DEVELOPMENT AND APPLICATIONS OF MOVING LEAST SQUARE RITZ METHOD IN SCIENCE AND ENGINEERING COMPUTATION

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Development and applications of Moving Least Square Ritz Method in Science and Engineering Computation

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STATEMENT OF AUTHENTICATION

I declare that this thesis submitted is, to the best of my knowledge and belief, original except as acknowledged in the text. I certify that this work is not submitted in candidature for any other degrees.

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_________________________                        ________/________/________
ABSTRACT

With the rapid development of computing technologies in the past three decades, the numerical methods have become indispensable tools for solving all kinds of science and engineering problems. Extensive researches have been carried out for the development of new numerical methods in the past few decades. The existing numerical methods have achieved great success in finding the numerical solutions for various theoretical and practical problems. A detailed literature review on the development and applications of several numerical methods in solid mechanics and electromagnetic field analysis is presented in the thesis. Despite the great achievements in this research area, there are always the needs to develop new numerical methods or to explore alternative techniques for the purpose of solving the complicate problems and improve the efficiency and accuracy of the existing or new numerical methods.

This thesis presents the development of a novel numerical method, the moving least square Ritz (MLS-Ritz) method, and its applications for solving science and engineering problems. The MLS-Ritz method is based on the moving least square (MLS) data interpolation technique and the Ritz minimization principle. The MLS technique is utilized to establish the Ritz trial functions for two-dimensional (2-D) and three-dimensional (3-D) cases. A point substitution approach is developed to enforce boundary conditions. The proposed MLS-Ritz method has the ability to expand the applicability of the conventional Ritz method and meshless method for analysing problems with complex geometries and multiple mediums.

The MLS-Ritz method is first applied to solve several solid mechanics problems. The free vibration of square and triangular plates is investigated by the MLS-Ritz
method. The characteristics of the MLS-Ritz method is examined through the
detailed convergence and comparison studies for selected cases. It shows that the
MLS-Ritz method is highly stable, accurate and efficient in solving such plate
vibration problems.

The MLS-Ritz method is also employed to study the challenging problem of free
vibration of rhombic plates with large skew angles. The domain decomposition
technique is developed and applied in this case to improve the convergence rate of
the computations. The study reveals that some of the previous studies on the
vibration of rhombic plates with large skew angles did not provide converged results.

The MLS-Ritz method is further applied to investigate the 3-D vibration behaviour
of isotropic elastic square and triangular plates. The MLS-Ritz method is efficient to
generate 3-D vibration frequencies for thick plates with high accuracy.

The applications of the MLS-Ritz method are also extended to the analysis of the
electromagnetic field problems. Three cases including electrical potential problems
in a uniform trough and with dielectric medium and a waveguide eigenvalue problem
are analysed and compared with solutions obtained by other methods. Comparison
studies show that excellent agreement is achieved for the three cases when
comparing with existing results in the open literature.

The future directions in the development of the MLS-Ritz method for science and
engineering computations are discussed.
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CHAPTER 1

INTRODUCTION

1.1 Significance and Background

Many scientific and engineering phenomena can be modelled mathematically by differential and integral equations with predefined initial and/or boundary conditions. Differential equations are one of the most widely adopted mathematical models to be used to simulate real-world problems. A differential equation is an equation involving one unknown function and its derivatives. When the function being studied involves two or more independent variables, the differential equation is usually a partial differential equation (PDE). As the functions of multiple variables are intrinsically more complicated than those of one variable, the PDEs can lead to some of the most challenging problems to be solved analytically or numerically.

The PDE was first derived by D’Alembert in 1744 in the study of fluids. He obtained another PDE in his investigation on the problem of a vibrating string in 1747. Since then, the establishment and solution of PDEs to describe physical problems have become one of the major developments and played a central role in almost all areas in science, engineering, economics, financial forecasting and other fields. For example, the differential form of the Maxwell’s equations is the most widely used expression for solving electromagnetic boundary-value problems. The Maxwell’s equations were established by combining electricity and magnetism into a unified electromagnetic theory, which rigorously formulate various complicated electro-
dynamical phenomena, such as the diffraction of light, wave propagation, plane wave scattering, etc.

