beam elements only and appropriate iteration methods will be used and evaluated for nonlinear problem during the research. Software based on the proposed model and solving procedure will be developed. Numerical solutions will be compared with existing results and parametric studies will be carried out. The computational method and results will be discussed and validated at the end of this project.
CHAPTER 3  MATHEMATICAL MODELLING

3.1 Linear Free Vibration of FG-CNT Composite Plates

3.1.1 Problem Definition

As shown in Figure 3.1.1(a), consider a CNT reinforced composite plate of length $L$, width $b$, and thickness $h$ along $x$, $y$ and $z$ axis, respectively. $\theta$ is the angle between the reinforcing CNT direction and the positive $x$ direction. Three types of aligned CNT reinforced plates are considered, namely the uniformly distributed CNT plates (UD-CNT), the functionally graded CNT plates type $\Lambda$ (FG$\Lambda$-CNT) which features CNT concentrated at bottom region, and the functionally graded CNT plates type $X$ (FG$X$-CNT) which features CNT concentrated at both top and bottom regions, as shown in Figure 3.1.1(b).

![Diagram of plate and CNT distribution](image)

Figure 3.1.1 Functionally graded carbon nanotube reinforced composite plates:
(a) Coordinate system of plate and (b) Cross sections of UD-, FGA- and FGX-CNT plates.

3.1.2 CNT Distribution and Engineering Constants

For the above three FG-CNT reinforced plates, The CNTs for the later two plates are...
assumed to be graded in the thickness direction. The three plates will have the same geometries and contain the same value of total weight of CNTs \( m_{tcnt} \), the same total volume of composite is \( Vol_{total} = Vol_{CNT} + Vol_{matrix} \), and with the same total CNTs volume fraction \( V_{tcnt} = Vol_{CNT}/Vol_{total} \). \( \rho_{cnt} \) is the density of CNT.

Without loss of generality, we introduce the distribution shape function \( f(z) \) and the mass distribution constant \( \Gamma \) of CNTs, and denote

\[
\int_{-\frac{h}{2}}^{\frac{h}{2}} f(z) \Gamma \, dz = m_{tcnt} \tag{3.1}
\]

The \( f(z) \) can be arbitrary functions of \( z \) as show below in Figure 3.1.2.

![Figure 3.1.2 Distribution shape functions of plates with arbitrary CNT distribution along thickness of the cross section](image)

Therefore

\[
\Gamma = \frac{m_{tcnt}}{\int_{-\frac{h}{2}}^{\frac{h}{2}} f(z) \, dz} \tag{3.2}
\]

and the mass distribution of CNTs along each ‘thin layer’ of the \( z \) coordinates is
\[ f(z)\Gamma = f(z) \frac{m_{tcnt}}{\int_{-\frac{h}{2}}^{\frac{h}{2}} f(z) \, dz} \]  

(3.3)

and the volume distribution of CNTs along each ‘thin layer’ of the \( z \) coordinates is

\[ V_{CNT}(z) = \frac{f(z)\Gamma}{\rho_{cnt}} = \frac{f(z)m_{tcnt}}{\rho_{cnt} \int_{-\frac{h}{2}}^{\frac{h}{2}} f(z) \, dz} \]  

(3.4)

the volume fraction of CNTs as a function of the \( z \) coordinates can be derived as

\[
\begin{align*}
V_{CNT}(z) &= \frac{V_{CNT}(z)}{Vol_{total}/h} = \frac{f(z)m_{tcnt}}{\rho_{cnt} \int_{-\frac{h}{2}}^{\frac{h}{2}} f(z) \, dz} \frac{h}{Vol_{total}} \\
&= \frac{f(z)h}{\int_{-\frac{h}{2}}^{\frac{h}{2}} f(z) \, dz} \frac{Vol_{CNT}}{Vol_{total}} = \frac{f(z)h}{\int_{-\frac{h}{2}}^{\frac{h}{2}} f(z) \, dz} \frac{V_{CNT}}{Vol_{total}} = \Theta V_{CNT} 
\end{align*}
\]  

(3.5)

in which

\[ \Theta = \frac{f(z)h}{\int_{-\frac{h}{2}}^{\frac{h}{2}} f(z) \, dz} \]  

(3.6)

is the volume fraction distribution function of CNTs.

For the latter two FG-CNT plates, the CNT distributions along the thickness of the plates can follow the power laws. For the FGA-CNT plates, the distribution shape function can be represented as (Shen, 2009b)

\[ f(z) = \left( \frac{h - 2z}{2h} \right)^k \quad \left( -\frac{h}{2} \leq z \leq \frac{h}{2} \right) \]  

(3.7)
and as shown in Figure 3.1.3.

![Figure 3.1.3 Distribution shape function of the FGA-CNT plates](image)

The volume fraction distribution function of CNTs can be derived as

\[
\Theta = \frac{f(z)h}{\int_{-\frac{h}{2}}^{\frac{h}{2}} f(z) \, dz} = \frac{\left(\frac{h - 2z}{2h}\right)^k h}{\int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\frac{h - 2z}{2h}\right)^k \, dz}
\]

\[
= \frac{\left(\frac{h - 2z}{2h}\right)^k h}{\left(\frac{1}{2h}\right)^k \left(\frac{1}{2z}\right) \int_{-\frac{h}{2}}^{\frac{h}{2}} (h - 2z)^k \, dz (h - 2z)} = \frac{-2(h - 2z)^{k+1}}{k + 1} \]

\[
= (k + 1) \left(\frac{h - 2z}{2h}\right)^k
\]

(3.8)

Substitute Eq. (3.8) into Eq. (3.5), the volume fraction of CNTs as a function of z coordinates is obtained

\[
V_{CNT}(z) = (k + 1) \left(\frac{h - 2z}{2h}\right)^k V_{tcnt} \quad \left(\frac{h}{2} \leq z \leq \frac{h}{2}\right)
\]

(3.9)

For the FGX-CNT plates, the distribution shape function can have the following form
\[ f(z) = \begin{cases} \left( \frac{2}{h} z \right)^k & 0 \leq z \leq \frac{h}{2} \\ \left( -\frac{2}{h} z \right)^k & -\frac{h}{2} \leq z < 0 \end{cases} \]  

(3.10)

which is shown in Figure 3.1.4.

![Figure 3.1.4 Distribution shape function of the FGX-CNT plates](image)

Substitute Eq. (3.10) into Eq. (3.6), the volume fraction distribution function of FGX-CNT plates is calculated as follows,

\[
\theta = \frac{\int_{-\frac{h}{2}}^{\frac{h}{2}} f(z) \, dz}{\frac{h}{2} \int_{-\frac{h}{2}}^{\frac{h}{2}} f(z) \, dz} \\
= \frac{\int_{0}^{\frac{h}{2}} \left( \frac{2}{h} z \right)^k \frac{h}{2} \left( \frac{2}{h} z \right)^k \, dz}{\int_{-\frac{h}{2}}^{0} \left( -\frac{2}{h} z \right)^k \frac{h}{2} \left( -\frac{2}{h} z \right)^k \, dz} \\
= \left( k + 1 \right) \frac{\left( 2z \right)^k}{h} \quad \left( 0 \leq z \leq \frac{h}{2} \right) \\
= \left( k + 1 \right) \frac{\left( -2z \right)^k}{h} \quad \left( -\frac{h}{2} \leq z < 0 \right) 
\]

(3.11)

The volume fraction of CNTs as a function of \( z \) coordinates can then be derived as
$$V_{CNT}(z) = \begin{cases} \frac{(\frac{2}{h}z)^k h}{2 \int_0^{\frac{h}{2}} \left(\frac{2}{h}z\right)^k dz} (k + 1) \left(\frac{2}{h}\right)^k V_{tcnt} & \left(0 \leq z \leq \frac{h}{2}\right) \\ \frac{(\frac{2}{h}z)^k h}{2 \int_0^{\frac{h}{2}} \left(\frac{2}{h}z\right)^k dz} (k + 1) \left(-\frac{2}{h}\right)^k V_{tcnt} & \left(-\frac{h}{2} \leq z < 0\right) \end{cases}$$

(3.12)

However from the practical perspective, only linear distribution is considered in the current study, i.e. $k = 1$. The CNT volume fraction of FGΛ-CNT plates while $k = 1$ is derived as

$$V_{CNT}(z) = \left(1 - \frac{2z}{h}\right)V_{tcnt} \quad \left(-\frac{h}{2} \leq z \leq \frac{h}{2}\right)$$

(3.13)

The corresponding volume fraction of FGX-CNTs with $k = 1$ is given by

$$V_{CNT}(z) = \begin{cases} \frac{4z}{h} V_{tcnt} & \left(0 \leq z \leq \frac{h}{2}\right) \\ -\frac{4z}{h} V_{tcnt} & \left(-\frac{h}{2} \leq z < 0\right) \end{cases}$$

(3.14)

For UD-CNT plates, the CNTs are uniformly dispersed along the thickness of the plates, which makes the CNTs volume fraction along the $z$ coordinates the same as the total CNT volume fraction

$$V_{CNT}(z) = V_{tcnt} \quad \left(-\frac{h}{2} \leq z \leq \frac{h}{2}\right)$$

(3.15)

In practical applications, the Mori-Tanaka Model (Mori & Tanaka, 1973) has been successfully applied to predict the properties of heterogeneous materials such as the micron fibre reinforced laminate composites. As a continuum mechanics method, based on Eshelby’s equivalent inclusion theory (Eshelby, 1957), the stress and strain differences due to the inclusion in the infinite medium can be equivalent to the eigenstrain exerted on the matrix. Thus, the equivalent engineering constants of the
composites can be solved by the equilibrium of average stress. For the CNT reinforced composites, as the inclusions are in the nanoscale, therefore the representative volume element (RVE) will be much smaller than its counterpart in the microscale normally encountered when using the Mori-Tanaka model. In the current study, the multiscaled approach has been adopted in order to link the RVE in the nanoscale with the extended rule of mixture.

The effective Young’s modulus and shear modulus of the CNT composite plates are determined based on matching the results from the molecular dynamics simulation with the extended rule of mixture. The expressions are as follows (Shen, 2009b)

\[
E_{11}(z) = \eta_1 V_{\text{CNT}}(z) E_{11}^{\text{cnt}} + V_m(z) E^m
\]  
\[
\frac{\eta_2}{E_{22}(z)} = \frac{V_{\text{CNT}}(z)}{E_{22}^{\text{cnt}}} + \frac{V_m(z)}{E^m}
\]  
\[
\frac{\eta_3}{G_{12}(z)} = \frac{V_{\text{CNT}}(z)}{G_{12}^{\text{cnt}}} + \frac{V_m(z)}{G^m}
\]  
\[G_{13}(z) = G_{12}(z)\]
\[G_{23}(z) = 1.2G_{12}(z)\]
\[\nu_{12}(z) = V_{\text{CNT}}(z)\nu_{12}^{\text{cnt}} + V_m(z)\nu^m\]
\[\nu_{21}(z) = \frac{\nu_{12}(z)}{E_{11}(z)} E_{22}(z)\]
\[\rho(z) = V_{\text{CNT}}(z)\rho^{\text{cnt}} + V_m(z)\rho^m\]
\[V_m(z) = 1 - V_{\text{CNT}}(z)\]

where \(E_{11}, E_{22}, G_{12}, G_{13}, G_{23}\) are Young’s moduli in the longitudinal and transverse directions and shear moduli of the nanocomposite, \(E_{11}^{\text{cnt}}, E_{22}^{\text{cnt}}\) and \(G_{12}^{\text{cnt}}\) are Young’s moduli in the longitudinal and transverse directions and shear modulus of CNT, \(E^m, G^m\) are Young’s modulus and shear modulus of matrix, \(\nu_{12}, \nu_{21}, \nu_{12}^{\text{cnt}}\) and \(\nu^m\) are Poisson’s ratios of nanocomposite, CNT and matrix, \(\rho, \rho^{\text{cnt}}\) and \(\rho^m\) are mass densities of nanocomposite, CNT and matrix, \(V_m\) is the matrix volume fraction, and \(\eta_1, \eta_2, \eta_3\) are CNT/matrix efficiency parameters, respectively.
3.1.3 Formulation

3.1.3.1 Displacement Field

Based on the third order shear deformation plate theory, the displacements of the plate along the $x$, $y$ and $z$ directions can be expressed as follows (Reddy, 2004b)

\[ u(x, y, z, t) = u_0(x, y, t) + z\phi_x(x, y, t) - \frac{4}{3h^2}z^3(\phi_x + \frac{\partial w_0}{\partial x}) \]  
(3.17a)

\[ v(x, y, z, t) = v_0(x, y, t) + z\phi_y(x, y, t) - \frac{4}{3h^2}z^3(\phi_y + \frac{\partial w_0}{\partial y}) \]  
(3.17b)

\[ w(x, y, z, t) = w_0(x, y, t) \]  
(3.17c)

in which $u_0$, $v_0$ and $w_0$ are the displacements along the $x$, $y$ and $z$ directions on the midsurface of the plate, and $\phi_x$ and $\phi_y$ are the rotations of the cross section along the $y$ and $x$ directions, respectively.

3.1.3.2 Strain-Displacement Relations

The linear strain-displacement relations of the plate are as follows

\[ \varepsilon_{xx} = \frac{\partial u}{\partial x} \]  
(3.18a)

\[ \varepsilon_{yy} = \frac{\partial v}{\partial y} \]  
(3.18b)

\[ \varepsilon_{zz} = \frac{\partial w}{\partial z} \]  
(3.18c)

\[ \varepsilon_{yz} = \frac{1}{2}(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}) = \frac{1}{2}\gamma_{yz} \]  
(3.18d)

\[ \varepsilon_{xz} = \frac{1}{2}(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}) = \frac{1}{2}\gamma_{xz} \]  
(3.18e)

\[ \varepsilon_{xy} = \frac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) = \frac{1}{2}\gamma_{xy} \]  
(3.18f)

The relationship of the engineering strains and the displacement of the mid-surface of the plate can be derived as follows
\[ \varepsilon_{xx} = \frac{\partial u_0}{\partial x} + z \left( \frac{\partial \phi_x}{\partial x} - \frac{4z^3}{3h^2} \left( \frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2} \right) \right) \] (3.19a)

\[ \varepsilon_{yy} = \frac{\partial v_0}{\partial y} + z \left( \frac{\partial \phi_y}{\partial y} - \frac{4z^3}{3h^2} \left( \frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w_0}{\partial y^2} \right) \right) \] (3.19b)

\[ \varepsilon_{zz} = 0 \] (3.19c)

\[ y_{yx} = \phi_y + \frac{\partial w_0}{\partial y} - \frac{4z^2}{h^2} \left( \phi_y + \frac{\partial w_0}{\partial y} \right) \] (3.19d)

\[ y_{xz} = \phi_x + \frac{\partial w_0}{\partial x} - \frac{4z^2}{h^2} \left( \phi_x + \frac{\partial w_0}{\partial x} \right) \] (3.19e)

\[ y_{xy} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + z \left( \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \right) - \frac{4z^3}{3h^2} \left( \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2 \frac{\partial^2 w_0}{\partial x \partial y} \right) \] (3.19f)

### 3.1.3.3 Elastic Constitutive Relations

As mentioned earlier, the thickness of most of the FG-CNT plates are relatively small compared to its length and width. Therefore the transverse normal stress \( \sigma_{zz} \) can be neglected. The plane stress-reduced constitutive relations should be used to reflect this situation. Invoking the Hooke’s law, the constitutive relationship in the global coordinates is derived as follows (Reddy, 2004b)

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{yz} \\
\sigma_{xz} \\
\sigma_{xy}
\end{bmatrix} =
\begin{bmatrix}
\tilde{Q}_{11} & \tilde{Q}_{12} & 0 & 0 & \tilde{Q}_{16} \\
\tilde{Q}_{12} & \tilde{Q}_{22} & 0 & 0 & \tilde{Q}_{26} \\
0 & 0 & \tilde{Q}_{44} & \tilde{Q}_{45} & 0 \\
0 & 0 & \tilde{Q}_{45} & \tilde{Q}_{55} & 0 \\
\tilde{Q}_{16} & \tilde{Q}_{26} & 0 & 0 & \tilde{Q}_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
y_{yx} \\
y_{xz} \\
y_{xy}
\end{bmatrix}
\] (3.20)

where

\[
\tilde{Q}_{11} = Q_{11} \cos^4 \theta + 2(Q_{12} + 2Q_{66}) \cos^2 \theta \sin^2 \theta + Q_{22} \sin^4 \theta \] (3.21a)

\[
\tilde{Q}_{12} = (Q_{11} + Q_{12} - 4Q_{66}) \cos^2 \theta \sin^2 \theta + Q_{12} (\sin^4 \theta + \cos^4 \theta) \] (3.21b)

\[
\tilde{Q}_{22} = Q_{22} \cos^4 \theta + 2(Q_{12} + 2Q_{66}) \cos^2 \theta \sin^2 \theta + Q_{11} \sin^4 \theta \] (3.21c)
\[ \bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})\cos^3 \theta \sin \theta \\
+ (2Q_{66} + Q_{12} - Q_{22})\cos \theta \sin^3 \theta \hspace{1cm} (3.21d) \]

\[ \bar{Q}_{26} = (Q_{12} - Q_{22} + 2Q_{66})\cos^3 \theta \sin \theta \\
+ (Q_{11} - Q_{12} - 2Q_{66})\cos \theta \sin^4 \theta \hspace{1cm} (3.21e) \]

\[ \bar{Q}_{66} = (Q_{11} + Q_{12} - 2Q_{12} - 2Q_{66})\cos^2 \theta \sin^2 \theta + Q_{66}(\cos^4 \theta + \sin^4 \theta) \hspace{1cm} (3.21f) \]

\[ \bar{Q}_{44} = Q_{44}\cos^2 \theta + Q_{55}\sin^2 \theta \hspace{1cm} (3.21g) \]

\[ \bar{Q}_{45} = (Q_{55} - Q_{44})\cos \theta \sin \theta \hspace{1cm} (3.21h) \]

\[ \bar{Q}_{55} = Q_{55}\cos^2 \theta + Q_{44}\sin^2 \theta \hspace{1cm} (3.21i) \]

and

\[ Q_{11(z)} = \frac{E_{11(z)}}{1 - \nu_{12(z)}\nu_{21(z)}} \hspace{1cm} (3.22a) \]

\[ Q_{12(z)} = \frac{\nu_{12(z)}E_{22(z)}}{1 - \nu_{12(z)}\nu_{21(z)}} \hspace{1cm} (3.22b) \]

\[ Q_{22(z)} = \frac{E_{22(z)}}{1 - \nu_{12(z)}\nu_{21(z)}} \hspace{1cm} (3.22c) \]

\[ Q_{44(z)} = G_{23(z)} \hspace{1cm} (3.22d) \]

\[ Q_{55(z)} = G_{13(z)} \hspace{1cm} (3.22e) \]

\[ Q_{66(z)} = G_{12(z)} \hspace{1cm} (3.22f) \]

introducing the following stiffness coefficients

\[ (A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}) = \int_{\frac{h}{2}}^{h} \bar{Q}_{ij}(1, z, z^2, z^3, z^4, z^6)dz \hspace{1cm} (3.23a) \]

(i, j = 1, 2, 6)

\[ (A_{ij}, D_{ij}, F_{ij}) = \int_{\frac{h}{2}}^{h} \bar{Q}_{ij}(1, z^2, z^4)dz \hspace{1cm} (3.23b) \]

(i, j = 4, 5)
and the stress resultants

\begin{align*}
\begin{cases}
N_{xx} \\
N_{yy} \\
N_{xy}
\end{cases} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \end{pmatrix} \, dz \\
M_{xx} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \end{pmatrix} z \, dz \\
M_{xy} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \end{pmatrix} z^2 \, dz \\
P_{xx} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \end{pmatrix} z^3 \, dz \\
Q_{xz} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{pmatrix} \sigma_{xz} \\ \sigma_{yz} \end{pmatrix} \, dz \\
Q_{yz} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{pmatrix} \sigma_{xz} \\ \sigma_{yz} \end{pmatrix} z^2 \, dz
\end{align*}

Substituting Eq. (3.20) to Eqs. (3.24a). The stress resultant - midsurface displacements relationship can be obtained as follows.

\begin{align*}
\begin{bmatrix} N_{xx} \\
N_{yy} \\
N_{xy} \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} & A_{16} \\
A_{12} & A_{22} & A_{26} \\
A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \frac{\partial u_0}{\partial x} \\
\frac{\partial v_0}{\partial y} \\
\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{bmatrix} \\
&\quad + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\
B_{12} & B_{22} & B_{26} \\
B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \frac{\partial \phi_x}{\partial x} \\
\frac{\partial \phi_y}{\partial y} \\
\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{bmatrix} \\
&\quad - c_1 \begin{bmatrix} E_{11} & E_{12} & E_{16} \\
E_{12} & E_{22} & E_{26} \\
E_{16} & E_{26} & E_{66} \end{bmatrix} \begin{bmatrix} \frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2} \\
\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w_0}{\partial y^2} \\
\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2 \frac{\partial^2 w_0}{\partial x \partial y} \end{bmatrix}
\end{align*}