The best way to solve a differential equation is to obtain its analytical solution. However, there are many situations where the analytical solution is either too difficult to obtain or does not exist at all. For instance, the domain under consideration may be so irregular that it is mathematically impossible to describe the boundary, or the configuration may be composed of several different materials whose regions are mathematically difficult to define. When the analytical solution of a problem is difficult to obtain or does not exist, it is necessary to approximate the solution of the problem numerically, commonly in combination with the analysis of simple special cases.

Numerical methods employ numbers to simulate mathematical processes that define the real-world problems. The astonishingly fast development of high speed digital computers has provided tremendous impetus to all numerical methods for solving the scientific and engineering problems. Complex problems which were beyond the reach of analysis in pre-computer days are now routinely tackled numerically. As the advancing of the technology and the demanding of the applications, various numerical methods and computational techniques have been developed in the past four decades.

At present, there are a large number of numerical methods, such as the finite element method, the finite difference method, the boundary element method, the meshless method and the Ritz method, are available in solving various scientific and engineering problems involving PDEs. Each of these existing methods has its own strengths and limitations. Many of the existing methods are still being developed and new numerical approaches are emerging due to the rapid development of science and engineering technology that requires more sophisticated numerical techniques to
meet the challenge. A detailed literature review on existing numerical methods for PDEs is presented in Chapter 2.

1.2 Scope and Objectives

The aim of this research project is to develop the moving least square Ritz (MLS-Ritz) method, an accurate, stable and flexible numerical approach for solving various boundary value problems, including solid mechanics and electromagnetic field problems.

In this study the applicability of the conventional Ritz method is expanded in science and engineering computations. The Ritz method is one of the earliest numerical methods and is still employed in many scientific and engineering applications. The Ritz method is simple, accurate and numerically stable in solving problems with simple geometries and/or boundary conditions. However, the applicability of the Ritz method has been greatly limited due to the difficulty in finding the Ritz trial function that satisfies the boundary conditions of a problem at hand. The proposed MLS-Ritz method will address this problem. In the proposed MLS-Ritz method, the moving least square technique is employed to establish the Ritz trial function and a point substitution technique is developed to enforce the boundary conditions of a given boundary value problem. The developed MLS-Ritz method will be able to analyse problems with domains of arbitrary shape, various boundary conditions and multiple mediums.

The MLS-Ritz method is essentially a meshless method. This study also addresses one of the difficulties in applying the meshless method. It is well-known that in the meshless method, the nominal value at a grid point is not the approximate function value, which introduces difficulties in enforcing the geometric boundary conditions.
The Lagrangian multiplier technique is commonly employed in the meshless method to process the boundary conditions. However, the Lagrangian multiplier technique increases the number of unknowns and it could also cause numerical instability in the meshless method. A point substitution technique is developed in this study to impose the boundary conditions of a given problem. This technique uses the approximate functional values at the grid points to process the boundary conditions. Therefore, the matrix size of the problem will not be increased and the resultant MLS-Ritz method will be highly stable and accurate.

To increase the versatility of the MLS-Ritz method, a domain decomposition technique in conjunction with the point substitution method is developed, enabling the MLS-Ritz to analyse problems of complex geometry and multi mediums.

Several solid mechanics and electromagnetic field problems are studied to verify the applicability and accuracy of the MLS-Ritz method. A difficult solid mechanics problem, i.e. the vibration analysis of skew plates with large skew angles, is investigated in details to explore the capability and limitation of the MLS-Ritz. Skew plates with large skew angles present a challenge to many numerical methods that fail to generate satisfactory results, due to the stress singularities at the obtuse corners in a skew plate. This study demonstrates that the proposed MLS-Ritz method is a robust numerical approach for such a challenging problem.