(3.25a)
\[
\begin{bmatrix}
M_{xx} \\
M_{yy} \\
M_{xy}
\end{bmatrix} = \begin{bmatrix}
B_{11} & B_{12} & B_{16} \\
B_{12} & B_{22} & B_{26} \\
B_{16} & B_{26} & B_{66}
\end{bmatrix} \begin{bmatrix}
\frac{\partial u_0}{\partial x} \\
\frac{\partial v_0}{\partial y} \\
\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
D_{11} & D_{12} & D_{16} \\
D_{12} & D_{22} & D_{26} \\
D_{16} & D_{26} & D_{66}
\end{bmatrix} \begin{bmatrix}
\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2} \\
\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w_0}{\partial y^2} \\
\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2 \frac{\partial^2 w_0}{\partial x \partial y}
\end{bmatrix}
\]

\[-c_1 \begin{bmatrix}
F_{11} & F_{12} & F_{16} \\
F_{12} & F_{22} & F_{26} \\
F_{16} & F_{26} & F_{66}
\end{bmatrix} \begin{bmatrix}
\frac{\partial \phi_x}{\partial x} \\
\frac{\partial \phi_y}{\partial y} \\
\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}
\end{bmatrix}
\] (3.25b)

\[
\begin{bmatrix}
P_{xx} \\
P_{yy} \\
P_{xy}
\end{bmatrix} = \begin{bmatrix}
E_{11} & E_{12} & E_{16} \\
E_{12} & E_{22} & E_{26} \\
E_{16} & E_{26} & E_{66}
\end{bmatrix} \begin{bmatrix}
\frac{\partial u_0}{\partial x} \\
\frac{\partial v_0}{\partial y} \\
\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
F_{11} & F_{12} & F_{16} \\
F_{12} & F_{22} & F_{26} \\
F_{16} & F_{26} & F_{66}
\end{bmatrix} \begin{bmatrix}
\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2} \\
\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w_0}{\partial y^2} \\
\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2 \frac{\partial^2 w_0}{\partial x \partial y}
\end{bmatrix}
\]

\[-c_1 \begin{bmatrix}
H_{11} & H_{12} & H_{16} \\
H_{12} & H_{22} & H_{26} \\
H_{16} & H_{26} & H_{66}
\end{bmatrix} \begin{bmatrix}
\frac{\partial \phi_x}{\partial x} \\
\frac{\partial \phi_y}{\partial y} \\
\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}
\end{bmatrix}
\] (3.25c)

\[
\begin{bmatrix}
Q_{yz} \\
Q_{xz}
\end{bmatrix} = \begin{bmatrix}
A_{44} & A_{45} \\
A_{45} & A_{55}
\end{bmatrix} \begin{bmatrix}
\phi_y + \frac{\partial w_0}{\partial y} \\
\phi_x + \frac{\partial w_0}{\partial x}
\end{bmatrix} - c_2 \begin{bmatrix}
D_{44} & D_{45} \\
D_{45} & D_{55}
\end{bmatrix} \begin{bmatrix}
\phi_y + \frac{\partial w_0}{\partial y} \\
\phi_x + \frac{\partial w_0}{\partial x}
\end{bmatrix}
\] (3.25d)
\[
\begin{bmatrix}
R_{yz} \\
R_{xz}
\end{bmatrix} = \begin{bmatrix}
D_{44} & D_{45} \\
D_{45} & D_{55}
\end{bmatrix} \begin{bmatrix}
\phi_y + \frac{\partial w_0}{\partial y} \\
\phi_x + \frac{\partial w_0}{\partial x}
\end{bmatrix} - c_2 \begin{bmatrix}
F_{44} & F_{45} \\
F_{45} & F_{55}
\end{bmatrix} \begin{bmatrix}
\phi_y + \frac{\partial w_0}{\partial y} \\
\phi_x + \frac{\partial w_0}{\partial x}
\end{bmatrix}
\] (3.25e)

where

\[
c_1 = \frac{4}{3h^2}, \quad c_2 = \frac{4}{h^2}
\] (3.26)

### 3.1.3.4 Variational Energy Formulation

The first variational terms of strains are of the following form

\[
\delta \varepsilon_{xx} = \frac{\partial \delta u_0}{\partial x} + z \frac{\partial \delta \phi_x}{\partial x} - 4z^3 \left( \frac{\partial \delta \phi_x}{\partial x} + \frac{\partial^2 \delta w_0}{\partial x^2} \right)
\] (3.27a)

\[
\delta \gamma_{xy} = \frac{\partial \delta u_0}{\partial y} + \frac{\partial \delta v_0}{\partial x} + z \left( \frac{\partial \delta \phi_x}{\partial y} + \frac{\partial \delta \phi_y}{\partial x} \right) - 4z^3 \left( \frac{\partial \delta \phi_x}{\partial y} + \frac{\partial \delta \phi_y}{\partial x} + 2 \frac{\partial^2 \delta w_0}{\partial x \partial y} \right)
\] (3.27b)

\[
\delta \gamma_{xz} = \delta \phi_x + \frac{\partial \delta w_0}{\partial x} - 4z^2 \left( \frac{\partial \delta \phi_x}{\partial x} + \frac{\partial \delta w_0}{\partial x} \right)
\] (3.27c)

\[
\delta \varepsilon_{yy} = \frac{\partial \delta v_0}{\partial y} + z \frac{\partial \delta \phi_y}{\partial y} - 4z^3 \left( \frac{\partial \delta \phi_y}{\partial y} + \frac{\partial^2 \delta w_0}{\partial y^2} \right)
\] (3.27d)

\[
\delta \gamma_{yz} = \delta \phi_y + \frac{\partial \delta w_0}{\partial y} - 4z^2 \left( \frac{\partial \delta \phi_y}{\partial y} + \frac{\partial \delta w_0}{\partial y} \right)
\] (3.27e)

\[
\delta \varepsilon_{zz} = 0
\] (3.27f)

The virtual strain energy \(\delta U\) is derived as follows

\[
\delta U = \int_{\Omega_0} \sigma : \delta \varepsilon \, dv
\] (3.28)
\[
\int_{\Omega_0} \left\{ \frac{h}{2} (\sigma_{xx} \delta \varepsilon_{xx} + \sigma_{yy} \delta \varepsilon_{yy} + \sigma_{zz} \delta \varepsilon_{zz} + \sigma_{yz} \delta \gamma_{yz} \\
+ \sigma_{xz} \delta \gamma_{xz} + \sigma_{xy} \delta \gamma_{xy}) \right\} dxdy
\]

substituting Eqs. (3.20) and (3.27) to Eq. (3.28)

\[
\delta U = \int_{\Omega_0} \left[ N_{xx} \frac{\partial \delta u_0}{\partial x} + M_{xx} \frac{\partial \delta \phi_x}{\partial x} - c_1 P_{xx} \left( \frac{\partial \delta \phi_x}{\partial x} + \frac{\partial^2 \delta w_0}{\partial x^2} \right) \\
+ N_{yy} \frac{\partial \delta v_0}{\partial y} + M_{yy} \frac{\partial \delta \phi_y}{\partial y} - c_1 P_{yy} \left( \frac{\partial \delta \phi_y}{\partial y} + \frac{\partial^2 \delta w_0}{\partial y^2} \right) \\
+ Q_{yz} \left( \delta \phi_y + \frac{\partial \delta w_0}{\partial y} \right) - c_2 R_{yz} \left( \delta \phi_y + \frac{\partial \delta w_0}{\partial y} \right) \\
+ Q_{xx} \left( \delta \phi_x + \frac{\partial \delta w_0}{\partial x} \right) - c_2 R_{xx} \left( \delta \phi_x + \frac{\partial \delta w_0}{\partial x} \right) \\
+ N_{xy} \left( \frac{\partial \delta u_0}{\partial y} + \frac{\partial \delta v_0}{\partial x} \right) + M_{xy} \left( \frac{\partial \delta \phi_x}{\partial y} + \frac{\partial \delta \phi_y}{\partial x} \right) \\
- c_1 P_{xy} \left( \frac{\partial \delta \phi_x}{\partial y} + \frac{\partial \delta \phi_y}{\partial x} + 2 \frac{\partial^2 \delta w_0}{\partial x \partial y} \right) \right] dxdy
\]

(3.29)

According to the D'Alembert's principle, the virtual kinetic energy \( \delta \mathcal{K} \) can be given as follows

\[
\int_{t_0}^{t_1} \delta \mathcal{K} = \int_{t_0}^{t_1} \left[ - \int_{v} \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} \cdot \delta \mathbf{u} \right] dt
\]

(3.30)

In the present study of free vibration, at certain time \( t_0 \), we have

\[
\delta \mathcal{K} = - \int_{v} \rho \left( \frac{\partial^2 u}{\partial t^2} \delta u + \frac{\partial^2 v}{\partial t^2} \delta v + \frac{\partial^2 w}{\partial t^2} \delta w \right) dxdydz
\]

(3.31)

Introducing the following inertia terms
\[ I_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(z)^i \, dz \quad (i = 0, 1, 2, 3, 4, 5, 6) \]  

(3.32)

and substituting Eq. (3.32) to Eq. (3.31), the virtual kinetic energy \( \delta K \) can be expressed as

\[
\delta K = -\int_{\mu_0} \left\{ \delta u_0 \left[ I_0 \ddot{u}_0 + I_1 \dot{\phi}_x - c_1 I_3 \left( \ddot{\phi}_x + \frac{\partial \ddot{w}_0}{\partial x} \right) \right] \\
+ \delta \phi_x \left[ I_1 \ddot{u}_0 + I_2 \ddot{\phi}_x - c_1 \left( I_3 \ddot{u}_0 + 2I_4 \ddot{\phi}_x + I_4 \frac{\partial \ddot{w}_0}{\partial x} \right) \right] \\
+ c_1^2 I_6 \left( \ddot{\phi}_x + \frac{\partial \ddot{w}_0}{\partial x} \right) \right] \\
+ \frac{\partial \delta w_0}{\partial x} \left[ -c_1 I_3 \ddot{u}_0 - c_1 I_4 \ddot{\phi}_x + c_1^2 I_6 \left( \ddot{\phi}_x + \frac{\partial \ddot{w}_0}{\partial x} \right) \right] \\
+ \delta v_0 \left[ I_0 \ddot{v}_0 + I_1 \dot{\phi}_y - c_1 I_3 \left( \ddot{\phi}_y + \frac{\partial \ddot{w}_0}{\partial y} \right) \right] \\
+ \delta \phi_y \left[ I_1 \ddot{v}_0 + I_2 \ddot{\phi}_y - c_1 \left( I_3 \ddot{v}_0 + 2I_4 \ddot{\phi}_y + I_4 \frac{\partial \ddot{w}_0}{\partial y} \right) \right] \\
+ c_1^2 I_6 \left( \ddot{\phi}_y + \frac{\partial \ddot{w}_0}{\partial y} \right) \right] \\
+ \frac{\partial \delta w_0}{\partial y} \left[ -c_1 I_3 \ddot{v}_0 - c_1 I_4 \ddot{\phi}_y + c_1^2 I_6 \left( \ddot{\phi}_y + \frac{\partial \ddot{w}_0}{\partial y} \right) \right] \\
+ I_0 w_0 \delta \ddot{w}_0 \right\} \, dx \, dy
\]  

(3.33)

Invoking the Hamilton’s principle

\[ \delta U - \delta K = 0 \]  

(3.34)

Eq. (3.34) is the equation of motion in the form of variational energy which will be solved in the next subsection.

### 3.1.3.5 p-Ritz Method

Assume that the Ritz trial functions \( \varphi, \xi \) and \( \psi \) are as follows, which satisfy the geometric boundary conditions
\[ u_0(x, y, t) = U(t) \sum_{q=0}^{p_s} \sum_{l=0}^{q} X_m \varphi^x_{ml}(x, y) \]

\[ v_0(x, y, t) = V(t) \sum_{q=0}^{p_s} \sum_{l=0}^{q} Y_m \varphi^y_{ml}(x, y) \]

\[ w_0(x, y, t) = W(t) \sum_{q=0}^{p_s} \sum_{l=0}^{q} Z_m \xi_m(x, y) \]

\[ \phi_x(x, y, t) = \Delta_x(t) \sum_{q=0}^{p_s} \sum_{l=0}^{q} \mathcal{R}_m^x \psi^x_m(x, y) \]

\[ \phi_y(x, y, t) = \Delta_y(t) \sum_{q=0}^{p_s} \sum_{l=0}^{q} \mathcal{R}_m^y \psi^y_m(x, y) \]

(3.35a)

and the variational terms of displacements are given as

\[ \delta u_0(x, y, t) = U(t) \sum_{q=0}^{p_s} \sum_{k=0}^{q} \delta X_k \varphi^x_K(x, y) \]

\[ \delta v_0(x, y, t) = V(t) \sum_{q=0}^{p_s} \sum_{k=0}^{q} \delta Y_k \varphi^y_K(x, y) \]

\[ \delta w_0(x, y, t) = W(t) \sum_{q=0}^{p_s} \sum_{k=0}^{q} \delta Z_k \xi_K(x, y) \]

\[ \delta \phi_x(x, y, t) = \Delta_x(t) \sum_{q=0}^{p_s} \sum_{k=0}^{q} \delta \mathcal{R}_k^x \psi^x_K(x, y) \]

\[ \delta \phi_y(x, y, t) = \Delta_y(t) \sum_{q=0}^{p_s} \sum_{k=0}^{q} \delta \mathcal{R}_k^y \psi^y_K(x, y) \]

(3.35b)

where \( U(t) = V(t) = W(t) = \Delta_x(t) = \Delta_y(t) = e^{i\omega t} \), \( \omega \) is the angular frequency, \( X_m, Y_m, Z_m, \mathcal{R}_m^x \) and \( \mathcal{R}_m^y \) are unknown coefficients, \( \delta X_k, \delta Y_k, \delta Z_k, \delta \mathcal{R}_k^x \) and \( \delta \mathcal{R}_k^y \) are the variational terms of the unknown coefficients. The Ritz trial function may take the following three types as proposed by Ritz originally: (1) Fourier sine
series; (2) ordinary polynomials; or (3) Legendre (orthogonal) polynomials (Leissa, 2005). Bhat studied the rectangular and polygonal plate vibration by using the characteristic orthogonal polynomials as the Ritz trial functions (Bhat, 1985, 1987). The orthogonal polynomials were generated by using a Gram-Schmidt process. By using orthogonal polynomials instead of simple polynomials, the spectral method has the merit of rapid convergence rate, accuracy for higher order vibration frequencies as well as reasonable computation cost. As the current study will be focused on the lower order frequency calculation, therefore the Ritz trial functions are taken the forms of the simple 2-D complete polynomials.

3.1.3.6 Non-Dimensional Quantities

Introduce the following nondimensional quantities

1) Spatial quantities

\[ \zeta = \frac{x}{L}, \quad \eta = \frac{y}{b}, \]
\[ U_0 = \frac{u_0}{L}, \quad V_0 = \frac{v_0}{b}, \quad W_0 = \frac{w_0}{h}, \]
\[ \Phi_\zeta = \frac{\phi_x L}{h}, \quad \Phi_\eta = \frac{\phi_y b}{h} \]  

(3.36a)

2) Stiffness quantities

\[ a_{pq} = \frac{h^2 A_{pq}}{D_m}, \quad b_{pq} = \frac{h B_{pq}}{D_m}, \quad d_{pq} = \frac{D_{pq}}{D_m}, \]
\[ e_{pq} = \frac{E_{pq}}{h D_m}, \quad f_{pq} = \frac{F_{pq}}{h^2 D_m}, \quad h_{pq} = \frac{H_{pq}}{h^4 D_m} \]
\[ D_m = \frac{E^m h^3}{12(1 - v^m)} \quad (p, q = 1,2,4,5,6) \]  

(3.36b)

3) Inertia quantities
Assume that the Ritz trial functions are the same name for nondimensional displacements, except that they are in nondimensional terms

\[
\begin{align*}
U_0 & \sim \varphi^x, \quad V_0 \sim \varphi^y, \quad W_0 \sim \xi, \quad \Phi_\xi \sim \psi^x, \quad \Phi_\eta \sim \psi^y \\
\{\varphi_m^x(\zeta, \eta), \varphi_m^y(\zeta, \eta), \xi_m(\zeta, \eta), \psi_m^x(\zeta, \eta), \psi_m^y(\zeta, \eta)\} \\
& = \zeta^i \eta^q \times LS \times RS \times TS \times BS \\
i & = 0, 1, 2, \ldots q \quad q = 0, 1, 2, \ldots p_s
\end{align*}
\]

\[
\begin{align*}
\{\varphi_k^x(\zeta, \eta), \varphi_k^y(\zeta, \eta), \xi_k(\zeta, \eta), \psi_k^x(\zeta, \eta), \psi_k^y(\zeta, \eta)\} \\
& = \zeta^k \eta^q \times LS \times RS \times TS \times BS \\
k & = 0, 1, 2, \ldots q \quad q = 0, 1, 2, \ldots p_s
\end{align*}
\] (3.37)

where \(LS, RS, TS\) and \(BS\) are the basic functions consisting of the boundary equations of the rectangular plate corresponding to the left edge, right edge, top edge and bottom edge, respectively, and are used to control the boundary conditions of the plate, which will be detailed in the subsequent Section 3.4.2.

Substituting Eqs. (3.35), (3.36), (3.37) to Eq. (3.34), the nondimensional eigenvalue equation can be derived as

\[
([\ddot{K}] - \bar{\omega}^2 [\bar{M}]) \{\ddot{C}\} = \{0\}
\] (3.38)

where the \(\bar{\omega} = \omega L b \sqrt{l_0/D_m}\) is the nondimensional frequency parameter, \([\ddot{K}]\) is the nondimensional stiffness matrix, \([\bar{M}]\) is the nondimensional mass matrix, and \(\{\ddot{C}\}\) is the vector of unknown Ritz coefficients, respectively. The elements of the \([\ddot{K}]\) and \([\bar{M}]\) matrix are listed in Appendix A.
3.2 Linear Free Vibration of FG-CNT Reinforced Beams

3.2.1 Problem Definition

As shown in Figure 3.2.1(a), consider a composite beam system of length $L$, width $b$, thickness $h$ along $x$, $y$ and $z$ directions, respectively. Three types of aligned CNT reinforced beams are considered, namely the uniformly distributed CNT beams (UD-CNT), functionally graded CNT beams type Λ (FGΛ-CNT) which features CNT concentrated at bottom region, and functionally graded CNT beams type X (FGX-CNT) which features CNT concentrated at both top and bottom regions, as shown in Figure 3.2.1(b). The reinforcing CNT alignments are along the length direction, which is the $x$ axis.

![Figure 3.2.1](image_url)

(a) UD-CNT (b) FGΛ-CNT (c) FGX-CNT

Figure 3.2.1 Functionally graded carbon nanotube reinforced composite beams: (a) coordinate system of beam and (b) cross sections of UD-, FGΛ- and FGX-CNT beams.

The equations of CNT distribution and engineering constants of the beams are the same as the ones of the plates. Readers may refer to section 3.1.2 for more details.
3.2.2 Formulation

The formulation of the third order beams is derived from the formulation of third order plates by reducing corresponding width effect. In the present study, assume that the length of the beam is significantly larger than the width as well as the thickness of the beam. The cross sections in the x-y plane remain undeformed during vibration, but can rotate against y axis. This implies that the translational and rotational displacement in the y direction are negligible, i.e. \( v(x, y, z, t) = \phi_y(x, y, t) = 0 \), and the displacement in the x direction and the transverse deflection of the beam can be assumed to be independent of y. In addition to that, the Poisson’s ratio effect in the x-y directions can be neglected.

3.2.2.1 Displacement Field

The displacements along the x and z directions in the third order beam theory are given by (Reddy, 2004b)

\[
\begin{align*}
    u(x, z, t) &= u_0(x, t) + z \phi_x(x, t) - \frac{4}{3h^2} z^3 (\phi_x + \frac{\partial w_0}{\partial x}) \\
    w(x, z, t) &= w_0(x, t)
\end{align*}
\]  

(3.39a)

in which \( u_0 \) and \( w_0 \) are the displacements along the x and z directions in the mid plane of the beam, and \( \phi_x \) is the rotation of the cross section against y axis.

3.2.2.2 Strain-Displacement Relations

The linear normal and shear strain and displacement relationships for the third order beam can be derived as follows

\[
\begin{align*}
    \varepsilon_{xx} &= \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} + z \frac{\partial \phi_x}{\partial x} - \frac{4z^3}{3h^2} \left( \frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2} \right) \\
    \gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \phi_x + \frac{\partial w_0}{\partial x} - \frac{4z^2}{h^2} \left( \phi_x + \frac{\partial w_0}{\partial x} \right)
\end{align*}
\]  

(3.40a)

(3.40b)

3.2.2.3 Elastic Constitutive Relations

As the CNT reinforcements of the beams are all along the x axis, therefore the
orthotropic shear coupling is neglected. From Eq. (3.25), it is conclude that

$$N_{yy} = N_{xy} = M_{yy} = M_{xy} = P_{yy} = P_{xy} = Q_{yz} = R_{yz} = 0$$ (3.41)

and the stress resultants are

$$N_{xx} = A_{11} \frac{\partial u_0}{\partial x} + B_{11} \frac{\partial \phi_x}{\partial x} - c_1 E_{11} \left( \frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2} \right)$$ (3.42a)

$$M_{xx} = B_{11} \frac{\partial u_0}{\partial x} + D_{11} \frac{\partial \phi_x}{\partial x} - c_1 F_{11} \left( \frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2} \right)$$ (3.42b)

$$P_{xx} = E_{11} \frac{\partial u_0}{\partial x} + F_{11} \frac{\partial \phi_x}{\partial x} - c_1 H_{11} \left( \frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2} \right)$$ (3.42c)

$$Q_{xz} = A_{55} \left( \phi_x + \frac{\partial w_0}{\partial x} \right) - c_2 D_{55} \left( \phi_x + \frac{\partial w_0}{\partial x} \right)$$ (3.42d)

$$R_{xz} = D_{55} \left( \phi_x + \frac{\partial w_0}{\partial x} \right) - c_2 F_{55} \left( \phi_x + \frac{\partial w_0}{\partial x} \right)$$ (3.42e)

### 3.2.2.4 Variational Energy Formulation

The variational terms of the strains can be presented as

$$\delta \varepsilon_{xx} = \frac{\partial \delta u_0}{\partial x} + z \frac{\partial \delta \phi_x}{\partial x} - \frac{4z^3}{3h^2} \left( \frac{\partial \delta \phi_x}{\partial x} + \frac{\partial^2 \delta w_0}{\partial x^2} \right)$$ (3.43a)

$$\delta \gamma_{xz} = \delta \phi_x + \frac{\partial \delta w_0}{\partial x} - \frac{4z^2}{h^2} \left( \delta \phi_x + \frac{\partial \delta w_0}{\partial x} \right)$$ (3.43b)

The virtual strain energy $\delta U$ can be derived as follows (Reddy, 2004b)

$$\delta U = \int_{L_0} \sigma : \delta \varepsilon \, dL$$

$$= \int_{A_0} \left\{ \int_{-h/2}^{h/2} \left( \sigma_{xx} \delta \varepsilon_{xx} + \sigma_{xz} \delta \gamma_{xz} \right) dz \right\} \, dx$$
\[
\int_{\Omega_0} \left\{ \frac{h}{h^2} \left[ \sigma_{xx} \left[ \frac{\partial \delta u_0}{\partial x} + z \frac{\partial \delta \phi_x}{\partial x} - \frac{4z^2}{5h^2} \left( \frac{\partial \delta \phi_x}{\partial x} + \frac{\partial^2 \delta w_0}{\partial x^2} \right) \right] \right.ight.
\]
\[
+ \sigma_{xx} \left[ \delta \phi_x + \frac{\partial \delta w_0}{\partial x} \right.
\]
\[
- \frac{4z^2}{h^2} \left( \delta \phi_x + \frac{\partial \delta w_0}{\partial x} \right) \left\} dz \right\} dx
\]
\[
\int_{\Omega_0} \left\{ N_{xx} \frac{\partial \delta u_0}{\partial x} + M_{xx} \frac{\partial \delta \phi_x}{\partial x} - c_1 P_{xx} \left( \frac{\partial \delta \phi_x}{\partial x} + \frac{\partial^2 \delta w_0}{\partial x^2} \right) \right.
\]
\[
+ Q_{xz} \left( \delta \phi_x + \frac{\partial \delta w_0}{\partial x} \right)
\]
\[
- c_2 R_{xz} \left( \delta \phi_x + \frac{\partial \delta w_0}{\partial x} \right) \left\} dx \right\}
\]

(3.44)

Virtual kinetic energy \( \delta \mathcal{K} \) is given by setting the terms related to \( \delta v_0 \), \( \frac{\partial \delta w_0}{\partial y} \) and \( \delta \phi_y \) to zero, as given below (Reddy, 2004b)

\[
\delta \mathcal{K} = -\int_{\Omega_0} \left\{ \delta u_0 \left[ I_0 \ddot{u}_0 + I_4 \dddot{\phi}_x - c_1 I_3 \left( \ddot{\phi}_x + \frac{\partial \delta w_0}{\partial x} \right) \right] \right.
\]
\[
+ \delta \phi_x \left[ I_1 \dddot{u}_0 + I_2 \dddot{\phi}_x \right.
\]
\[
- c_1 \left( I_5 \dddot{u}_0 + 2I_4 \dddot{\phi}_x + I_4 \frac{\partial \delta w_0}{\partial x} \right)
\]
\[
+ c_1^2 I_6 \left( \dddot{\phi}_x + \frac{\partial \delta w_0}{\partial x} \right) \left\] \right.
\]
\[
+ \frac{\partial \delta w_0}{\partial x} \left[ -c_1 I_3 \ddot{u}_0 - c_1 I_4 \dddot{\phi}_x \right.
\]
\[
+ c_1^2 I_6 \left( \dddot{\phi}_x + \frac{\partial \delta w_0}{\partial x} \right) + I_0 \dddot{w}_0 \delta w_0 \left\} \right\} dx \]

(3.45)

invoking the Hamilton’s principle, the variational energy equation is derived

\[
\delta U - \delta \mathcal{K} = 0
\]

(3.46)
3.2.2.5 p-Ritz Method

The displacement fields and their variational terms for the beam in harmonic vibration can be expressed as follows.

\[
\begin{align*}
  u_0(x, t) &= U(t) \sum_{i=1}^{N} \chi_i \varphi_i(x) \\
  w_0(x, t) &= W(t) \sum_{i=1}^{N} \xi_i \xi_i(x) \\
  \phi_x(x, t) &= \Delta(t) \sum_{i=1}^{N} \mathcal{R}_i \psi_i(x)
\end{align*}
\]

(3.47a)

and

\[
\begin{align*}
  \delta u_0(x, t) &= U(t) \sum_{k=1}^{N} \delta \chi_k \varphi_k(x) \\
  \delta w_0(x, t) &= W(t) \sum_{k=1}^{N} \delta \xi_k \xi_k(x) \\
  \delta \phi_x(x, t) &= \Delta(t) \sum_{k=1}^{N} \delta \mathcal{R}_k \psi_k(x)
\end{align*}
\]

(3.47b)

where \( U(t) = W(t) = \Delta(t) = e^{i \omega t} \), \( \omega \) is the natural frequency parameter, \( \chi_i, \xi_i \) and \( \mathcal{R}_i \) are unknown coefficients, \( \delta \chi_k, \delta \xi_k \) and \( \delta \mathcal{R}_k \) are the variational terms of the unknown coefficients. The Ritz trial functions are taken the forms of 1-D polynomials.

3.2.2.6 Non-Dimensional Quantities

Introduce the following nondimensional quantities

1) Spatial quantities
\[
\zeta = \frac{x}{L}, \quad \eta = \frac{L}{h'}, \quad U_0 = \frac{u_0}{L}, \quad W_0 = \frac{w_0}{h}, \quad \phi_\zeta = \frac{\phi_x L}{h}
\]

(3.48a)

2) Stiffness quantities

\[
\begin{align*}
a_{11} &= \frac{h^2 A_{11}}{D_m}, & a_{55} &= \frac{h^2 A_{55}}{D_m}, & b_{11} &= \frac{h B_{11}}{D_m}, & d_{55} &= \frac{D_{55}}{D_m}, \\
e_{11} &= \frac{E_{11}}{h D_m}, & f_{11} &= \frac{F_{11}}{h^2 D_m}, & f_{55} &= \frac{F_{55}}{h^2 D_m}, & h_{11} &= \frac{H_{11}}{h^4 D_m}
\end{align*}
\]

(3.48b)

3) Inertia quantities

\[
\begin{align*}
i_1 &= \frac{I_1}{h^2 I_0}, & i_2 &= \frac{I_2}{h^2 I_0}, & i_3 &= \frac{I_3}{h^3 I_0}, & i_4 &= \frac{I_4}{h^4 I_0}, \\
i_6 &= \frac{I_6}{h^6 I_0}
\end{align*}
\]

(3.48c)

Similar to the p-Ritz method in plate theory, assume the nondimensional Ritz trial functions are as follows

\[
\{\varphi_i(\zeta), \xi_i(\zeta), \psi_i(\zeta)\} = \zeta^{(i-1)} \times LS \times RS \quad i = 1, 2, \ldots, N
\]

(3.49a)

\[
\{\varphi_k(\zeta), \xi_k(\zeta), \psi_k(\zeta)\} = \zeta^{(k-1)} \times LS \times RS \quad k = 1, 2, \ldots, N
\]

(3.49b)

where LS and RS are the basic functions consisting of the boundary equations of the beam corresponding to the left edge and right edge, respectively, and are used to control the boundary conditions of the beam (Liew, Wang, Xiang, & Kitipornchai, 1998), which will be detailed in the subsequent Section 3.4.2.

Substituting Eqs. (3.47), (3.48), (3.49) into Eq. (3.47), a generalized eigenvalue equation for the vibration of CNT reinforced beam is derived

\[
([\bar{K}] - \bar{\omega}^2[\bar{M}])\{\ddot{C}\} = [0]
\]

(3.50)
in which $[\bar{K}]$ is the stiffness matrix, $[\bar{M}]$ is the mass matrix and $\{\bar{C}\}$ is the vector of unknown Ritz coefficients, respectively. The natural frequency of the beam $\bar{\omega}$ is of the form of $\bar{\omega} = \omega L^2 \sqrt{I_0/D_m}$. The elements of the symmetric nondimensional stiffness matrix $[\bar{K}]$ and mass matrix $[\bar{M}]$ are listed in Appendix B.

3.3 Nonlinear Free Vibration of FG-CNT Reinforced Beams

In the current nonlinear formulation, the strain-displacement relationship of von Kármán sense is included which takes the mid-plane stretching into account. The resulting nonlinear variational equation cannot be solved directly like the linear cases discussed in the previous sections. When solving by the p-Ritz method, a proper iteration method should be used in order to let the nonlinear results converge.

In addition to that, there is a unique phenomenon in the nonlinear analysis of the FG-CNT composites. In the free vibration analysis, during a full free vibration cycle, the strain energy in the beam at maximum downwards amplitude $w_{max}^-$ should be equal to the strain energy at the maximum upwards amplitude $w_{max}^+$. This is because no energy input or loss occurs during the full vibration cycle. Therefore for UD- and FGX-CNT beams, because of the symmetrical distribution of CNT in the thickness direction, it is obvious that for a given maximum downward vibration amplitude $w_{max}^-$, the value of the maximum upward vibration amplitude $w_{max}^+$ is equal to the value of $w_{max}^-$. However for FGA-CNT beams, due to the asymmetric distribution of CNTs and also the plane $x-y$ lays at the mid-surface of the beam, for a given maximum downward vibration amplitude $w_{max}^-$, the value of the maximum upward vibration amplitude $w_{max}^+$ is not equal to the value of $w_{max}^-$. Therefore the eigenvalues of the downward half vibration cycle are different from the eigenvalues obtained for the upward half vibration cycle. From the strain energy balance point of view, for a full cycle of vibration, we can observe from numerical tests that the value of $w_{max}^-$ is less than the value of $w_{max}^+$ for FGA-CNT beams. This similar phenomenon was also observed for a cracked functionally graded beams by Kitipornchai (Kitipornchai, Ke, Yang, & Xiang, 2009) and Singh and Rao (Singh & Rao, 1998).
The formulation of geometric nonlinear free vibration of FG-CNTs beams is detailed in the following section.

### 3.3.1 Problem Definition

For the above reason, in addition to the three types of beams investigated in Section 3.2, a functionally graded CNT beam type V (FGV-CNT) has been added for further consideration as shown in Figure 3.3.1. It is essentially a mirrored beam of the FGA-CNT beam against $x$-$y$ plane. The coordinates, geometrics and CNT reinforcing angle of the beams are the same as the previous studied ones in Section 3.2.

![Figure 3.3.1 Functionally graded carbon nanotube reinforced composite beams: cross sections of FGV-CNT beams.](image)

As a mirrored beam of the FGA-CNT beam, the distribution shape function of FGV-CNT beam is

$$f(z) = \left(\frac{h + 2z}{2h}\right)^k \quad \left(-\frac{h}{2} \leq z \leq \frac{h}{2}\right)$$

(3.51)

The CNT volume fraction of FGV-CNT beams while $k = 1$ is derived as

$$V_{CNT}(z) = \left(1 + \frac{2z}{h}\right)V_{tcnt} \quad \left(-\frac{h}{2} \leq z \leq \frac{h}{2}\right)$$

(3.52)

### 3.3.2 Formulation

#### 3.3.2.1 Strain-Displacement Relations

Under the displacement field of the third order beam theory, the normal strain $\varepsilon_{xx}$ and shear strain $\gamma_{xz}$ with von Kármán nonlinearity can be derived as follows
\[ \varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \]
\[ = \frac{\partial u_0}{\partial x} + z \frac{\partial \phi_x}{\partial x} - 4z^3 \left( \frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2} \right) + \frac{1}{2} \left( \frac{\partial w_0}{\partial x} \right)^2 \]  (3.53a)
\[ \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \]
\[ = \phi_x + \frac{\partial w_0}{\partial x} - \frac{4z^2}{h^2} \left( \phi_x + \frac{\partial w_0}{\partial x} \right) \]  (3.53b)

### 3.3.2.2 Elastic Constitutive Relations

The force resultants \((N_{xx}, M_{xx})\) and higher order force resultants \((P_{xx}, Q_{xz}, R_{xz})\) are defined as follows (Reddy, 2004b)

\[ N_{xx} = A_{11} \frac{\partial u_0}{\partial x} + B_{11} \frac{\partial \phi_x}{\partial x} - c_1 E_{11} \left( \frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2} \right) \]
\[ + \frac{1}{2} A_{11} \left( \frac{\partial w_0}{\partial x} \right)^2 \]  (3.54a)
\[ M_{xx} = B_{11} \frac{\partial u_0}{\partial x} + D_{11} \frac{\partial \phi_x}{\partial x} - c_1 F_{11} \left( \frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2} \right) \]
\[ + \frac{1}{2} B_{11} \left( \frac{\partial w_0}{\partial x} \right)^2 \]  (3.54b)
\[ P_{xx} = E_{11} \frac{\partial u_0}{\partial x} + F_{11} \frac{\partial \phi_x}{\partial x} - c_1 H_{11} \left( \frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2} \right) + \frac{1}{2} E_{11} \left( \frac{\partial w_0}{\partial x} \right)^2 \]  (3.54c)
\[ Q_{xz} = A_{55} \left( \phi_x + \frac{\partial w_0}{\partial x} \right) - c_2 D_{55} \left( \phi_x + \frac{\partial w_0}{\partial x} \right) \]  (3.54d)
\[ R_{xz} = D_{55} \left( \phi_x + \frac{\partial w_0}{\partial x} \right) - c_2 F_{55} \left( \phi_x + \frac{\partial w_0}{\partial x} \right) \]  (3.54e)

### 3.3.2.3 Variational Energy Formulation

The variational terms of the strains is given as

\[ \delta \varepsilon_{xx} = \frac{\partial \delta u_0}{\partial x} + z \frac{\partial \delta \phi_x}{\partial x} - \frac{4z^3}{3h^2} \left( \frac{\partial \delta \phi_x}{\partial x} + \frac{\partial^2 \delta w_0}{\partial x^2} \right) + \frac{\partial w_0}{\partial x} \frac{\partial \delta w_0}{\partial x} \]  (3.55a)
\[
\delta y_{xz} = \delta \phi_x + \frac{\partial \delta w_0}{\partial x} - \frac{4z^2}{h^2} \left( \delta \phi_x + \frac{\partial \delta w_0}{\partial x} \right)
\]  
(3.55b)

According to the variational method, the nonlinear virtual strain energy \( \delta \mathcal{U} \) at certain time \( t_0 \) is given by Eq. (3.56). The Virtual kinetic energy \( \delta \mathcal{K} \) is the same as the one in Section 3.2.2.4, Eq. (3.45)

\[
\delta \mathcal{U} = \int_{-L}^{L} \left[ N_{xx} \left( \frac{\partial \delta u_0}{\partial x} + \frac{\partial \delta w_0}{\partial x} \right) + M_{xx} \frac{\partial \delta \phi_x}{\partial x} + c_1 P_{xx} \left( \frac{\partial \delta \phi_x}{\partial x} + \frac{\partial^2 \delta w_0}{\partial x^2} \right) + Q_{xx} \frac{\partial \delta \phi_x}{\partial x} \right] \, dx
\]  
(3.56)

invoking the Hamilton’s principle, the nonlinear equation is derived

\[
\delta \mathcal{U} - \delta \mathcal{K} = 0
\]  
(3.57)

In order to solve the nonlinear equation (3.57), a direct iteration approach is adopted. However if the p-Ritz method is directly applied to Eq. (3.57), the resulting stiffness matrix will be nonsymmetrical. For the purpose of utilizing the symmetric eigensolver, the above equation needs to be transformed to symmetric stiffness matrix formulation before the p-Ritz method can be applied. One simple way is to split the product of linear portion of the resultant \( N_{xx} \) and the variational terms of \( w_0 \) into two equal parts, and consider one of the two parts of the linear portion as known values from the previous iteration (Reddy, 2004a).

### 3.3.2.4 p-Ritz Method

The non-dimensional quantities and Ritz procedures are the same as the ones in Section 3.2.2.6. A generalized eigenvalue equation for the nonlinear vibration of CNT reinforced beam is derived as
\[
([\bar{K}] - \bar{\omega}^2 [\bar{M}])\{\bar{C}\} = [0]
\]  
(3.58)

Notice that the nondimensional stiffness matrix \([\bar{K}]\) is comprised of linear and nonlinear matrices

\[
[\bar{K}] = [\bar{K}]_l + [\bar{K}]_{nl}
\]  
(3.59)

The elements of linear stiffness matrix \([\bar{K}]_l\) and the mass matrix \([\bar{M}]\) are the same as the ones listed in APPENDIX B. The elements of nonlinear stiffness matrix \([\bar{K}]_{nl}\) are listed in APPENDIX C.

3.3.2.5 Solution Procedure

A direct iterative approach is adopted in solving the nonlinear eigenvalue equation given in Eq. (3.57). At the beginning of the iteration, assume that the last step solution of the eigenvalue \(\{\bar{\omega}\}_r^r\) and eigenvector \(\{\bar{C}\}_r^r\) are known, either be the solution from linear eigen system or the solution from the last iteration. The eigenvector \(\{\bar{C}\}_r^r\) is scaled up so that the maximum downward vibration amplitude of the transverse displacement \(\bar{w}_0\) is equal to the required downward nonlinear vibration amplitude \(\bar{w}_{max}^-\). The new stiffness matrix \([\bar{K}]_r^{r+1}\) is calculated with the nonlinear terms involving the scaled eigenvector \(\{\bar{C}\}_r^r\). The new eigenvalue \(\{\bar{\omega}\}_r^{r+1}\) and eigenvector \(\{\bar{C}\}_r^{r+1}\) are then solved based on the updated eigenvalue system. The above iteration process is repeated until the relative difference between the eigenvalues \(\{\bar{\omega}\}_r^{r+1}\) and \(\{\bar{\omega}\}_r^r\) is less than 0.1%. As mentioned earlier, for UD- and FGX-CNT beams, because of the symmetrical distribution of CNT in the thickness direction, the converged eigenvalues are the final nonlinear frequency parameter.

For FGA- and FGV-CNT beams, an extra step needs to be implemented. Once the downward vibration mode shape corresponding to the maximum downward vibration amplitude \(\bar{w}_{max}^-\) is determined, the amplitude \(\bar{w}_{max}^+\) and eigenvalues for the upward vibration half cycle \(\bar{\omega}^+\) should be calculated based on the strain energy balance of the system, in the current study, the iteration convergence criterion is set to be less that
0.05%. The strain energy can be obtained as follows,

\[ \mathcal{U} = \{\tilde{C}\}^T [\tilde{K}] \{\tilde{C}\} \quad (3.60) \]

The final nonlinear full cycle frequency parameter can be derived as follows

\[ \bar{\omega} = \frac{2\pi}{\bar{\omega}^+ + \bar{\omega}^-} = \frac{2\omega^+ + \omega^-}{\omega^+ + \omega^-} \quad (3.61) \]

For detailed solving procedure one can also refer to Ke et al. (Ke, et al., 2010)

It is worth pointing out that by using the direct iteration procedure, the above computation is quite time consuming, especially when the nonlinear amplitudes are relatively large. In addition, the convergences become difficult to attain when beam vibrating in higher amplitudes. To overcome this problem, a combined acceleration technique has been developed and adopted in the current study. The computation domain of nonlinear amplitude is discretized according to the Chebyshev points of the first kind. Then the fixed point iteration with the Steffensen's method (Farnum, 1991) has been adopted to accelerate the speed and chance of convergence. The energy balance iteration was based on the Newton-Raphson method.

### 3.4 Comparison of the Third Order and First Order Theories

#### 3.4.1 Displacements Fields

The displacements of the first order shear deformation theory are as follows.