It is also demonstrated that the MLS-Ritz method can be applied to two-dimensional (2-D) as well as three-dimensional (3-D) problems. A 3-D vibration analysis is carried out for square plates and triangular plates with various combinations of boundary conditions.
1.3 Thesis Outline

Chapter 2 provides a general literature review on various numerical methods in solving problems defined by partial differential equations, especially for problems in solid mechanics and electromagnetic field analyses. The development and the advantages and disadvantages of some of the numerical methods are discussed.

Chapter 3 presents the moving least square interpolation scheme for 2-D and 3-D functions. A brief introduction to the least square and moving least square techniques is given. A particular weight function is proposed and the characteristics of the weight function in relation to the moving least square interpolation is studied. The shape functions for 2-D and 3-D function interpolations are derived. The numerical evaluation of the shape functions and their derivatives is discussed.

Chapter 4 develops the moving least square Ritz (MLS-Ritz) method for the analysis of 2-D solid mechanics problems. The formulations for the Kirchhoff thin plate are presented and the total potential energy functional for vibration of thin plates is derived. Based on the MLS interpolation scheme proposed in Chapter 3, the Ritz trial function for the transverse displacement of a plate is established. The geometric boundary conditions of the plate are enforced through a proposed point substitution method. The MLS-Ritz method is applied to study the free vibration of rectangular and triangular thin plates. The convergence and accuracy of the MLS-Ritz method against the number of MLS grid points, the radius of support and the number of Gaussian points are discussed in details.

The vibration of skew plates with large skew angles is investigated and presented in Chapter 5. This is a challenging problem and many numerical methods fail to generate accurate frequencies due to the stress singularities at the obtuse corners of
the skew plates. The stability and accuracy of the MLS-Ritz method in solving such difficult problem are presented in this chapter.

Chapter 6 presents the application of the MLS-Ritz method for the analysis of 3-D solid mechanics problems. The free vibration of 3-D square and right-angled triangular plates is studied in this chapter. The implementation of the MLS-Ritz method and the enforcement of the boundary conditions for the 3-D problems are discussed. The convergence and accuracy of the MLS-Ritz method are studied in details when applied for the solutions of the 3-D plate vibration problems.

Chapter 7 investigates the applications of the MLS-Ritz method for the analysis of 2-D electromagnetic field problems. The electromagnetic field theory is briefly presented. The numerical solutions of two static electrical field problems and a waveguide problem are obtained by the MLS-Ritz method. The convergence and accuracy of the MLS-Ritz method in analysing electromagnetic field problems are discussed.

Chapter 8 summarizes the findings and major achievements made through this study. Some future directions of the MLS-Ritz method are proposed.

Appendix A provides a list of articles that have been published or are in press in refereed journals and conference proceedings during the conduction of this work.
CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

With the rapid development of computing technology in the last four decades, numerous numerical methods have been developed for solving various scientific and engineering problems. Among existing numerical methods, the most popular and powerful one is the finite element method (FEM). Many of the other methods are also important and play vital roles in science and engineering computations, such as the finite difference method (FDM), the boundary element method (BEM), the Ritz method, the meshless method, the differential quadrature method (DQM), etc. This chapter will present a review on various numerical methods and their applications in solving solid mechanics and electromagnetic field problems. More attention is given to the Ritz method as the current research work is to develop a moving least square Ritz method and apply it to solve several solid mechanics and electromagnetic field problems.

2.2 Finite Element Method

The FEM is one of the most effective numerical methods widely used for finding approximate solutions of differential equations. The FEM with its versatility and
applicability has dominated the research and practical applications in computational mechanics and electromagnetics for the past few decades. The FEM has been continually developed and is now employed in almost all areas of science and engineering disciplines.

The brief history of the development of the FEM can be found in Logan’s book [1]. The FEM began in the 1940s in the field of structural engineering by Hrennikoff [2]. In 1943, Courant [3] proposed setting up the solution of stresses in a variational form and then introduced the concept of shape function. In 1953, Levy [4] developed the flexibility or force method. Furthermore, he suggested the stiffness or displacement method. In the development process of the FEM, Argyris and Kelsey [5], Turner, Clough, Martin and Topp [6] were the major original contributors. The name “finite element” was coined in the paper by Clough [7] in 1960 when both triangular and rectangular elements were used for plane stress analysis. The method was applied originally for strain calculation in the area of structural mechanics. Melosh presented his work [8] in 1963 which demonstrated that the FEM can be set up in terms of a variational formulation. Since then, the FEM was gradually employed by researchers to solve non-structural problems.