The displacements for plates

\[ u(x, y, z, t) = u_0(x, y, t) + z\phi_x(x, y, t) \quad (3.62a) \]
\[ v(x, y, z, t) = v_0(x, y, t) + z\phi_y(x, y, t) \quad (3.62b) \]
\[ w(x, y, z, t) = w_0(x, y, t) \quad (3.62c) \]
and the displacements for beams

\[ u(x, z, t) = u_0(x, t) + z\phi(x, t) \quad (3.63a) \]
\[ w(x, z, t) = w_0(x, t) \quad (3.63b) \]

Comparing the displacement field of the third order theory and the first order theory, one can conclude that the third order theory can be reduced to the first order theory by setting the third order stiffness and inertia terms to zero, meanwhile accounting for the shear correction factor \( \kappa \). i.e. for beams, the formulation of the first order theory can be obtained by setting

\[ D_{55} = E_{11} = F_{55} = F_{11} = H_{11} = 0 \quad (3.64a) \]
\[ l_3 = l_4 = l_6 = 0 \quad (3.64b) \]

and

\[ A_{55} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \kappa \bar{Q}_{55} dz \quad (3.65) \]

For plates, the formulation of first order theory can be obtained by setting

\[ E_{mn} = F_{mn} = H_{mn} = 0, \quad (m, n = 1, 2, 6) \quad (3.66a) \]
\[ D_{mn} = F_{mn} = 0, \quad (m, n = 4, 5) \quad (3.66b) \]
\[ l_3 = l_4 = l_6 = 0 \quad (3.66c) \]

and

\[ A_{mn} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \kappa \bar{Q}_{mn} dz \quad (m, n = 4, 5). \quad (3.67) \]
3.4.2 Discussion of Boundary Conditions

In the p-Ritz method, only the essential boundary conditions need to be enforced while the natural boundary conditions will be approached when the number of trial function terms increases. Therefore, it is essential in the Ritz method to examine the convergence of the solutions with respect to the number of trial function terms.

The following types of essential boundary conditions are considered in the current study for plates:

**Hard Clamped (HC)**

\[ U_N = U_T = W_0 = \phi_N = \phi_T = \frac{\partial W_0}{\partial N} = 0 \]

**Soft Clamped (SC)**

\[ U_N = U_T = W_0 = \phi_N = \phi_T = 0 \]

**Pinned (P)**

\[ U_N = U_T = W_0 = 0 \]

**Roller (R)**

\[ U_T = W_0 = \phi_T = 0 \]

**Free (F)**

no constraint on \( U, W \) and \( \phi \)

and the corresponding natural boundary conditions are:

**Hard Clamped (HC)**

no constraint on resultants

**Soft Clamped (SC)**

\[ P_{NN} = 0 \]

**Pinned (P)**

\[ P_{NN} = M_{NN} = M_{NT} = 0 \]

**Roller (R)**

\[ N_{NN} = M_{NN} = P_{NN} = 0 \]

**Free (F)**

\[ N_{NN} = N_{NT} = \bar{V}_N = 0 \]

\[ M_{NN} = M_{NT} = P_{NN} = 0 \]

where \( \bar{V}_N \) is the equivalent shear stress resultant which detailed by Reddy (Reddy, 2004). The subscript \( N \) denotes the normal direction and \( T \) denotes the tangential direction to the edge of the plates. For example, if the left side of the rectangular plate is hard-clamped (HC), then the following restraints on the nondimensional displacements will be applied: \( U_0 = V_0 = W_0 = \phi_\zeta = \phi_\eta = \frac{\partial W_0}{\partial \zeta} = 0 \). If the left side is Pinned (P), then: \( U_0 = V_0 = W_0 = 0, \; M_{\zeta\zeta} = M_{\zeta\eta} = M_{\eta\eta} = 0 \).

For beams the following types of essential boundary conditions are considered:
Hard Clamped (HC) \[ U_N = W_0 = \Phi_N = \frac{\partial W_0}{\partial N} = 0 \]

Soft Clamped (SC) \[ U_N = W_0 = \Phi_N = 0 \]

Pinned (P) \[ U_N = W_0 = 0 \]

Roller (R) \[ W_0 = 0 \]

Free (F) \[ \text{no constraint on } U, W \text{ and } \Phi \]

and the corresponding natural boundary conditions are:

Hard Clamped (HC) \[ \text{no constraint on resultants} \]

Soft Clamped (SC) \[ P_{NN} = 0 \]

Pinned (P) \[ M_{NN} = P_{NN} = 0 \]

Roller (R) \[ N_{NN} = M_{NN} = P_{NN} = 0 \]

Free (F) \[ N_{NN} = M_{NN} = P_{NN} = Q_{NZ} = 0 \]

For the beams, the basic functions \( LS \) and \( RS \) of the above boundary conditions are listed below in Table 3.4.1. For the plates, the basic functions \( LS, RS, TS \) and \( BS \) of the above boundary conditions are listed in Table 3.4.2 and Table 3.4.3.

Note that if the soft-clamped (SC) boundary condition is applied, in the first order shear deformation theory, the nondimensional displacements \( U, V, W \) of plate at the boundaries are zero. However, in the third order theory, the nondimensional displacements \( U, V, W \) at the boundaries equal to the following expression, where \( \epsilon \) is the nondimensional variable in the thickness direction, which corresponds to the dimensional variable \( z \).

\[
U = -\frac{4}{3} \epsilon^3 \frac{\partial W_0}{\partial \zeta} \quad (3.68a)
\]

\[
V = -\frac{4}{3} \epsilon^3 \frac{\partial W_0}{\partial \eta} \quad (3.68b)
\]

\[
W = 0 \quad (3.68c)
\]

Due the nonzero of \( \frac{\partial W_0}{\partial \zeta} \) and \( \frac{\partial W_0}{\partial \eta} \) at the vicinity of boundary when the edge is soft-clamped (SC), it is expected that there are discrepancies between the results.
obtained using the first order and the third order theories, and they have been observed and discussed in the following parametric studies. From the physical point of view, it is expected that the third order theory together with the soft-clamped (SC) boundary condition will be able to give a rudimental estimation when the boundary condition is not ideal, for example, the degradation of hard-clamp itself and tendency of rotating against the edge when the plate/beam vibrates.

Table 3.4.1 The Ritz boundary basic functions LS and RS for beams

<table>
<thead>
<tr>
<th>Displacements</th>
<th>LS</th>
<th>RS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hard-Clamped (HC): ( U_0 = )</td>
<td>( W_0 = \Phi_\zeta = \frac{\partial W_0}{\partial \zeta} = 0 )</td>
<td></td>
</tr>
<tr>
<td>( U_0 )</td>
<td>( \zeta + 1/2 )</td>
<td>( \zeta - 1/2 )</td>
</tr>
<tr>
<td>( W_0 )</td>
<td>( (\zeta + 1/2)^2 )</td>
<td>( (\zeta - 1/2)^2 )</td>
</tr>
<tr>
<td>( \Phi_\zeta )</td>
<td>( \zeta + 1/2 )</td>
<td>( \zeta - 1/2 )</td>
</tr>
<tr>
<td>Soft-Clamped (SC): ( U_0 = )</td>
<td>( W_0 = \Phi_\zeta = 0 )</td>
<td></td>
</tr>
<tr>
<td>( U_0 )</td>
<td>( \zeta + 1/2 )</td>
<td>( \zeta - 1/2 )</td>
</tr>
<tr>
<td>( W_0 )</td>
<td>( \zeta + 1/2 )</td>
<td>( \zeta - 1/2 )</td>
</tr>
<tr>
<td>( \Phi_\zeta )</td>
<td>( \zeta + 1/2 )</td>
<td>( \zeta - 1/2 )</td>
</tr>
<tr>
<td>Pinned (P): ( U_0 = W_0 = 0 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( U_0 )</td>
<td>( \zeta + 1/2 )</td>
<td>( \zeta - 1/2 )</td>
</tr>
<tr>
<td>( W_0 )</td>
<td>( \zeta + 1/2 )</td>
<td>( \zeta - 1/2 )</td>
</tr>
<tr>
<td>( \Phi_\zeta )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Roller (R): ( W_0 = 0 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( U_0 )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( W_0 )</td>
<td>( \zeta + 1/2 )</td>
<td>( \zeta - 1/2 )</td>
</tr>
<tr>
<td>( \Phi_\zeta )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Free (F): no constraint</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( U_0 )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( W_0 )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \Phi_\zeta )</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Table 3.4.2 The Ritz boundary basic functions $LS$ and $RS$ for plates

<table>
<thead>
<tr>
<th>Displacement</th>
<th>$LS$</th>
<th>$RS$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Hard-Clamped (HC):</strong></td>
<td>$U_0 = V_0 = W_0 = \phi_\zeta = \phi_\eta = \frac{\partial w_0}{\partial \zeta} = 0$</td>
<td></td>
</tr>
<tr>
<td>$U_0$</td>
<td>$\zeta + 1/2$</td>
<td>$\zeta - 1/2$</td>
</tr>
<tr>
<td>$V_0$</td>
<td>$\zeta + 1/2$</td>
<td>$\zeta - 1/2$</td>
</tr>
<tr>
<td>$W_0$</td>
<td>$(\zeta + 1/2)^2$</td>
<td>$(\zeta - 1/2)^2$</td>
</tr>
<tr>
<td>$\phi_\zeta$</td>
<td>$\zeta + 1/2$</td>
<td>$\zeta - 1/2$</td>
</tr>
<tr>
<td>$\phi_\eta$</td>
<td>$\zeta + 1/2$</td>
<td>$\zeta - 1/2$</td>
</tr>
<tr>
<td><strong>Soft-Clamped (SC):</strong></td>
<td>$U_0 = V_0 = W_0 = \phi_\zeta = \phi_\eta = 0$</td>
<td></td>
</tr>
<tr>
<td>$U_0$</td>
<td>$\zeta + 1/2$</td>
<td>$\zeta - 1/2$</td>
</tr>
<tr>
<td>$V_0$</td>
<td>$\zeta + 1/2$</td>
<td>$\zeta - 1/2$</td>
</tr>
<tr>
<td>$W_0$</td>
<td>$\zeta + 1/2$</td>
<td>$\zeta - 1/2$</td>
</tr>
<tr>
<td>$\phi_\zeta$</td>
<td>$\zeta + 1/2$</td>
<td>$\zeta - 1/2$</td>
</tr>
<tr>
<td>$\phi_\eta$</td>
<td>$\zeta + 1/2$</td>
<td>$\zeta - 1/2$</td>
</tr>
<tr>
<td><strong>Pinned (P):</strong></td>
<td>$U_0 = V_0 = W_0 = 0$</td>
<td></td>
</tr>
<tr>
<td>$U_0$</td>
<td>$\zeta + 1/2$</td>
<td>$\zeta - 1/2$</td>
</tr>
<tr>
<td>$V_0$</td>
<td>$\zeta + 1/2$</td>
<td>$\zeta - 1/2$</td>
</tr>
<tr>
<td>$W_0$</td>
<td>$\zeta + 1/2$</td>
<td>$\zeta + 1/2$</td>
</tr>
<tr>
<td>$\phi_\zeta$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\phi_\eta$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td><strong>Roller (R):</strong></td>
<td>$V_0 = W_0 = \phi_\eta = 0$</td>
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</tr>
<tr>
<td>$U_0$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$V_0$</td>
<td>$\zeta + 1/2$</td>
<td>$\zeta - 1/2$</td>
</tr>
<tr>
<td>$W_0$</td>
<td>$\zeta + 1/2$</td>
<td>$\zeta - 1/2$</td>
</tr>
<tr>
<td>$\phi_\zeta$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\phi_\eta$</td>
<td>$\zeta + 1/2$</td>
<td>$\zeta - 1/2$</td>
</tr>
<tr>
<td><strong>Free (F):</strong></td>
<td><em>no constraint</em></td>
<td></td>
</tr>
<tr>
<td>$U_0$</td>
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<td>1</td>
</tr>
<tr>
<td>$V_0$</td>
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</tr>
<tr>
<td>$W_0$</td>
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</tr>
<tr>
<td>$\phi_\zeta$</td>
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</tr>
<tr>
<td>$\phi_\eta$</td>
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</table>
Table 3.4.3 The Ritz boundary basic functions $TS$ and $BS$ for plates

<table>
<thead>
<tr>
<th>Displacement</th>
<th>BS</th>
<th>TS</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Hard-Clamped (HC):</strong></td>
<td>$U_0 = V_0 = W_0 = \Phi_\zeta = \Phi_\eta = \frac{\partial W_0}{\partial \eta} = 0$</td>
<td></td>
</tr>
<tr>
<td>$U_0$</td>
<td>$\eta + 1/2$</td>
<td>$\eta - 1/2$</td>
</tr>
<tr>
<td>$V_0$</td>
<td>$\eta + 1/2$</td>
<td>$\eta - 1/2$</td>
</tr>
<tr>
<td>$W_0$</td>
<td>$(\eta + 1/2)^2$</td>
<td>$(\eta - 1/2)^2$</td>
</tr>
<tr>
<td>$\Phi_\zeta$</td>
<td>$\eta + 1/2$</td>
<td>$\eta - 1/2$</td>
</tr>
<tr>
<td>$\Phi_\eta$</td>
<td>$\eta + 1/2$</td>
<td>$\eta - 1/2$</td>
</tr>
<tr>
<td><strong>Soft-Clamped (SC):</strong></td>
<td>$U_0 = V_0 = W_0 = \Phi_\zeta = \Phi_\eta = 0$</td>
<td></td>
</tr>
<tr>
<td>$U_0$</td>
<td>$\eta + 1/2$</td>
<td>$\eta - 1/2$</td>
</tr>
<tr>
<td>$V_0$</td>
<td>$\eta + 1/2$</td>
<td>$\eta - 1/2$</td>
</tr>
<tr>
<td>$W_0$</td>
<td>$\eta + 1/2$</td>
<td>$\eta - 1/2$</td>
</tr>
<tr>
<td>$\Phi_\zeta$</td>
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<td>$\eta - 1/2$</td>
</tr>
<tr>
<td>$\Phi_\eta$</td>
<td>$\eta + 1/2$</td>
<td>$\eta - 1/2$</td>
</tr>
<tr>
<td><strong>Pinned (P):</strong></td>
<td>$U_0 = V_0 = W_0 = 0$</td>
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</tr>
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<td>$U_0$</td>
<td>$\eta + 1/2$</td>
<td>$\eta - 1/2$</td>
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<tr>
<td>$V_0$</td>
<td>$\eta + 1/2$</td>
<td>$\eta - 1/2$</td>
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<tr>
<td>$W_0$</td>
<td>$\eta + 1/2$</td>
<td>$\eta - 1/2$</td>
</tr>
<tr>
<td>$\Phi_\zeta$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\Phi_\eta$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td><strong>Roller (R):</strong></td>
<td>$U_0 = W_0 = \Phi_\zeta = 0$</td>
<td></td>
</tr>
<tr>
<td>$U_0$</td>
<td>$\eta + 1/2$</td>
<td>$\eta - 1/2$</td>
</tr>
<tr>
<td>$V_0$</td>
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<td>1</td>
</tr>
<tr>
<td>$W_0$</td>
<td>$\eta + 1/2$</td>
<td>$\eta - 1/2$</td>
</tr>
<tr>
<td>$\Phi_\zeta$</td>
<td>$\eta + 1/2$</td>
<td>$\eta - 1/2$</td>
</tr>
<tr>
<td>$\Phi_\eta$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td><strong>Free (F):</strong></td>
<td><em>no constraint</em></td>
<td></td>
</tr>
<tr>
<td>$U_0$</td>
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<td>1</td>
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<tr>
<td>$V_0$</td>
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<tr>
<td>$W_0$</td>
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</tr>
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<td>$\Phi_\zeta$</td>
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</tr>
<tr>
<td>$\Phi_\eta$</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
CHAPTER 4  RESULTS AND DISCUSSION

4.1 Material properties
The following material constants for (10, 10) SWCNT and PMMA matrix are used in this study unless otherwise stated (Shen & Xiang, 2012): $E_{11}^{cnt} = 5646.6$ GPa, $E_{22}^{cnt} = 7080$ GPa, $G_{12}^{cnt} = 1944.5$ GPa, $E^m = 2.5$ GPa, $v_{12}^{cnt} = 0.175$, $v^m = 0.3$, $\rho^{cnt} = 2100$ kg/m$^3$, and $\rho^m = 1190$ kg/m$^3$, respectively. By matching with the results from molecular dynamics simulations (Griebel & Hamaekers, 2004; Han & Elliott, 2007), the CNT efficiency parameters are given in Table 1 (Shen & Xiang, 2012). Note that the SWCNT density is taken as 2100 kg/m$^3$ in this study (Lu et al., 2006) which is different from the one adopted by Shen (Shen & Xiang, 2012) with SWCNT density being 1400 kg/m$^3$.

By matching with the results from MD simulations (Griebel & Hamaekers, 2004; Han & Elliott, 2007), the CNT efficiency parameters are given in Table 4.1.1.

<table>
<thead>
<tr>
<th>CNT efficiency parameters</th>
<th>$V_{tcnt}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.12</td>
</tr>
<tr>
<td>$\eta_1$</td>
<td>0.137</td>
</tr>
<tr>
<td>$\eta_2$</td>
<td>1.022</td>
</tr>
<tr>
<td>$\eta_3$</td>
<td>0.715</td>
</tr>
</tbody>
</table>

4.2 Linear Free Vibration of Plates

4.2.1 Convergence and Comparison Studies
A convergence study is carried out to determine the number of polynomial terms $p_s$ required in the computations of FG-CNT plates. The study is performed on a FGA-CNT square plate with all sides hard-clamped (HC-HC-HC-HC). The length to thickness ratio $L/h$ is 10, total CNT volume fraction $V_{tcnt}$ is 0.17 with CNT reinforcing angle $\theta = 0^\circ$. The frequency parameters $\bar{\omega} = \omega L^2 \sqrt{\rho_m/(E_m h^2)}$ of the first ten vibration modes are given in Table 4.2.1. It is found that the degree set of 60
two-dimensional complete polynomial space and more are needed to produce relatively well converged frequencies for the FG-CNT plates. However, the computation time for \( p_s \) of more than 7 degree set are significantly longer. Hence, in the current study, the degree set 7 is used for all subsequent computations.

<table>
<thead>
<tr>
<th>Mode sequence</th>
<th>( p_s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>21.8096</td>
</tr>
<tr>
<td>2</td>
<td>29.3804</td>
</tr>
<tr>
<td>3</td>
<td>43.3817</td>
</tr>
<tr>
<td>4</td>
<td>45.1138</td>
</tr>
<tr>
<td>5</td>
<td>48.1987</td>
</tr>
<tr>
<td>6</td>
<td>49.3046</td>
</tr>
<tr>
<td>7</td>
<td>60.5059</td>
</tr>
<tr>
<td>8</td>
<td>63.9210</td>
</tr>
<tr>
<td>9</td>
<td>65.7425</td>
</tr>
<tr>
<td>10</td>
<td>69.8700</td>
</tr>
</tbody>
</table>

It is worth pointing out that although a series with higher complete polynomial terms can generally produce convergent solutions, a series with excessive Ritz polynomial terms results in larger but often ill-conditioned matrix and gives rise to numerical instability. This is mainly due to digits truncation of the computer and was previously observed in other publications (Huang & Leissa, 2009; Qatu & Leissa, 1991). Using computers with higher precision such as the 64-bit or 128-bit computers may alleviate the numerical instability. However, in order to solve higher frequency vibration problems, other computation methods may be used, such as the spectral method.

Table 4.2.2 plots the eight flexure mode shapes among the above first ten modes when \( p_s = 7 \), \( m \) and \( n \) denotes the half-wave numbers in the \( x (\zeta) \) and \( y (\eta) \) direction. Notice that the 6th and the 9th modes are the transverse shear dominated vibration.
modes, therefore are not given in Table 4.2.2. As the material property is orthotropic, in comparison with the isotropic material, it is observed that the contour line of the fundamental mode shape is oval instead of circle. Also it can be seen that the sequences of the mode shape is different from the ones for isotropic square plate, such as the mode shape (1,3) appears after mode (1,2) and before mode (2,1), and the mode shape (1,4) and (3,1) appear last in the listed results.

Table 4.2.2 Flexural mode shapes of square FGA-CNT plate (HC-HC-HC-HC, \(L/h = 10, \ V_{\text{cnt}} = 0.17, \ \theta = 0^\circ\))

<table>
<thead>
<tr>
<th>((m,n))</th>
<th>(\bar{\omega})</th>
<th>Mode Shape</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,1)</td>
<td>21.7012</td>
<td><img src="image1" alt="Mode Shape Image" /></td>
</tr>
<tr>
<td>(1,2)</td>
<td>29.1750</td>
<td><img src="image2" alt="Mode Shape Image" /></td>
</tr>
</tbody>
</table>
(1,3)  43.1178

(2,1)  43.1850

(2,2)  47.7105
(2,3) 57.8185

(1,4) 61.4191

(3,1) 68.9367
To verify the current formulation, the fundamental frequency parameters $\bar{\omega} = \omega L^2 \sqrt{\rho/(E_2 h^2)}$ of laminated cross-ply are presented in Table 4.2.3 together with the results from Reddy (Reddy, 2004b). The study is performed on a laminated square cross-ply plate ($0^\circ/90^\circ/90^\circ/0^\circ$) with all sides roller supported (R-R-R-R). The present results agree well with the published results, except that there is a typo in the published results.

Table 4.2.3 Comparison of frequency parameters of cross-ply plate ($0^\circ/90^\circ/90^\circ/0^\circ$) with existing results of different theories

<table>
<thead>
<tr>
<th>$E_1/E_2$</th>
<th>$L/h$</th>
<th>Present</th>
<th></th>
<th>Ref (Reddy, 2004b)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>3rd order</td>
<td>1st order</td>
<td>3rd order</td>
<td>1st order</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>6.560</td>
<td>6.569</td>
<td>6.560</td>
<td>6.570</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>7.243</td>
<td>7.247</td>
<td>7.243</td>
<td>7.247</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>8.272</td>
<td>8.298</td>
<td>8.272</td>
<td>8.298</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>12.218</td>
<td>12.239</td>
<td>12.218</td>
<td>12.238</td>
</tr>
</tbody>
</table>

As shown in Table 4.2.4, the first six frequency parameters $\bar{\omega} = \omega L^2 \sqrt{\rho/(E_m h^2)}$ of FGV-CNT and FGX-CNT plates are compared with the results from the existing paper (Zhu, et al., 2012) based on the first order shear deformation theory. The material properties in the compared literature are adopted for comparison studies, and the boundary conditions are equivalent to roller supports at all sides (R-R-R-R). The studied FGV-CNT square plate has the length to thickness ratio $L/h = 50$, total CNT volume fraction $V_{tcnt} = 0.14$ with CNT reinforcing angle $\theta = 0^\circ$. The shear correction factor for the first order theory in the present study is taken as 5/6. The current results match well against the compared results.

The above convergence and comparison studies verify the correctness and accuracy of the current formulation and solution procedure which will be applied to obtain results presented in the following sections.
Table 4.2.4 Comparison of frequency parameters of FG-CNT square plate (R-R-R-R,  $L/h = 50$, $V_{cnt} = 0.14$, $\theta = 0^\circ$)

<table>
<thead>
<tr>
<th>Modes (m,n)</th>
<th>FGV-CNT</th>
<th>FGX-CNT</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,1)</td>
<td>17.965</td>
<td>17.995</td>
</tr>
<tr>
<td>(1,2)</td>
<td>22.539</td>
<td>22.643</td>
</tr>
<tr>
<td>(1,3)</td>
<td>34.046</td>
<td>34.660</td>
</tr>
<tr>
<td>(1,4)</td>
<td>54.530</td>
<td>54.833</td>
</tr>
<tr>
<td>(2,1)</td>
<td>65.897</td>
<td>66.552</td>
</tr>
<tr>
<td>(2,2)</td>
<td>68.296</td>
<td>68.940</td>
</tr>
</tbody>
</table>

4.2.2 Analysis of First and Third Order Plate Solutions

Table 4.2.5 lists the comparison of fundamental frequency parameters $\bar{\omega} = \omega b^2 \sqrt{p_m/(E_m h^2)}$ of CNT plates with different end support conditions between the first order and the third order plate theories. The beam $L/h$ ratio is set to be 10 and the total CNT volume fraction is 0.17. It is noticed that the first order plate results are substantially larger than the corresponding third order results by up to 16%, 15% and 17% for UD-CNT, FGA-CNT and FGX-CNT plates with SC-SC-SC-SC boundary conditions, respectively. On the other hand, the frequency parameters for HC-HC-HC-HC CNT plates based on the first order plate theory are smaller than the corresponding ones based on the third order plate theory by 3.5%, 2.8% and 3.5% for plates with UD-, FGA- and FGX- CNT distributions. The differences are negligible for plates with P-P-P-P support conditions. In order to further investigate the aforementioned differences when the SC-SC-SC-SC boundary condition is imposed, the fundamental mode shapes based on both theories have been plotted in Figure 4.2.1 and Figure 4.2.2. Figure 4.2.1 illustrates the cross sections of the mode shape based on the third order theory along (a) the $y$ axis and (b) $x$ axis of the FGA-CNT plate. Figure 4.2.2 shows the cross sections of the mode shape at the same position based on the first order theory. It can be seen that the mode shape of the first order theory at both $x$ and $y$ directions are slightly slender than that of the third order theory. Furthermore, comparing to the mode shape obtained using the third order theory, the slope based on the first order theory varies more severely at the vicinity of the boundary area (see Figure 4.2.1 (b) and Figure 4.2.2 (b)). The mode shape differences

66
between the first order and the third order theories may attribute to the differences of
the in-plane displacements at the boundary area which are mentioned in Section
3.4.2.

Table 4.2.5 Comparison of fundamental frequency parameters of square CNT composite
plates using different plate theories

<table>
<thead>
<tr>
<th></th>
<th>3rd order</th>
<th>1st order</th>
<th>( \frac{\omega_{3\text{rd}}}{\omega_{1\text{st}}} - 1 \times 100 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>UD</td>
<td>P-P-P-P</td>
<td>15.5132</td>
<td>15.4805</td>
</tr>
<tr>
<td>HC</td>
<td>HC-HC-HC-HC</td>
<td>22.5250</td>
<td>21.7474</td>
</tr>
<tr>
<td>FGA</td>
<td>P-P-P-P</td>
<td>15.2231</td>
<td>15.2173</td>
</tr>
<tr>
<td>HC</td>
<td>HC-HC-HC-HC</td>
<td>21.7012</td>
<td>21.0867</td>
</tr>
<tr>
<td>FGX</td>
<td>P-P-P-P</td>
<td>16.9996</td>
<td>17.0611</td>
</tr>
<tr>
<td>SC</td>
<td>SC-SC-SC-SC</td>
<td>19.0766</td>
<td>22.3360</td>
</tr>
<tr>
<td>HC</td>
<td>HC-HC-HC-HC</td>
<td>23.4574</td>
<td>22.6419</td>
</tr>
</tbody>
</table>

Figure 4.2.1 Normalized fundamental flexural mode shape cross sections along (a) y axis and
(b) x axis based on third order shear deformation theory of the FGA-CNT plate with
SC-SC-SC-SC boundary condition
Figure 4.2.2 Normalized fundamental flexural mode shape cross sections along (a) y axis and 
(b) x axis based on first order shear deformation theory of the FGA-CNT plate with 
SC-SC-SC-SC boundary condition

4.2.3 Parametric Studies

The following parametric studies are all based on the third order plate theory. Figure 
4.2.3 presents the relationship between the fundamental frequency parameters 
\( \bar{\omega} = \omega b^2 \sqrt{\rho_m/(E_m h^2)} \) against the plate thickness ratio \( h/b \) for square CNT 
composite plates. The plates have the same total CNT volume fraction \( V_{tcnt} = 0.12 \),
different CNT distributions and support conditions. As expected, the frequency 
parameters decrease as the plate thickness ratio \( h/b \) increases, and the frequency 
parameters increase when the plate support conditions change from pin to 
soft-clamped to hard-clamped. It is also observed that the frequency parameters of the 
plates with FGX distribution have the largest values, followed by the ones with UD 
distribution, while the plates with FGA distribution always have the smallest 
frequency parameters.

Figure 4.2.4 shows the relationship between the fundamental frequency parameters 
\( \bar{\omega} = \omega b^2 \sqrt{\rho_m/(E_m h^2)} \) against the plate aspect ratio \( L/b \) for rectangular CNT 
composite plates. The total CNT volume fraction \( V_{tcnt} = 0.12 \), \( b/h = 20 \) are used for 
all plates with different CNT distributions and support conditions. It is observed that 
for all plates in Figure 4.2.4, the fundamental frequency parameters 
\( \bar{\omega} = \omega b^2 \sqrt{\rho_m/(E_m h^2)} \) decrease as the plate aspect ratio \( L/b \) varies from 0.5 to 3. 
Again, plates with FGX distribution have the largest frequency parameters, followed
by plates with UD distribution and FG Λ distribution. The influence of the CNT distribution diminishes when the plate aspect ratio becomes large. It is also observed that the differences of the soft-clamped and hard-clamped conditions are negligible as the plate aspect ratio approaches 3.

Table 4.2.6 presents the effect of CNT reinforcing angle $\theta$ on the fundamental frequency parameters $\bar{\omega} = \omega b^2 \sqrt{\rho_m / (E_m h^2)}$ for square CNT composite plates with total CNT volume fraction $V_{t\text{cnt}} = 0.28$. The plate thickness ratio $h/b$ is set to be 0.1 and the plate is either pin supported, soft-clamped or hard-clamped on all edges. It is again observed that plates with FGX distribution have the largest frequency parameters for all angles in Table 4.2.6, followed by plates with UD and FG Λ distributions.

An interesting phenomenon can be seen from Table 4.2.6. The fundamental frequency parameters based on the third order theory decrease when the angle $\theta$ varies from 0° to 45° except for the case of SC-SC-SC-SC boundary, where the fundamental frequency parameters for the plates with CNT reinforcing angle of 45° are actually larger than ones with the angle of 30°. However, if the first order theory is used as shown in Table 4.2.7 for the case of SC-SC-SC-SC boundary, no such observation can be made. The fundamental frequency parameters $\bar{\omega} = \omega b^2 \sqrt{\rho_m / (E_m h^2)}$ for all plates uniformly decrease with the increase of angle $\theta$, and reach minimum at the angle of 45°. Again, this may be caused by the discrepancy of theory differences as detailed in Section 3.4.2.
Figure 4.2.3 Fundamental frequency parameters versus plate thickness ratio $h/b$ for CNT composite plates with various end support conditions and CNT distributions.

Figure 4.2.4 Fundamental frequency parameters versus plate aspect ratio $L/b$ for CNT composite plates with various support conditions and CNT distributions.
Table 4.2.6 Fundamental frequency parameters of square CNT composite plates with different CNT reinforcing angles

<table>
<thead>
<tr>
<th>CNT Distribution</th>
<th>Boundary Conditions</th>
<th>CNT Reinforcing Angle ( \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0°</td>
</tr>
<tr>
<td>FGX</td>
<td>P-P-P-P</td>
<td>18.4766</td>
</tr>
<tr>
<td>UD</td>
<td>P-P-P-P</td>
<td>17.2635</td>
</tr>
<tr>
<td>FGA</td>
<td>P-P-P-P</td>
<td>17.0973</td>
</tr>
<tr>
<td>FGX</td>
<td>HC-HC-HC-HC</td>
<td>24.6550</td>
</tr>
<tr>
<td>UD</td>
<td>HC-HC-HC-HC</td>
<td>23.9735</td>
</tr>
<tr>
<td>FGA</td>
<td>HC-HC-HC-HC</td>
<td>23.4523</td>
</tr>
</tbody>
</table>

Table 4.2.7 Fundamental frequency parameters of square CNT composite plates of SC-SC-SC-SC boundary conditions based on first order theory

<table>
<thead>
<tr>
<th>CNT Distribution</th>
<th>Boundary Conditions</th>
<th>CNT Reinforcing Angle ( \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0°</td>
</tr>
<tr>
<td>FGX</td>
<td>SC-SC-SC-SC</td>
<td>23.8251</td>
</tr>
</tbody>
</table>

4.3 Linear Free Vibration of Beams

4.3.1 Convergence and Comparison Studies

To validate the accuracy of the Ritz method and determine the number of Ritz polynomial terms \( N \) used in the computations, a convergence study is carried out for the vibration of a FGA-CNT composite beam with hard-clamped conditions at both ends (HC-HC). The \( L/h \) ratio is taken to be 10 and the total CNT volume fraction \( V_{cnt} \) is 0.12. The frequency parameter \( \bar{\omega} = \omega L^2 \sqrt{\rho_m/(E_m h^2)} \) is used in Figure 4.3.1 and \( \bar{\omega}_{15} \) is the corresponding frequency parameter with the number of Ritz polynomial terms \( N = 15 \). It is observed that the frequency parameters converge
rapidly when the number of Ritz terms increases. For the first six vibration modes, the results converge when the number of Ritz terms $N$ is greater than 13. We choose $N = 15$ for all subsequent computations. Figure 4.3.2 plots the first five frequency mode shapes of this FGΛ-CNT beam when $N = 15$ which shows that the p-Ritz solution approach can produce correct mode shapes for the beam.

Figure 4.3.1 Convergence of frequency parameters for a FGΛ-CNT composite beam

Figure 4.3.2 The first five frequency mode shapes of FGΛ-CNT beam with Ritz terms $N = 15$

(HC-HC, $L/h = 10$, $V_{tcnt} = 0.12$)
Comparison studies are performed against existing results to validate the correctness of the current method. As mentioned earlier, the third order shear deformation theory can be reduced to first order shear deformation theory by setting the high order stiffness and inertia components to zero, meanwhile accounting for the shear correct factor, which is 5/6 in the present study. In order to compare with the existing results, different nondimensional parameters and material properties have been adopted in accordance with the published results.

To verify the current solutions based on the third order beam theory, the fundamental frequency parameters $\bar{\omega} = \omega l^2 \sqrt{\rho_m/(E_m h^2)}$ of Al-Al$_2$O$_3$ functionally graded beams are presented in Table 4.3.1 and Table 4.3.2 together with the solutions by Simsek (Simsek, 2010). The FG power-law exponent $k$ is taken be to 0 to 10, the beam $L/h$ ratio is set to be 20 and the boundary conditions of the beams are HC-HC and HC-F, respectively. Excellent agreement between the current first and third order beam solutions and the ones by Simsek (Simsek, 2010) is observed.

Table 4.3.1 Comparisons of the fundamental frequency parameters with the power-law exponent for FG Al-Al$_2$O$_3$ beam (HC-HC, $L/h = 20$)

<table>
<thead>
<tr>
<th>Theory</th>
<th>Present study</th>
<th>Ref. (Simsek, 2010)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1$^{\text{st}}$ order</td>
<td>3$^{\text{rd}}$ order</td>
</tr>
<tr>
<td>$k=0$</td>
<td>12.22211</td>
<td>12.22272</td>
</tr>
<tr>
<td>$k=0.2$</td>
<td>11.38093</td>
<td>11.38349</td>
</tr>
<tr>
<td>$k=0.5$</td>
<td>10.42427</td>
<td>10.42687</td>
</tr>
<tr>
<td>$k=2$</td>
<td>8.60318</td>
<td>8.59683</td>
</tr>
<tr>
<td>$k=5$</td>
<td>8.16887</td>
<td>8.14372</td>
</tr>
<tr>
<td>$k=10$</td>
<td>7.91163</td>
<td>7.88478</td>
</tr>
</tbody>
</table>
Table 4.3.2 Comparisons of the fundamental frequency parameters with the power-law exponent for FG Al-Al₂O₃ beam (HC-F, \(L/h = 20\))

<table>
<thead>
<tr>
<th>Theory</th>
<th>Present study</th>
<th>Ref. (Simsek, 2010)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1st order</td>
<td>3rd order</td>
</tr>
<tr>
<td>(k=0)</td>
<td>1.94959</td>
<td>1.94959</td>
</tr>
<tr>
<td>(k=0.2)</td>
<td>1.81409</td>
<td>1.81414</td>
</tr>
<tr>
<td>(k=0.5)</td>
<td>1.66030</td>
<td>1.66030</td>
</tr>
<tr>
<td>(k=1)</td>
<td>1.50103</td>
<td>1.50103</td>
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<tr>
<td>(k=2)</td>
<td>1.36969</td>
<td>1.36956</td>
</tr>
<tr>
<td>(k=5)</td>
<td>1.30373</td>
<td>1.30336</td>
</tr>
<tr>
<td>(k=10)</td>
<td>1.26495</td>
<td>1.26453</td>
</tr>
</tbody>
</table>

Table 4.3.3 and Table 4.3.4 present the comparison of the first three frequency parameters \(\bar{\omega} = \omega L\sqrt{p_m[(1 - (\nu_m)^2)]/E_m}\) of aligned FG-CNT and UD-CNT beams based on the first order theory. The current first order theory results are in good agreement with the ones from (Yas & Samadi, 2012).

Table 4.3.3 Comparisons of the first three frequency parameters for aligned FG-CNT and UD-CNT beams (P-P, \(L/h = 15\))

<table>
<thead>
<tr>
<th>(V_{tcnt})</th>
<th>Beam type</th>
<th>(\omega_1)</th>
<th>(\omega_2)</th>
<th>(\omega_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.12</td>
<td>Present - 1st order</td>
<td>UD</td>
<td>0.9739</td>
<td>2.8640</td>
</tr>
<tr>
<td></td>
<td>(Yas &amp; Samadi, 2012)</td>
<td>UD</td>
<td>0.9753</td>
<td>2.8728</td>
</tr>
<tr>
<td></td>
<td>Present - 1st order</td>
<td>FG-A</td>
<td>0.9440</td>
<td>2.6353</td>
</tr>
<tr>
<td></td>
<td>(Yas &amp; Samadi, 2012)</td>
<td>FG-A</td>
<td>0.9453</td>
<td>2.6424</td>
</tr>
<tr>
<td></td>
<td>Present - 1st order</td>
<td>FG-X</td>
<td>1.1129</td>
<td>3.0701</td>
</tr>
<tr>
<td></td>
<td>(Yas &amp; Samadi, 2012)</td>
<td>FG-X</td>
<td>1.1150</td>
<td>3.0814</td>
</tr>
<tr>
<td>0.17</td>
<td>Present - 1st order</td>
<td>UD</td>
<td>1.1977</td>
<td>3.6128</td>
</tr>
<tr>
<td></td>
<td>(Yas &amp; Samadi, 2012)</td>
<td>UD</td>
<td>1.1999</td>
<td>3.6276</td>
</tr>
<tr>
<td></td>
<td>Present - 1st order</td>
<td>FG-A</td>
<td>1.1588</td>
<td>3.2967</td>
</tr>
<tr>
<td></td>
<td>(Yas &amp; Samadi, 2012)</td>
<td>FG-A</td>
<td>1.1609</td>
<td>3.3084</td>
</tr>
<tr>
<td></td>
<td>Present - 1st order</td>
<td>FG-X</td>
<td>1.3795</td>
<td>3.9096</td>
</tr>
<tr>
<td></td>
<td>(Yas &amp; Samadi, 2012)</td>
<td>FG-X</td>
<td>1.3830</td>
<td>3.9293</td>
</tr>
<tr>
<td>$V_{tcnt}$</td>
<td>Beam type</td>
<td>$\omega_1$</td>
<td>$\omega_2$</td>
<td>$\omega_3$</td>
</tr>
<tr>
<td>-----------</td>
<td>-----------</td>
<td>------------</td>
<td>------------</td>
<td>------------</td>
</tr>
<tr>
<td>0.12</td>
<td>Present - 1$^{st}$ order</td>
<td>UD</td>
<td>1.5034</td>
<td>3.1228</td>
</tr>
<tr>
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<td>UD</td>
<td>1.5085</td>
<td>3.1353</td>
</tr>
<tr>
<td></td>
<td>Present - 1$^{st}$ order</td>
<td>FG-Λ</td>
<td>1.4025</td>
<td>2.9882</td>
</tr>
<tr>
<td></td>
<td>Ref. (Yas &amp; Samadi, 2012)</td>
<td>FG-Λ</td>
<td>1.4068</td>
<td>2.9997</td>
</tr>
<tr>
<td></td>
<td>Present - 1$^{st}$ order</td>
<td>FG-X</td>
<td>1.5936</td>
<td>3.2486</td>
</tr>
<tr>
<td></td>
<td>Ref. (Yas &amp; Samadi, 2012)</td>
<td>FG-X</td>
<td>1.6000</td>
<td>3.2629</td>
</tr>
<tr>
<td>0.17</td>
<td>Present - 1$^{st}$ order</td>
<td>UD</td>
<td>1.9057</td>
<td>3.9968</td>
</tr>
<tr>
<td></td>
<td>Ref. (Yas &amp; Samadi, 2012)</td>
<td>UD</td>
<td>1.9144</td>
<td>4.0187</td>
</tr>
<tr>
<td></td>
<td>Present - 1$^{st}$ order</td>
<td>FG-Λ</td>
<td>1.7649</td>
<td>3.8109</td>
</tr>
<tr>
<td></td>
<td>Ref. (Yas &amp; Samadi, 2012)</td>
<td>FG-Λ</td>
<td>1.7721</td>
<td>3.8312</td>
</tr>
<tr>
<td></td>
<td>Present - 1$^{st}$ order</td>
<td>FG-X</td>
<td>2.0386</td>
<td>4.1851</td>
</tr>
<tr>
<td></td>
<td>Ref. (Yas &amp; Samadi, 2012)</td>
<td>FG-X</td>
<td>2.0498</td>
<td>4.2111</td>
</tr>
<tr>
<td>0.28</td>
<td>Present - 1$^{st}$ order</td>
<td>UD</td>
<td>2.1440</td>
<td>4.4137</td>
</tr>
<tr>
<td></td>
<td>Ref. (Yas &amp; Samadi, 2012)</td>
<td>UD</td>
<td>2.1618</td>
<td>4.4556</td>
</tr>
<tr>
<td></td>
<td>Present - 1$^{st}$ order</td>
<td>FG-Λ</td>
<td>2.0343</td>
<td>4.2994</td>
</tr>
<tr>
<td></td>
<td>Ref. (Yas &amp; Samadi, 2012)</td>
<td>FG-Λ</td>
<td>2.0504</td>
<td>4.3414</td>
</tr>
<tr>
<td></td>
<td>Present - 1$^{st}$ order</td>
<td>FG-X</td>
<td>2.2937</td>
<td>4.6537</td>
</tr>
<tr>
<td></td>
<td>Ref. (Yas &amp; Samadi, 2012)</td>
<td>FG-X</td>
<td>2.3169</td>
<td>4.7051</td>
</tr>
</tbody>
</table>

The convergence and comparison study confirms that the current formulation is able to calculate the converged first five modes for the linear free vibration of FG-CNT beams based on both the first order and the third order theories. The theory
comparison and parametric studies will be presented in the following sections.

### 4.3.2 Analysis of First and Third Order Beam Solutions

Table 4.3.5 lists the comparison of fundamental frequency parameters \( \bar{\omega} = \omega L^2 \sqrt{\rho_m/(E_m h^2)} \) of CNT beams with different end support conditions between the first order and the third order beam theories. The beam \( L/h \) ratio is set to be 10 and the total CNT volume fraction \( V_{tcnt} \) is 0.17. It is noticed that the first order beam results are substantially larger than the corresponding third order beam results by up to 18\%, 17.6\% and 16.9\% for FGX-CNT, UD-CNT and FGA-CNT beams with SC-SC boundary conditions, respectively. On the other hand, the first order frequency parameters for HC-HC CNT beams are smaller than the corresponding ones based on the third order beam theory by 6.2\%, 5.9\% and 5.1\% for beams with FGX-, UD- and FGA-CNT distributions. The differences are negligible for beams with P-P support conditions.

<table>
<thead>
<tr>
<th>CNT distribution</th>
<th>Boundary conditions</th>
<th>1st order ( \bar{\omega}^{1st} )</th>
<th>3rd order ( \bar{\omega}^{3rd} )</th>
<th>( \frac{\bar{\omega}^{3rd} - \bar{\omega}^{1st}}{\bar{\omega}^{3rd}} \times 100 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>FGX-CNT</td>
<td>P-P</td>
<td>13.7208</td>
<td>13.7339</td>
<td>0.10</td>
</tr>
<tr>
<td></td>
<td>SC-SC</td>
<td>17.4782</td>
<td>14.9535</td>
<td>-16.88</td>
</tr>
<tr>
<td></td>
<td>HC-HC</td>
<td>17.5563</td>
<td>18.4950</td>
<td>5.08</td>
</tr>
<tr>
<td>FGX-CNT</td>
<td>P-P</td>
<td>15.3246</td>
<td>15.2797</td>
<td>-0.29</td>
</tr>
<tr>
<td></td>
<td>SC-SC</td>
<td>18.6042</td>
<td>15.7718</td>
<td>-17.96</td>
</tr>
<tr>
<td></td>
<td>HC-HC</td>
<td>18.7000</td>
<td>19.9291</td>
<td>6.17</td>
</tr>
<tr>
<td>UD-CNT</td>
<td>P-P</td>
<td>13.9703</td>
<td>14.0138</td>
<td>0.31</td>
</tr>
<tr>
<td></td>
<td>SC-SC</td>
<td>18.0081</td>
<td>15.3073</td>
<td>-17.64</td>
</tr>
<tr>
<td></td>
<td>HC-HC</td>
<td>18.0953</td>
<td>19.2214</td>
<td>5.86</td>
</tr>
</tbody>
</table>

The nominal shear strains \( (\phi_\zeta - dW_0/d\zeta) \) along the beam based on the first order
and third order beam theories for P-P, SC-SC and HC-HC FGA-CNT beams with $V_{t\text{cnt}} = 0.17$ and $L/h = 10$ are plotted in Figure 4.3.3 to Figure 4.3.5. In Figure 4.3.3, we observe that the nominal shear strains from the first and third order beam theories are similar for the P-P CNT beam which leads to the frequency parameters based on the two beam theories being close to each other. For the SC-SC CNT beam (see Figure 4.3.4), large discrepancies of the first order and third order nominal shear strains are observed near the vicinity of the beam supports and more importantly the trend of the nominal shear strains are also different at the end supports. The differences in the shear strain distributions may be caused by the nonzero of the translational displacement at the boundary as stated in Section 3.4.2, which may lead to a “softer” response at the boundary. Figure 4.3.5 shows that the first order and third order shear strains for the HC-HC CNT beam have similar trend along the beam with some discrepancies around the supports which may explain the less substantial differences of the first and third order frequency parameters of this beam than the SC-SC counterpart.

Figure 4.3.3 Nominal shear strains along the FGA-CNT beam (P-P, $V_{t\text{cnt}} = 0.17$, $L/h = 10$)
Figure 4.3.4 Nominal shear strains along the FGA-CNT beam (SC-SC, $V_{cnt} = 0.17, L/h = 10$)

Figure 4.3.5 Nominal shear strains along the FGA-CNT beam (HC-HC, $V_{cnt} = 0.17, L/h = 10$)
In Figure 4.3.6 to Figure 4.3.8, the fundamental frequency mode shapes of the aforementioned FGA-CNT beam based on the first and third order theory are presented. As shown in Figure 4.3.6, the mode shapes of the FGA-CNT beam with P-P boundary condition based on the first order and third order theory are overlapping with each other. In Figure 4.3.7, for the beam with SC-SC boundary condition, the mode shape obtained using the first order theory is slightly slender than the one obtained using the third order theory, similar to the case for plates with SC-SC-SC-SC support condition (see Section 4.2.2). On the contrary, for the beam with HC-HC boundary condition shown in Figure 4.3.8, the mode shape of the third order theory is more slender.

Figure 4.3.6 Mode shapes comparison of the FGA-CNT beam based on the first order and third order theory (P-P, $V_{cnt} = 0.17$, $L/h = 10$)
Figure 4.3.7 Mode shapes comparison of the FGA-CNT beam based on the first order and third order theory (SC-SC, $V_{tcnt} = 0.17$, $L/h = 10$)

Figure 4.3.8 Mode shapes comparison of the FGA-CNT beam based on the first order and third order theory (HC-HC, $V_{tcnt} = 0.17$, $L/h = 10$)

Table 4.3.6 presents the comparison of fundamental frequency parameters $\bar{\omega} = \omega L^2 \sqrt{\rho_m/(E_m h^2)}$ of UD and FGA-CNT beams with different $L/h$ ratios between the first order and the third order beam theories. The beam boundary condition is SC-SC.
and the total CNT volume fraction \( V_{cnt} \) is 0.12. It can be seen that the difference of the frequency parameters based on the first order theory and the third order theory increases first when \( L/h \) value varies from 10 to 20 and then decreases as \( L/h \) value further increases. This frequency parameter difference for UD-CNT beams is slightly more pronounced than the one for FGA-CNT beams.

Table 4.3.6 Fundamental frequency parameters for FGA-CNT, and UD-CNT beams with different \( L/h \) ratios (SC-SC, \( V_{cnt} = 0.12 \))

<table>
<thead>
<tr>
<th>CNT distribution</th>
<th>( L/h )</th>
<th>Present study</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1(^{st}) order</td>
</tr>
<tr>
<td>FGA-CNT</td>
<td>10</td>
<td>13.8615</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>22.1104</td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>26.1369</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>28.1569</td>
</tr>
<tr>
<td>UD-CNT</td>
<td>10</td>
<td>14.2567</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>23.9827</td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>29.4610</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>32.4801</td>
</tr>
</tbody>
</table>

4.3.3 Parametric Studies

New vibration frequency parameters based on the third order shear deformation beam theory are presented in this subsection.

Figure 4.3.9 presents the relationship between the fundamental frequency parameters and the beam \( L/h \) ratio for the CNT composite beams with different CNT distributions and beam support conditions. The CNT volume fraction is fixed at 0.12. It is observed that the frequency parameters of the beams for the FGX distribution have the largest values, followed by the ones with UD distribution, while the beams with the FGA distribution always have the smallest frequency parameters. The fundamental frequency parameters for all beams increase as the beam \( L/h \) ratio increases. However, the rate of frequency parameter variation is different for beams with different support conditions. The frequency parameters of beams with HC-HC boundary conditions have a more pronounced increase than the ones of cantilever beams (HC-F).
Table 4.3.7 lists the first five frequency parameters $\tilde{\omega} = \frac{\omega L^2 \sqrt{\rho_m/(E_m h^2)}}{\rho_m}$ for CNT composite beams with different boundary conditions. The beam $L/h$ ratio is fixed at 12 and the volume fraction $V_{tcnt}$ is set to be 0.17 for all the beams under consideration. It is observed that with the boundary conditions changing from HC-HC, HC-P, SC-SC, P-P to HC-F, the frequency parameters of the first three modes decrease rapidly. The boundary condition influence is less pronounced to the fourth and fifth mode frequency parameters. Taking FGA-CNT beam for instance, the fundamental frequency of HC-HC beam is nearly four times of that of HC-F beam. However, the differences for the fourth and fifth mode frequencies of HC-HC and HC-F FGA-CNT beams are about 28% and 22%, respectively. Similar phenomena can be observed for UD- and FGX-CNT beams.

![Figure 4.3.9 Fundamental frequency parameters of beams with different L/h ratios.](image)
Table 4.3.7 The first five frequency parameters of beams with different boundary conditions of the first five modes \((L/h = 12, \nu_{CNT} = 0.17)\)

<table>
<thead>
<tr>
<th>CNT</th>
<th>Mode</th>
<th>HC-HC</th>
<th>HC-P</th>
<th>SC-SC</th>
<th>P-P</th>
<th>HC-F</th>
</tr>
</thead>
<tbody>
<tr>
<td>UD-CNT</td>
<td>(\bar{\omega}_1)</td>
<td>21.9660</td>
<td>18.5184</td>
<td>17.5174</td>
<td>15.2313</td>
<td>6.1690</td>
</tr>
<tr>
<td></td>
<td>(\bar{\omega}_2)</td>
<td>45.7118</td>
<td>43.4266</td>
<td>41.0867</td>
<td>40.9916</td>
<td>25.4398</td>
</tr>
<tr>
<td></td>
<td>(\bar{\omega}_3)</td>
<td>73.2042</td>
<td>70.4037</td>
<td>67.8506</td>
<td>67.7421</td>
<td>53.3180</td>
</tr>
<tr>
<td></td>
<td>(\bar{\omega}_4)</td>
<td>102.3188</td>
<td>98.8987</td>
<td>96.0513</td>
<td>95.6058</td>
<td>81.1924</td>
</tr>
<tr>
<td></td>
<td>(\bar{\omega}_5)</td>
<td>133.6532</td>
<td>129.3521</td>
<td>126.1578</td>
<td>125.2682</td>
<td>110.8875</td>
</tr>
<tr>
<td>FGA-CNT</td>
<td>(\bar{\omega}_1)</td>
<td>20.8855</td>
<td>17.5239</td>
<td>16.9794</td>
<td>14.8671</td>
<td>5.3048</td>
</tr>
<tr>
<td></td>
<td>(\bar{\omega}_2)</td>
<td>44.2152</td>
<td>41.6616</td>
<td>39.1507</td>
<td>38.6692</td>
<td>23.5585</td>
</tr>
<tr>
<td></td>
<td>(\bar{\omega}_3)</td>
<td>71.5699</td>
<td>68.6289</td>
<td>65.9460</td>
<td>65.9102</td>
<td>50.8264</td>
</tr>
<tr>
<td></td>
<td>(\bar{\omega}_4)</td>
<td>100.7714</td>
<td>97.2483</td>
<td>93.9110</td>
<td>93.7691</td>
<td>78.8812</td>
</tr>
<tr>
<td></td>
<td>(\bar{\omega}_5)</td>
<td>132.1871</td>
<td>127.8463</td>
<td>124.4675</td>
<td>123.7403</td>
<td>108.6575</td>
</tr>
<tr>
<td>FGX-CNT</td>
<td>(\bar{\omega}_1)</td>
<td>22.9422</td>
<td>19.7582</td>
<td>18.0511</td>
<td>16.9068</td>
<td>7.1027</td>
</tr>
<tr>
<td></td>
<td>(\bar{\omega}_2)</td>
<td>47.4026</td>
<td>45.1824</td>
<td>42.8999</td>
<td>42.8821</td>
<td>27.2740</td>
</tr>
<tr>
<td></td>
<td>(\bar{\omega}_3)</td>
<td>75.5686</td>
<td>72.5406</td>
<td>70.2505</td>
<td>69.7038</td>
<td>55.7497</td>
</tr>
<tr>
<td></td>
<td>(\bar{\omega}_4)</td>
<td>105.6945</td>
<td>101.8341</td>
<td>99.1237</td>
<td>98.1513</td>
<td>83.9485</td>
</tr>
<tr>
<td></td>
<td>(\bar{\omega}_5)</td>
<td>138.6025</td>
<td>133.6850</td>
<td>130.7380</td>
<td>129.0220</td>
<td>114.5832</td>
</tr>
</tbody>
</table>

Table 4.3.8 presents the fundamental frequency parameters \(\bar{\omega} = \omega L^2\sqrt{\rho_m/(E_m h^2)}\) for the CNT composite beams with CNT volume fraction varying from 0.12, 0.17 to 0.28 while the beam \(L/h\) ratio is fixed at 12. As discussed previously, the beams with HC-HC boundary conditions are most sensitive to the CNT volume fraction change. The FGX-CNT beams have the largest frequency parameters while the FGA-CNT beams have the smallest. Given the same boundary conditions and with the CNT volume fraction increasing from 0.12 to 0.17, the frequency parameters \(\bar{\omega}\) are observed to have a significant jump ranging from 20% to 25%. However, the rates of increase tend to ease after the volume fraction exceeds 0.17. This observation has two implications. On one hand, it verifies that adding small amount of CNTs can have extraordinary effect on the vibration properties of the beams. On the other hand, further increase of the CNT volume fraction in the beams may not lead to a corresponding improvement on the beam vibration properties as anticipated.
Table 4.3.8 Fundamental frequency parameters of beams with different boundary conditions
and \( V_{tcnt} (L/h = 12) \)

<table>
<thead>
<tr>
<th>CNT distribution</th>
<th>( V_{tcnt} )</th>
<th>Boundary conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.12</td>
<td>17.5604 14.9142 13.9946 12.4402 5.1035</td>
</tr>
<tr>
<td>UD-CNT</td>
<td>0.17</td>
<td>21.9660 18.5184 17.5174 15.2313 6.1690</td>
</tr>
<tr>
<td></td>
<td>0.28</td>
<td>23.5944 20.2387 18.7914 17.2125 7.2053</td>
</tr>
<tr>
<td></td>
<td>0.17</td>
<td>20.8855 17.5239 16.9794 14.8671 5.3048</td>
</tr>
<tr>
<td></td>
<td>0.28</td>
<td>22.8133 19.4828 18.4555 16.9276 6.2718</td>
</tr>
<tr>
<td>FGX-CNT</td>
<td>0.12</td>
<td>18.4169 15.9405 14.5196 13.7748 5.8559</td>
</tr>
<tr>
<td></td>
<td>0.17</td>
<td>22.9422 19.7582 18.0511 16.9068 7.1027</td>
</tr>
<tr>
<td></td>
<td>0.28</td>
<td>24.0387 21.0154 19.0413 18.5193 8.0984</td>
</tr>
</tbody>
</table>

4.4 Nonlinear Free Vibration of Beams

4.4.1 Convergence and Comparison Studies

To validate the accuracy of the Ritz method and determine the number of Ritz polynomial terms \( N \) used in the computations, a convergence study is carried out for the vibration of a FGA-CNT composite beam with hard-clamped conditions at both ends (HC-HC). The \( L/h \) ratio is taken to be 10 and the total CNT volume fraction \( V_{tcnt} \) is 0.12. The linear frequency parameters are taken the form of \( \Omega_l = \omega L^2 \sqrt{\rho_m/(E_m h^2)} \) in the subsequent numerical studies. As shown in Table 4.4.1, it is observed that the frequency parameters converge rapidly when the number of Ritz terms increases from 4 to 9. For the first three vibration modes, the results converge well when the number of Ritz terms \( N \) reaches 8. As in the present study for nonlinear vibration of the CNT beams, only the fundamental vibration mode is considered. We choose \( N = 8 \) for all subsequent computations for both linear and nonlinear frequencies.

Figure 4.4.1 and Figure 4.4.2 plot the fundamental frequency mode shape of the FGA-CNT and the FGV-CNT beams, respectively. Both beams have P-P supports, with \( L/h = 10 \) and \( V_{tcnt} = 0.12 \). The maximum downward amplitude \( W_{\text{max}} \) is 0.3. From the mode shapes, it can be seen that during a full vibration cycle, the upwards
amplitude is higher than 0.3 for FGA-CNT beam while lower than 0.3 for FGV-CNT beam.

Table 4.4.1 Convergence of frequency parameters $\Omega_i$ for a FGA-CNT composite beam ($L/h = 10$, $V_{t\text{cnt}} = 0.12$).

<table>
<thead>
<tr>
<th>Mode</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>50.3855</td>
<td>50.2407</td>
<td>50.1652</td>
<td>50.1299</td>
<td>50.0811</td>
<td>50.0807</td>
</tr>
<tr>
<td>4</td>
<td>94.6938</td>
<td>72.4708</td>
<td>71.5698</td>
<td>71.0340</td>
<td>70.9482</td>
<td>70.8462</td>
</tr>
<tr>
<td>5</td>
<td>160.5932</td>
<td>140.0681</td>
<td>99.6955</td>
<td>97.1531</td>
<td>94.4709</td>
<td>94.2251</td>
</tr>
</tbody>
</table>

Figure 4.4.1 Fundamental nonlinear frequency mode shape of FGA-CNT beam based on the third order theory (P-P, $L/h = 10$, $V_{t\text{cnt}} = 0.12$, $W_{\text{max}} = 0.3$)
Comparison studies have been performed against existing published results to validate the numerical approach used in this study. The material properties used in the comparison studies are taken from Ke et al. (Ke, et al., 2010) In the present research, the shear correction factor $\kappa$ for the first order shear deformation beam theory is taken as $5/6$ and is the same as those adopted in the compared literature (Ke, et al., 2010)

Table 4.4.2 presents the nonlinear to linear frequency parameter ratios $\Omega_{nl}/\Omega_l$ for UD-CNT beams with CNT total volume fraction $V_{cnt} = 0.17$. The beam $L/h$ ratio is taken to be 10. The maximum nondimensional amplitude is defined as $W_{\text{max}} = w_{\text{max}}/h$, where $w_{\text{max}}$ is the maximum downward displacement in the beam. The solutions from the current study are in good agreement with the ones by Ke et al. (Ke, et al., 2010) based on the first order beam theory and for three different support conditions (P-P, SC-SC and SC-P).

The convergence and comparison studies verified the correctness and efficiency of the current solution approach which will be applied to generate the solutions for the parametric studies in the following sections.
Table 4.4.2 Nonlinear to linear frequency parameter ratios $\Omega_{nl}/\Omega_{l}$ for UD-CNT beams with different boundary conditions ($L/h=10$, $V_{cnt}=0.17$)

<table>
<thead>
<tr>
<th>Boundary conditions</th>
<th>Sources</th>
<th>$W_{max}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0.1</td>
</tr>
<tr>
<td>P-P</td>
<td>Present study</td>
<td>1.0261</td>
</tr>
<tr>
<td>(Ke, et al., 2010)</td>
<td></td>
<td>1.0259</td>
</tr>
<tr>
<td>SC-SC</td>
<td>Present study</td>
<td>1.0136</td>
</tr>
<tr>
<td>(Ke, et al., 2010)</td>
<td></td>
<td>1.0136</td>
</tr>
<tr>
<td>SC-P</td>
<td>Present study</td>
<td>1.0188</td>
</tr>
<tr>
<td>(Ke, et al., 2010)</td>
<td></td>
<td>1.0187</td>
</tr>
</tbody>
</table>

4.4.2 Analysis of First and Third Order Beam Solutions

Table 4.4.3 lists the linear fundamental frequency parameters $\Omega_{l}$ and the nonlinear to linear frequency parameter ratios $\Omega_{nl}/\Omega_{l}$ of FGA-CNT and FGV-CNT beams with different end support conditions based on the first order and the third order beam theories. The beam $L/h$ ratio is set to be 10 and the total CNT volume fraction is 0.12. The linear fundamental frequencies for FGA- and FGV-CNT beams with the same support conditions and beam theories are identical.

Firstly, for the nonlinear to linear frequency ratios $\Omega_{nl}/\Omega_{l}$, there are no significant differences between the first and third order results for beams with P-P boundary conditions. But for beams with SC-SC boundary conditions, the ratios obtained from the third order theory are larger than the results from the first order theory. However, for beams with HC-HC boundary conditions the results based on the third order theory are slightly smaller than the ones from the first order theory.

Secondly, we observe the influence of carbon nanotube distributions on the nonlinear vibration of the beams. For HC-HC beams, the nonlinear to linear frequency ratios $\Omega_{nl}/\Omega_{l}$ are almost identical for FGA and FGV distributions based on the first order or the third order theory. For the P-P case, the nonlinear to linear frequency ratios $\Omega_{nl}/\Omega_{l}$ show moderate differences for FGA and FGV distributions when either the first order or the third order theory is used. However, for SC-SC beams, the results based on the first order theory are almost identical, while the results based on the third
order theory show moderate discrepancies for FGA and FGV distributions. Again, it shows the nonzero of $\partial W_0 / \partial \zeta$ in the third order theory can contribute to a “softer” response of the nonlinear FG-CNT beam vibration.

Table 4.4.3 Nonlinear to linear frequency parameter ratios $\Omega_{nl} / \Omega_l$ of FGA- and FGV-CNT beams with different boundary conditions ($L/h = 10$, $V_{cnt} = 0.12$)

<table>
<thead>
<tr>
<th>Boundary Theory</th>
<th>$\Omega_l$</th>
<th>Beam type</th>
<th>$W_{max}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.1</td>
</tr>
<tr>
<td>P-P 3rd order</td>
<td>11.1601</td>
<td>FGA</td>
<td>1.0102</td>
</tr>
<tr>
<td></td>
<td></td>
<td>FGV</td>
<td>1.0066</td>
</tr>
<tr>
<td>P-P 1st order</td>
<td>11.1295</td>
<td>FGA</td>
<td>1.0104</td>
</tr>
<tr>
<td></td>
<td></td>
<td>FGV</td>
<td>1.0067</td>
</tr>
<tr>
<td>SC-SC 3rd order</td>
<td>11.9480</td>
<td>FGA</td>
<td>1.0253</td>
</tr>
<tr>
<td></td>
<td></td>
<td>FGV</td>
<td>1.0217</td>
</tr>
<tr>
<td>SC-SC 1st order</td>
<td>13.8612</td>
<td>FGA</td>
<td>1.0212</td>
</tr>
<tr>
<td></td>
<td></td>
<td>FGV</td>
<td>1.0212</td>
</tr>
<tr>
<td>HC-HC 3rd order</td>
<td>14.8064</td>
<td>FGA</td>
<td>1.0206</td>
</tr>
<tr>
<td></td>
<td></td>
<td>FGV</td>
<td>1.0206</td>
</tr>
<tr>
<td>HC-HC 1st order</td>
<td>14.0623</td>
<td>FGA</td>
<td>1.0222</td>
</tr>
<tr>
<td></td>
<td></td>
<td>FGV</td>
<td>1.0222</td>
</tr>
</tbody>
</table>

Table 4.4.4 presents the linear fundamental frequency parameters $\Omega_l$ and the nonlinear to linear frequency parameter ratios $\Omega_{nl} / \Omega_l$ for UD- and FGX-CNT beams. The beam geometry and the total CNT volume fraction are the same as the FGA-CNT and FGV-CNT beams presented in Table 4.4.3. Similar to their asymmetric counterparts in Table 4.4.3, the nonlinear to linear frequency ratios based on the first and third order beam theories are close to each other for UD- and FGX-CNT beams with P-P support conditions. The nonlinear to linear frequency ratios based on the first order beam theory are smaller than the ones based on the third order beam theory for UD- and FGX-CNT beams with SC-SC boundary conditions, while the first order beam results are slightly larger than the third order beam results for UD- and FGX-CNT beams with HC-HC support conditions.
Table 4.4.4 Nonlinear to linear frequency parameter ratios $\Omega_{nl}/\Omega_t$ of UD- and FGX-CNT beams ($L/h = 10$, $V_{tcnt} = 0.12$).

<table>
<thead>
<tr>
<th>Boundary</th>
<th>Theory</th>
<th>$\Omega_t$</th>
<th>Beam type</th>
<th>$W_{max}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.1</td>
</tr>
<tr>
<td>P-P</td>
<td>3rd order</td>
<td>11.3732</td>
<td>UD</td>
<td>1.0321</td>
</tr>
<tr>
<td></td>
<td>1st order</td>
<td>12.3850</td>
<td>FGX</td>
<td>1.0273</td>
</tr>
<tr>
<td>SC-SC</td>
<td>3rd order</td>
<td>12.2067</td>
<td>UD</td>
<td>1.0289</td>
</tr>
<tr>
<td></td>
<td>1st order</td>
<td>12.6733</td>
<td>FGX</td>
<td>1.0268</td>
</tr>
<tr>
<td>HC-HC</td>
<td>3rd order</td>
<td>15.3294</td>
<td>UD</td>
<td>1.0192</td>
</tr>
<tr>
<td></td>
<td>1st order</td>
<td>15.9807</td>
<td>FGX</td>
<td>1.0178</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.0211</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.0202</td>
</tr>
</tbody>
</table>

From the results in Table 4.4.3 and Table 4.4.4, we can observe that the nonlinear to linear frequency ratios $\Omega_{nl}/\Omega_t$ for beams with $L/h = 10$ and $V_{tcnt} = 0.12$ subject to P-P and HC-HC support conditions based on the third order beam theory show slightly less significant nonlinearity than their first order beam counterparts. However, the trend is reversed for the beams with SC-SC boundary conditions.

### 4.4.3 Parametric Studies

The following parametric studies are all based on the third order beam theory. Table 4.4.5 shows the nonlinear to linear frequency parameter ratios $\Omega_{nl}/\Omega_t$ of FGA-CNT beams with different beam $L/h$ ratios. The boundary conditions are SC-SC and the total CNT volume fraction is 0.17. The results show that for a given amplitude, the nonlinear to linear frequency ratios $\Omega_{nl}/\Omega_t$ decrease as the $L/h$ ratio increases.

Figure 4.4.3 and Figure 4.4.4 present the nonlinear to linear frequency parameter ratios $\Omega_{nl}/\Omega_t$ against the nondimensional maximum amplitude for FGA- and FGV-CNT beams. For both beams, the beam $L/h$ is set to be 10, and the total volume fraction is taken to be 0.17 and 0.28, respectively. Results in Figure 4.4.3 and Figure
4.4.4 show that for a given downward amplitude, the frequency parameter ratios $\Omega_{nl}/\Omega_l$ are symmetrical for FGA- and FGV-CNT beams with HC-HC boundary conditions, but unsymmetrical for FGA- and FGV-CNT beams with P-P and SC-SC boundary conditions, which are also observed from the results in Table 4.4.3 with the CNT volume fraction of 0.12. FGA-CNT beams show stronger nonlinear effect than FGV-CNT beams with SC-SC and P-P support conditions.

Table 4.4.5 Nonlinear to linear frequency parameter ratios $\Omega_{nl}/\Omega_l$ for FGA-CNT beams with different length to depth ratio $L/h$ (SC-SC, $V_{tcnt} = 0.17$).

<table>
<thead>
<tr>
<th>$L/h$</th>
<th>$\Omega_l$</th>
<th>$W_{max}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0.1</td>
</tr>
<tr>
<td>10</td>
<td>14.9585</td>
<td>1.0228</td>
</tr>
<tr>
<td>20</td>
<td>23.2781</td>
<td>1.0101</td>
</tr>
<tr>
<td>30</td>
<td>28.1484</td>
<td>1.0073</td>
</tr>
<tr>
<td>40</td>
<td>31.0400</td>
<td>1.0063</td>
</tr>
</tbody>
</table>

In the current formulation, the midsurface of the beam is located at the centre of the beam thickness, which is not at the physical neutral surface of the beam (Rafiee, Yang, & Kitipornchai, 2013). For FGA- and FGV-CNT beams with HC-HC support conditions, the required energy for the beams to reach a maximum given downward amplitude is the same. Therefore, their nonlinear effect is considered at the same level, which leads to the same nonlinear frequencies for both FGA- and FGV-CNT beams at the maximum given download amplitude. However, for FGA- and FGV-CNT beams with SC-SC or P-P support conditions, the energy required for the beams moving downward to a maximum given amplitude is different. The required energy for the FGA-CNT beam is larger than the one for FGV-CNT beam, which leads to a larger nonlinear frequency for the FGA-CNT beam than the one for the FGV-CNT beam.

It is observed from Figure 4.4.3 that for FGV-CNT beams with CNT volume fraction of 0.17, the nonlinear effect for the beam with SC-SC support conditions are always stronger than the one for the beams with HC-HC and P-P support conditions as the nondimensional downward amplitude $W_{max}^{-}$ varies from 0.1 to 0.5. However, for
FGA-CNT beams, the nonlinear effect for the beam with P-P supports is smaller than the one for the beams with HC-HC and SC-SC supports when the downward amplitude $W_{max}$ is less than 0.2. Further increase of the downward amplitude leads to the nonlinear effect of the P-P beam outstrips the one for the HC-HC beam at $W_{max} > 0.3$ and the SC-SC beam at $W_{max} > 0.4$, respectively.

Figure 4.4.3 Nonlinear to linear frequency ratios $\Omega_{nl}/\Omega_l$ versus nondimensional downward maximum amplitude $W_{max}$ for CNT beams with FGΛ and FGV distributions ($L/h = 10$, $V_{cnt} = 0.17$).

Figure 4.4.4 presents the nonlinear to linear frequency ratios $\Omega_{nl}/\Omega_l$ for FGΛ- and FGV-CNT beams with beam $L/h$ ratio of 10 and CNT volume fraction of 0.28. Similar to the FGV-CNT beams shown in Figure 4.4.3, the SC-SC case shows the strongest nonlinear effect, followed by the HC-HC and P-P cases. However, the nonlinear effect for FGA-CNT beams in Figure 4 shows different trend when comparing with the cases in Figure 4.4.3. The SC-SC beam in Figure 4.4.4 has the strongest nonlinear effect, while the nonlinear effect of the P-P beam is more significant than the one for the HC-HC beam when $W_{max} > 0.2$. 

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Figure 4.4.4 Nonlinear to linear frequency ratios $\Omega_{nl}/\Omega_l$ versus nondimensional downward maximum amplitude $\tilde{W}_{max}$ for CNT beams with FGΛ and FGV distributions ($L/h = 10, V_{cent} = 0.28$).

Table 4.4.6 lists the nonlinear to linear frequency parameter ratios $\Omega_{nl}/\Omega_l$ for FGΛ- and FGV-CNT beams with asymmetric boundary conditions (SC -P and HC -P). The beam $L/h$ is taken to be 10 and the CNT volume fraction $V_{cent}$ is set to be 0.28. As observed for beams with symmetric boundary conditions, the FGΛ-CNT beams with SC-P and HC-P support conditions show stronger nonlinear effect than the FGV-CNT beams.

Table 4.4.6 Nonlinear to linear frequency parameter ratio $\Omega_{nl}/\Omega_l$ for FGΛ- and FGV-CNT beams ($L/h = 10, V_{cent} = 0.28$).

<table>
<thead>
<tr>
<th>Boundary Theory</th>
<th>Beam type</th>
<th>$W_{max}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0.1</td>
</tr>
<tr>
<td>SC-P 3rd order</td>
<td>FGΛ</td>
<td>1.0212</td>
</tr>
<tr>
<td></td>
<td>FGV</td>
<td>1.0159</td>
</tr>
<tr>
<td>HC-P 3rd order</td>
<td>FGΛ</td>
<td>1.0239</td>
</tr>
<tr>
<td></td>
<td>FGV</td>
<td>1.0201</td>
</tr>
</tbody>
</table>
Table 4.4.7 lists the nonlinear to linear frequency parameter ratios $\frac{\Omega_{nl}}{\Omega_l}$ for the UD- and FGX-CNT beams with different boundary conditions. The $L/h$ is 10 and total CNT volume fraction is 0.28. It is observed that the linear frequency parameters increases for both UD- and FGX-CNT beams when the boundary conditions of the beams change from P-P, SC-P, SC-SC, HC-P to HC-HC, respectively. However, the reverse trend is observed for the nonlinear effect of the beams when the beam support conditions vary in the aforementioned sequence for a given amplitude.

Table 4.4.7 Nonlinear to linear frequency parameter ratios $\frac{\Omega_{nl}}{\Omega_l}$ for UD- and FGX-CNT beams ($L/h = 10$, $V_{cnt} = 0.28$).

<table>
<thead>
<tr>
<th>Boundary type</th>
<th>Beam type</th>
<th>$\Omega_l$</th>
<th>$W_{max}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0.1</td>
<td>0.2</td>
</tr>
<tr>
<td>P-P</td>
<td>UD</td>
<td>15.5884</td>
<td>1.0365</td>
</tr>
<tr>
<td>SC-P</td>
<td>UD</td>
<td>15.9249</td>
<td>1.0353</td>
</tr>
<tr>
<td>SC-SC</td>
<td>UD</td>
<td>16.3487</td>
<td>1.0340</td>
</tr>
<tr>
<td>HC-P</td>
<td>UD</td>
<td>17.8528</td>
<td>1.0293</td>
</tr>
<tr>
<td>HC-HC</td>
<td>UD</td>
<td>20.5389</td>
<td>1.0228</td>
</tr>
<tr>
<td>P-P</td>
<td>FGX</td>
<td>16.4624</td>
<td>1.0328</td>
</tr>
<tr>
<td>SC-P</td>
<td>FGX</td>
<td>16.5152</td>
<td>1.0326</td>
</tr>
<tr>
<td>SC-SC</td>
<td>FGX</td>
<td>16.5943</td>
<td>1.0326</td>
</tr>
<tr>
<td>HC-P</td>
<td>FGX</td>
<td>18.3698</td>
<td>1.0278</td>
</tr>
<tr>
<td>HC-HC</td>
<td>FGX</td>
<td>20.8298</td>
<td>1.0224</td>
</tr>
</tbody>
</table>
CHAPTER 5 CONCLUSIONS

5.1 Conclusions on Current Studies
As a novel type of composite material, the functionally graded carbon nanotubes (FG-CNT) reinforced composites emerge as a potential candidate for the next generation engineering material which can be tailored to possess multifunctional properties or to response “smartly” to the external stimuli. In the current study, the results of the molecular dynamics (MD) simulation on the FG-CNT composites have been reviewed, and the material properties of the FG-CNT composite are obtained using a micromechanical approach. The linear and geometric nonlinear free vibrations of FG-CNT beams as well as the linear free vibration of FG-CNT plates have been investigated. Based on the first and third order shear deformation theory, the variational formulation on the free vibration of FG-CNT beams and plates have been derived, and then solved by using the p-Ritz method.

Convergence studies and comparison on existing publication results show the accuracy and the efficiency of the current formulation and solution procedure. When using different shear deformation theories (i.e. the first order and third order theories), the analysis for FG-CNT beams/plates show that substantial differences between frequency parameters involving soft clamped (SC) boundary conditions, while there are negligible differences for beams/plates with Pinned (P) boundary conditions and slight differences for beams/plates with hard clamped (HC) boundary conditions. Further study is needed to determine which theory (first order versus third order) gives more accurate solutions for the FG-CNT composites with clamped support conditions when comparing with results generated from the 3D elasticity theory.

Extensive parametric studies have been carried out exclusively based on the third order shear deformation theory. The influence of CNT volume fractions, CNT distributions, boundary conditions, CNT reinforcing angle, geometries of the beams/plates, nonlinear vibration amplitudes on the beams/plates vibration frequency parameters are investigated.
The following observations have been made for linear vibration of FG-CNT beams. 1) Adding small amount of CNTs can have extraordinary effect on the vibration properties of the beams. On the other hand, further increase of the CNT volume fraction in the beams may not lead to a corresponding improvement on the beam vibration properties as anticipated. 2) The frequency parameters of the beams for the FGX distribution have the largest values, followed by the ones with UD distribution, while the beams with the FGA distribution always have the smallest frequency parameters. 3) With the boundary conditions changing from HC-HC, HC-P, SC-SC, P-P to HC-F, the frequency parameters of the first three modes of the beam decrease rapidly. The boundary condition influence is less pronounced to the fourth and fifth mode frequency parameters. 4) The fundamental frequency parameters for all beams increase as the beam $L/h$ ratio increases. However, the rate of frequency parameter variation is different for beams with different support conditions.

For a given downward amplitude of nonlinear vibration of FG-CNT beams, it is observed that 1) the frequency parameter ratios $\Omega_{nl}/\Omega_l$ are symmetrical for FGA- and FGV-CNT beams with HC-HC boundary conditions, but unsymmetrical for FGA- and FGV-CNT beams with P-P and SC-SC boundary conditions. 2) FGA-CNT beams show stronger nonlinear effect than FGV-CNT beams with SC-SC and P-P support conditions at certain downward amplitude. 3) For a given amplitude, the nonlinear effect of the beams decreases when the beam support conditions vary in the sequence of P-P, SC-P, SC-SC, HC-P, HC-HC.

Comparing to the linear vibration of FG-CNT beams, similar aforementioned conclusions can also be made for linear vibration of FG-CNT plates. Other conclusions for plates are as follows: 1) the fundamental frequency parameters decrease as the plate aspect ratio $L/b$ increase. 2) The influence of the CNT distribution and the differences of the soft-clamped and hard-clamped conditions diminish when the plate aspect ratio $L/b$ becomes large. 3) The fundamental frequency parameters based on the third order theory decrease when the angle $\theta$ varies from $0^\circ$ to $45^\circ$ except for the case of SC-SC-SC-SC boundary. However, if the first order theory is used, the fundamental frequency parameters uniformly decrease with the increase of angle $\theta$ between $0^\circ$ - $45^\circ$ for all boundary conditions.
5.2 Suggestion for Future Studies

The study of the FG-CNT composites is truly a multidisciplinary subject, and there are many research topics waiting for further exploration. From the structural mechanics point of view, the following future studies may be suggested.

As shown in current studies, there are discrepancies between results of the first order and the third order shear deformation theories when the clamped boundary conditions (SC and HC) are involved. Therefore, comparison studies of the current results against the results obtained from the 3D elasticity theory, the commercial finite element codes, or the experimental studies are need to be carried out in order to truly understand the vibration behaviour of the FG-CNT composites.

Other than the structures formed only by FG-CNT composites, hybrid laminated FG-CNT composites are also envisioned, for example, to have FG-CNT face sheets with sandwiched core made of isotropic material, or to have face sheets made of piezoelectric materials laminated with FG-CNT composite cores. The thermal mechanical behaviour of such composites may also be an interesting research topic.

Structures under thermal and electrical field are usually encountered in the MEMS studies. Besides, the phenomenal size effect is also a challenging issue when systems become smaller. The current formulation may be modified to accommodate these topics in the future.

In addition to the numerical analysis, vibration experiment is also an important research area for the studying of FG-CNT structures. As an inverse problem, the results obtained from the experiments can be used to calibrate the real physical parameters and verify the mathematical model used in the theoretical analysis. The experiments may also disclose shortcomings in the numerical analysis and contribute to the numerical modelling accordingly.

The material property of the FG-CNT composite can be determined by an appropriate test. However, if one can establish the relationship between the macroscopic properties and the microscopic material constitution, the
micromechanical approach can be utilised to predict the overall material properties of the FG-CNT composites. Successfully prediction of the material properties of FG-CNT composites via multi-scale modelling not only can save the effort of material testing, but also enables the material design and simulation. As an emerging research topic, the computational material science on the CNT reinforced composites is certainly under the spotlight from many research communities.
APPENDIX A

This section listed the elements of the nondimensional stiffness matrix and mass matrix of linear free vibration of the FG-CNT plates.

The geometric ratios of the plates are

\[
\frac{L}{h} = L_{sh} \quad \text{(A.1)}
\]
\[
\frac{b}{h} = b_{sh} \quad \text{(A.2)}
\]
\[
\beta = \frac{L}{b} = \frac{L_{sh}}{b_{sh}} \quad \text{(A.3)}
\]

The elements of the \([\bar{K}]\) matrix are as follows

\[
K_{11}^{ik} = L_{sh}^2 b_{sh}^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[ \left( a_{11} \frac{\partial \varphi_i^x}{\partial \zeta} + \beta a_{16} \frac{\partial \varphi_i^x}{\partial \eta} \right) \frac{\partial \varphi_k^x}{\partial \zeta} \right. \\
+ \beta \left( a_{16} \frac{\partial \varphi_i^x}{\partial \zeta} + \beta a_{66} \frac{\partial \varphi_i^x}{\partial \eta} \right) \frac{\partial \varphi_k^x}{\partial \eta} \right] d\zeta d\eta \quad \text{(A.4)}
\]

\[
K_{12}^{ik} = L_{sh}^2 b_{sh}^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[ \frac{1}{\beta} \left( \beta a_{12} \frac{\partial \varphi_i^y}{\partial \eta} + a_{16} \frac{\partial \varphi_i^y}{\partial \zeta} \right) \frac{\partial \varphi_k^x}{\partial \zeta} \right. \\
+ \left( \beta a_{26} \frac{\partial \varphi_i^y}{\partial \eta} + a_{66} \frac{\partial \varphi_i^y}{\partial \zeta} \right) \frac{\partial \varphi_k^x}{\partial \eta} \right] d\zeta d\eta \quad \text{(A.5)}
\]

\[
K_{13}^{ik} = \left( -\frac{4}{3} L_{sh} b_{sh} \right) \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[ \left( \frac{1}{\beta} e_{11} \frac{\partial^2 \xi_i}{\partial \zeta^2} + \beta e_{12} \frac{\partial^2 \xi_i}{\partial \eta^2} \right) \frac{\partial \varphi_k^x}{\partial \zeta} \right. \\
+ 2e_{16} \frac{\partial^2 \xi_i}{\partial \zeta^2} \frac{\partial \varphi_k^x}{\partial \eta} \right] d\zeta d\eta \quad \text{(A.6)}
\]
\[
K_{14}^{ik} = L_{sh} b_{sh} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[ \frac{1}{\beta} \left( b_{11} - \frac{4}{3} e_{11} \right) \frac{\partial \psi_i^x}{\partial \zeta} \frac{\partial \phi_k^x}{\partial \zeta} + \left( b_{16} - \frac{4}{3} e_{16} \right) \frac{\partial \psi_i^x}{\partial \eta} \frac{\partial \phi_k^y}{\partial \zeta} + \left( b_{16} - \frac{4}{3} e_{16} \right) \frac{\partial \psi_i^y}{\partial \eta} \frac{\partial \phi_k^x}{\partial \zeta} \right] d\zeta d\eta
\]
\[
= L_{sh} b_{sh} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[ \frac{1}{\beta} \left( b_{12} - \frac{4}{3} e_{12} \right) \frac{\partial \psi_i^y}{\partial \eta} \frac{\partial \phi_k^x}{\partial \zeta} + \beta \left( b_{66} - \frac{4}{3} e_{66} \right) \frac{\partial \psi_i^x}{\partial \zeta} \frac{\partial \phi_k^y}{\partial \eta} \right] d\zeta d\eta
\]

(A.7)

\[
K_{15}^{ik} = L_{sh} b_{sh} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[ \beta \left( a_{22} - \frac{1}{\beta} a_{26} \frac{\partial \phi_i^x}{\partial \zeta} \right) \frac{\partial \phi_k^y}{\partial \eta} + \frac{1}{\beta} \left( a_{26} - \frac{1}{\beta} a_{66} \frac{\partial \psi_i^y}{\partial \zeta} \right) \frac{\partial \phi_k^y}{\partial \eta} \right] d\zeta d\eta
\]
\[
= L_{sh} b_{sh} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[ \beta \left( b_{12} - \frac{4}{3} e_{12} \right) \frac{\partial \psi_i^y}{\partial \eta} \frac{\partial \phi_k^x}{\partial \zeta} + \beta \left( b_{26} - \frac{4}{3} e_{26} \right) \frac{\partial \psi_i^y}{\partial \eta} \frac{\partial \phi_k^x}{\partial \zeta} \right] d\zeta d\eta
\]

(A.8)

\[
K_{22}^{ik} = L_{sh} b_{sh}^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[ \frac{1}{\beta} \left( 1 \right) \frac{\partial^2 \xi_i}{\partial \zeta^2} + \beta \left( a_{26} - \frac{1}{\beta} a_{66} \frac{\partial \psi_i^y}{\partial \zeta} \right) \frac{\partial \phi_k^x}{\partial \eta} \right] d\zeta d\eta
\]
\[
= L_{sh} b_{sh}^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[ \frac{1}{\beta} \left( 1 \right) \frac{\partial^2 \xi_i}{\partial \zeta^2} + \beta \left( b_{26} - \frac{4}{3} e_{26} \right) \frac{\partial \psi_i^y}{\partial \eta} \frac{\partial \phi_k^x}{\partial \zeta} + \beta \left( b_{26} - \frac{4}{3} e_{26} \right) \frac{\partial \psi_i^y}{\partial \eta} \frac{\partial \phi_k^x}{\partial \zeta} \right] d\zeta d\eta
\]

(A.9)

\[
K_{23}^{ik} = \left( -\frac{4}{3} L_{sh} b_{sh} \right) \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[ \frac{1}{\beta} \left( 1 \right) \frac{\partial^2 \xi_i}{\partial \zeta^2} + \beta \left( a_{26} - \frac{1}{\beta} a_{66} \frac{\partial \psi_i^y}{\partial \zeta} \right) \frac{\partial \phi_k^x}{\partial \eta} \right] d\zeta d\eta
\]

(A.10)

\[
K_{24}^{ik} = L_{sh} b_{sh} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[ \frac{1}{\beta} \left( b_{12} - \frac{4}{3} e_{12} \right) \frac{\partial \psi_i^x}{\partial \zeta} \frac{\partial \phi_k^y}{\partial \eta} \right] d\zeta d\eta
\]
\[
= L_{sh} b_{sh} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[ \frac{1}{\beta} \left( b_{16} - \frac{4}{3} e_{16} \right) \frac{\partial \psi_i^x}{\partial \eta} \frac{\partial \phi_k^y}{\partial \zeta} + \beta \left( b_{66} - \frac{4}{3} e_{66} \right) \frac{\partial \psi_i^x}{\partial \eta} \frac{\partial \phi_k^y}{\partial \zeta} \right] d\zeta d\eta
\]

(A.11)
\[
K_{25}^{ik} = L_{sh}b_{sh} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[ \beta \left( b_{22} - \frac{4}{3} e_{22} \right) \frac{\partial \psi_i^\prime \partial \varphi_k^\prime}{\partial \eta} + \left( b_{26} - \frac{4}{3} e_{26} \right) \frac{\partial \psi_i^\prime \partial \varphi_k^\prime}{\partial \zeta} + \frac{1}{\beta} \left( b_{66} - \frac{4}{3} e_{66} \right) \frac{\partial \psi_i^\prime \partial \varphi_k^\prime}{\partial \zeta} \right] d\zeta d\eta 
\]  
\[
K_{33}^{ik} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \frac{16}{9} \right) \left[ \frac{1}{\beta} \left( \frac{1}{\beta} h_{11} \frac{\partial^2 \xi_i}{\partial \zeta^2} + \beta h_{12} \frac{\partial^2 \xi_i}{\partial \eta^2} + 2 h_{16} \frac{\partial^2 \xi_i}{\partial \zeta \partial \eta} \right) \frac{\partial^2 \xi_k}{\partial \zeta^2} + \beta \left( \frac{1}{\beta} h_{22} \frac{\partial^2 \xi_i}{\partial \eta^2} + \beta h_{26} \frac{\partial^2 \xi_i}{\partial \zeta \partial \eta} + 2 \frac{1}{\beta} h_{16} \frac{\partial^2 \xi_i}{\partial \zeta \partial \eta} + \beta \frac{1}{\beta} h_{26} \frac{\partial^2 \xi_i}{\partial \zeta \partial \eta} \right) \frac{\partial^2 \xi_k}{\partial \zeta \partial \eta} \right] d\zeta d\eta 
\]
\[
K_{34}^{ik} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( -\frac{4}{3} \right) \left[ \frac{1}{\beta^2} \left( f_{11} - \frac{4}{3} h_{11} \right) \frac{\partial \psi_i^{\xi'} \partial^2 \xi_k}{\partial \xi \partial \zeta^2} \right. \\
+ \frac{1}{\beta} \left( f_{16} - \frac{4}{3} h_{16} \right) \frac{\partial \psi_i^{\xi'} \partial^2 \xi_k}{\partial \eta \partial \zeta^2} \right. \\
\left. + \beta \left( f_{26} - \frac{4}{3} h_{26} \right) \frac{\partial \psi_i^{\xi'} \partial^2 \xi_k}{\partial \eta \partial \eta^2} + \left( f_{12} - \frac{4}{3} h_{12} \right) \frac{\partial \psi_i^{\xi'} \partial^2 \xi_k}{\partial \xi \partial \eta^2} \right. \\
\left. + \frac{1}{\beta} \left( 2f_{16} - \frac{8}{3} h_{16} \right) \frac{\partial \psi_i^{\xi'} \partial^2 \xi_k}{\partial \zeta \partial \eta \partial \eta} \right. \\
\left. + \left( 2f_{66} - \frac{8}{3} h_{66} \right) \frac{\partial \psi_i^{\xi'} \partial^2 \xi_k}{\partial \eta \partial \zeta \partial \eta} \right] \\
+ L_{sh} b_{sh} \left[ (a_{45} - 8d_{45} + 16f_{45}) \psi_i^{\xi'} \frac{\partial \xi_k}{\partial \eta} \right. \\
\left. + \frac{1}{\beta} (a_{55} - 8d_{55} + 16f_{55}) \psi_i^{\xi'} \frac{\partial \xi_k}{\partial \zeta} \right] \, d\zeta \, d\eta
\] (A.14)

\[
K_{35}^{ik} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( -\frac{4}{3} \right) \left[ \left( f_{12} - \frac{4}{3} h_{12} \right) \frac{\partial \psi_i^{\eta'} \partial^2 \xi_k}{\partial \eta \partial \xi^2} + \frac{1}{\beta} \left( f_{16} - \frac{4}{3} h_{16} \right) \frac{\partial \psi_i^{\eta'} \partial^2 \xi_k}{\partial \eta \partial \zeta^2} \right. \\
\left. + \beta^2 \left( f_{22} - \frac{4}{3} h_{22} \right) \frac{\partial \psi_i^{\eta'} \partial^2 \xi_k}{\partial \eta^2 \partial \eta^2} \right. \\
\left. + \beta \left( f_{26} - \frac{4}{3} h_{26} \right) \frac{\partial \psi_i^{\eta'} \partial^2 \xi_k}{\partial \eta \partial \eta^2} \right. \\
\left. + \beta \left( 2f_{26} - \frac{8}{3} h_{26} \right) \frac{\partial \psi_i^{\eta'} \partial^2 \xi_k}{\partial \eta \partial \zeta \partial \eta} \right. \\
\left. + \left( 2f_{66} - \frac{8}{3} h_{66} \right) \frac{\partial \psi_i^{\eta'} \partial^2 \xi_k}{\partial \eta \partial \zeta \partial \eta} \right] \\
+ L_{sh} b_{sh} \left[ \beta (a_{44} - 8d_{44} + 16f_{44}) \psi_i^{\eta'} \frac{\partial \xi_k}{\partial \eta} \right. \\
\left. + (a_{45} - 8d_{45} + 16f_{45}) \psi_i^{\eta'} \frac{\partial \xi_k}{\partial \zeta} \right] \, d\zeta \, d\eta
\] (A.15)
\[ K_{ik}^{44} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[ \frac{1}{\beta^2} \left( 1 - \frac{8}{3} f_{11} + \frac{16}{9} h_{11} \right) \frac{\partial \psi_i^x \partial \psi_k^x}{\partial \zeta \partial \zeta} \\
+ \frac{1}{\beta} \left( d_{16} - \frac{8}{3} f_{16} + \frac{16}{9} h_{16} \right) \frac{\partial \psi_i^x \partial \psi_k^x}{\partial \eta \partial \zeta} \\
+ \frac{1}{\beta} \left( d_{16} - \frac{8}{3} f_{16} + \frac{16}{9} h_{16} \right) \frac{\partial \psi_i^x \partial \psi_k^x}{\partial \zeta \partial \eta} \\
+ \left( d_{66} - \frac{8}{3} f_{66} + \frac{16}{9} h_{66} \right) \frac{\partial \psi_i^x \partial \psi_k^x}{\partial \eta \partial \eta} \\
+ L_{sh} b_{sh} \frac{1}{\beta} (a_{55} - 8d_{55} + 16f_{55}) \psi_i^x \psi_k^x \right] d\zeta d\eta \tag{A.16} \]

\[ K_{ik}^{45} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[ \left( d_{12} - \frac{8}{3} f_{12} + \frac{16}{9} h_{12} \right) \frac{\partial \psi_i^y \partial \psi_k^x}{\partial \eta \partial \zeta} \\
+ \frac{1}{\beta} \left( d_{16} - \frac{8}{3} f_{16} + \frac{16}{9} h_{16} \right) \frac{\partial \psi_i^y \partial \psi_k^x}{\partial \zeta \partial \zeta} \\
+ \beta \left( d_{26} - \frac{8}{3} f_{26} + \frac{16}{9} h_{26} \right) \frac{\partial \psi_i^y \partial \psi_k^x}{\partial \eta \partial \eta} \\
+ \left( d_{66} - \frac{8}{3} f_{66} + \frac{16}{9} h_{66} \right) \frac{\partial \psi_i^y \partial \psi_k^x}{\partial \zeta \partial \eta} \\
+ L_{sh} b_{sh} (a_{45} - 8d_{45} + 16f_{45}) \psi_i^y \psi_k^x \right] d\zeta d\eta \tag{A.17} \]

\[ K_{ik}^{55} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[ \beta^2 \left( d_{22} - \frac{8}{3} f_{22} + \frac{16}{9} h_{22} \right) \frac{\partial \psi_i^y \partial \psi_k^y}{\partial \eta \partial \eta} \\
+ \beta \left( d_{26} - \frac{8}{3} f_{26} + \frac{16}{9} h_{26} \right) \frac{\partial \psi_i^y \partial \psi_k^y}{\partial \eta \partial \zeta} \\
+ \beta \left( d_{26} - \frac{8}{3} f_{26} + \frac{16}{9} h_{26} \right) \frac{\partial \psi_i^y \partial \psi_k^y}{\partial \zeta \partial \eta} \\
+ \left( d_{66} - \frac{8}{3} f_{66} + \frac{16}{9} h_{66} \right) \frac{\partial \psi_i^y \partial \psi_k^y}{\partial \zeta \partial \eta} \\
+ L_{sh} b_{sh} \beta (a_{44} - 8d_{44} + 16f_{44}) \psi_i^y \psi_k^y \right] d\zeta d\eta \tag{A.18} \]
The elements of the $\tilde{M}$ matrix are

\[
M_{11}^k = L_{sh}^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \varphi_i^x \varphi_k^x d\zeta d\eta
\]

(A.19)

\[
M_{12}^k = 0
\]

(A.20)

\[
M_{13}^k = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} -\frac{4}{3} i_3 \frac{\partial \xi_i}{\partial \zeta} \varphi_k^x d\zeta d\eta
\]

(A.21)

\[
M_{14}^k = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[(i_1 - \frac{4}{3} i_3) \psi_i^x \varphi_k^x \right] d\zeta d\eta
\]

(A.22)

\[
M_{15}^k = 0
\]

(A.23)

\[
M_{22}^k = b_{sh}^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \varphi_i^y \varphi_k^y d\zeta d\eta
\]

(A.24)

\[
M_{23}^k = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} -\frac{4}{3} i_3 \frac{\partial \xi_i}{\partial \eta} \varphi_k^y d\zeta d\eta
\]

(A.25)

\[
M_{24}^k = 0
\]

(A.26)

\[
M_{25}^k = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(i_1 - \frac{4}{3} i_3\right) \psi_i^y \varphi_k^y d\zeta d\eta
\]

(A.27)

\[
M_{33}^k = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{16}{9} \frac{1}{L_{sh}^2} i_6 \frac{\partial \xi_i}{\partial \zeta} \frac{\partial \xi_k}{\partial \zeta} + \frac{16}{9} \frac{1}{b_{sh}^2} i_6 \frac{\partial \xi_i}{\partial \eta} \frac{\partial \xi_k}{\partial \eta} + \xi_i \xi_k \right) d\zeta d\eta
\]

(A.28)

\[
M_{34}^k = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(-\frac{4}{3} i_4 + \frac{16}{9} i_6 \right) \psi_i^x \frac{\partial \xi_k}{\partial \zeta} d\zeta d\eta
\]

(A.29)

\[
M_{35}^k = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(-\frac{4}{3} i_4 + \frac{16}{9} i_6 \right) \psi_i^y \frac{\partial \xi_k}{\partial \eta} d\zeta d\eta
\]

(A.30)

\[
M_{44}^k = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(i_2 - \frac{8}{3} i_4 + \frac{16}{9} i_6 \right) \psi_i^x \psi_k^x d\zeta d\eta
\]

(A.31)

\[
M_{45}^k = 0
\]

(A.32)

\[
M_{55}^k = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(i_2 - \frac{8}{3} i_4 + \frac{16}{9} i_6 \right) \psi_i^y \psi_k^y d\zeta d\eta
\]

(A.33)
APPENDIX B

This section listed the elements of the nondimensional stiffness matrix and mass matrix of linear free vibration of the FG-CNT plates.

The geometric ratio of the beams is

$$\eta = \frac{L}{h} \quad \text{(B.1)}$$

The elements of symmetric nondimensional linear stiffness matrix $[\bar{K}]$ are listed below

\[
\begin{align*}
K^{ik}_{11} & = \frac{1}{2} \left( \eta^4 a_{11} \frac{\partial \phi_i}{\partial \zeta} \frac{\partial \phi_k}{\partial \zeta} \right) d\zeta \quad \text{(B.2)} \\
K^{ik}_{12} & = \frac{1}{2} \left( -\frac{4}{3} \eta^2 e_{11} \frac{\partial^2 \xi_i}{\partial \zeta^2} \frac{\partial \phi_k}{\partial \zeta} \right) d\zeta \quad \text{(B.3)} \\
K^{ik}_{13} & = \frac{1}{2} \left( \eta^2 b_{11} \frac{\partial \psi_i}{\partial \zeta} \frac{\partial \phi_k}{\partial \zeta} - \frac{4}{3} \eta^2 e_{11} \frac{\partial \psi_i}{\partial \zeta} \frac{\partial \phi_k}{\partial \zeta} \right) d\zeta \quad \text{(B.4)} \\
K^{ik}_{22} & = \frac{1}{2} \left( 16 \frac{h_{11}}{9} \frac{\partial^2 \xi_i}{\partial \zeta^2} \frac{\partial^2 \xi_k}{\partial \zeta^2} + \eta^2 a_{55} \frac{\partial \xi_i}{\partial \zeta} \frac{\partial \xi_k}{\partial \zeta} \\
& \quad - 8\eta^2 d_{55} \frac{\partial \xi_i}{\partial \zeta} \frac{\partial \xi_k}{\partial \zeta} + 16\eta^2 f_{55} \frac{\partial \xi_i}{\partial \zeta} \frac{\partial \xi_k}{\partial \zeta} \right) d\zeta \quad \text{(B.5)} \\
K^{ik}_{23} & = \frac{1}{2} \left( -\frac{4}{3} f_{11} \frac{\partial \psi_i}{\partial \zeta} \frac{\partial^2 \xi_k}{\partial \zeta^2} + 16 \frac{h_{11}}{9} \frac{\partial \psi_i}{\partial \zeta} \frac{\partial^2 \xi_k}{\partial \zeta^2} \\
& \quad + \eta^2 a_{55} \psi_i \frac{\partial \xi_k}{\partial \zeta} - 8\eta^2 d_{55} \psi_i \frac{\partial \xi_k}{\partial \zeta} \\
& \quad + 16\eta^2 f_{55} \psi_i \frac{\partial \xi_k}{\partial \zeta} \right) d\zeta \quad \text{(B.6)}
\end{align*}
\]
$$\bar{K}_{33}^{ik} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \frac{\partial \psi_i}{\partial \zeta} \frac{\partial \psi_k}{\partial \zeta} - \frac{8}{3} f_{11} \frac{\partial \psi_i}{\partial \zeta} \frac{\partial \psi_k}{\partial \zeta} + \frac{16}{9} h_{11} \frac{\partial \psi_i}{\partial \zeta} \frac{\partial \psi_k}{\partial \zeta} \right. $$

$$\left. + \eta^2 a_{55} \psi_i \psi_k - 8 \eta^2 d_{55} \psi_i \psi_k \right) d\zeta$$

(B.7)

The elements of symmetric nondimensional mass matrix are as follows

$$\bar{M}^{ik}_{11} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \eta^2 \varphi_i \varphi_k d\zeta$$

(B.8)

$$\bar{M}^{ik}_{12} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( -\frac{4}{3} i_3 \frac{\partial \xi_i}{\partial \zeta} \varphi_k \right) d\zeta$$

(B.9)

$$\bar{M}^{ik}_{13} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( i_1 \psi_i \psi_k - \frac{4}{3} i_3 \psi_i \varphi_k \right) d\zeta$$

(B.10)

$$\bar{M}^{ik}_{22} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[ \frac{16}{9} \left( \frac{1}{\eta} \right)^2 i_6 \frac{\partial \xi_i}{\partial \zeta} \frac{\partial \xi_k}{\partial \zeta} + \xi_i \xi_k \right] d\zeta$$

(B.11)

$$\bar{M}^{ik}_{23} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[ -\frac{4}{3} \left( \frac{1}{\eta} \right)^2 i_4 \psi_i \frac{\partial \xi_k}{\partial \zeta} + \frac{16}{9} \left( \frac{1}{\eta} \right)^2 i_6 \psi_i \frac{\partial \xi_k}{\partial \zeta} \right] d\zeta$$

(B.12)

$$\bar{M}^{ik}_{33} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[ \frac{1}{\eta} \left( \frac{1}{\eta} \right)^2 i_2 \psi_i \psi_k - \frac{8}{3} \left( \frac{1}{\eta} \right)^2 i_4 \psi_i \psi_k + \frac{16}{9} \left( \frac{1}{\eta} \right)^2 i_6 \psi_i \psi_k \right] d\zeta$$

(B.13)
APPENDIX C

This section listed the elements of the symmetric nondimensional nonlinear stiffness matrix

\[
\begin{align*}
\left[ \bar{K}_{12}^{ik} \right]_{nl} &= \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \eta^2 a_{11} \frac{\partial w_0}{\partial \zeta} \frac{\partial \xi_i}{\partial \zeta} \frac{\partial \varphi_k}{\partial \zeta} \right) d\zeta \\
\left[ \bar{K}_{22}^{ik} \right]_{nl} &= \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \eta^2 a_{11} \frac{\partial u_0}{\partial \zeta} \frac{\partial \xi_i}{\partial \zeta} \frac{\partial \xi_k}{\partial \zeta} + b_{11} \frac{\partial \varphi_x}{\partial \zeta} \frac{\partial \xi_i}{\partial \zeta} \frac{\partial \xi_k}{\partial \zeta} \\
&\quad \quad - \frac{4}{3} e_{11} \frac{\partial \varphi_x}{\partial \zeta} \frac{\partial \xi_i}{\partial \zeta} \frac{\partial \xi_k}{\partial \zeta} - \frac{4}{3} e_{11} \frac{\partial w_0}{\partial \zeta} \frac{\partial \xi_i}{\partial \zeta} \frac{\partial \xi_k}{\partial \zeta} \frac{\partial \varphi_x}{\partial \zeta} \right) d\zeta \\
\left[ \bar{K}_{23}^{ik} \right]_{nl} &= \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( b_{11} \frac{\partial w_0}{\partial \zeta} \frac{\partial \psi_j}{\partial \zeta} \frac{\partial \xi_k}{\partial \zeta} - \frac{4}{3} e_{11} \frac{\partial w_0}{\partial \zeta} \frac{\partial \psi_j}{\partial \zeta} \frac{\partial \xi_k}{\partial \zeta} \right) d\zeta
\end{align*}
\]
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