Further extension of the method was made possible by Gallagher et al. [9], Gallagher and Padlog [10], Zienkiewicz et al. [11], Zienkiewicz and Parekh [12], Szabo and Lee [13], and Belytschko [14, 15]. Thereafter, enormous advances have been made in the applications of the FEM to solve complicated scientific and engineering problems. For example, the FEM was used for the first time in late 60s for the solution of electrostatic fields in a waveguide [16-18] and today the application of FEM in electrical and electronic engineering is increasing exponentially in dealing with various analysis and design problems in these areas [19, 20]. The same situation exists in solid and structural mechanics where the FEM is employed to solve various problems including plates, shells, solids and beam-columns, etc. [21-24]. The FEM is also been applied to solve heat transfer and fluid flow problems [25-28]. Indeed, it is
useful in virtually every field of science and engineering because of its flexibility and applicability to a variety of different scientific and engineering problems.

The implementation of the finite element technique is normally made in conjunction with a mathematical scheme such as the variational principle, virtual work or the method of weighted residuals [19, 29, 30]. The FEM combines several mathematical concepts to produce a system of linear or non-linear equations. It is a numerical procedure for solving physical problems governed by a differential equation or an energy theorem.

The technique of the FEM involves modelling the computational domain using small interconnected elements, the finite elements. Every interconnected element is directly or indirectly linked to every other element through the common interfaces, including nodes and/or boundary lines and/or surfaces. The behaviour of a given node is determined in terms of the properties of every other element in the calculated domain. The equations describing the behaviour of each element can be expressed in a simple form. Combining the individual element equations to form one large system of simultaneous equations results in a series of algebraic equations best expressed in matrix notation [30].

Two characteristics that distinguish the FEM from other numerical procedures are: (1) the method utilizes an integral formulation to generate a system of algebraic equations; and (2) the method uses continuous piecewise smooth functions for approximating the unknown quantity or quantities within an element.

There are three versions of the FEM, i.e. the $h$-version, $p$-version and the $h-p$ version FEM [31-34], respectively. The $h$-version is the standard one and has been thoroughly studied. In the $h$-version, the polynomial degree $p$ used in the interpolation of the elements is fixed, and usually lower order polynomials are
employed, typically $p = 1, 2$. The $h$-version FEM relies on finer meshes to achieve the required accuracy.

In contrast, the $p$-version fixes the mesh. It requires lesser elements to achieve the same accuracy (compared with $h$-version) by increasing the degree $p$ of the elements uniformly or selectively. However, the processing time for each element in the $p$-version FEM is considerably longer than that in the $h$-version FEM. Both versions have their own advantages and disadvantages and one needs to decide which one is most suitable for the problem at hand.

The $h$-$p$ version is a combination of both the $h$- and $p$-versions [31]. In the $h$-$p$ version FEM, the error in the solution is controlled by both the diameter $h$ of the largest element and the polynomial order $p$. The $p$ and the $h$-$p$ version FE methods are particularly effective in solving elliptical equations associated with solid mechanics problems [33].

Despite the general applicability of the FEM, some drawbacks of the method need to be addressed. The interconnecting structure of the elements via nodes is called a finite-element mesh. The need for the finite-element mesh leads to some distinct disadvantage of the finite element method. Generation of a mesh for a complex geometry is in itself a difficult and time consuming task. It can be difficult and time consuming to build, check and change the models. For example, for problems with simple geometries, the FEM may not be as effective as the finite difference, Ritz or finite strip methods. The FEM has difficulty to reach the required accuracy in the analysis of plates with discontinuous boundaries. The difficulty lies in the stress concentration developed at the discontinuity. When analysing problems with large deformation such as in metal forming, the FEM needs to regenerate the meshes constantly to capture the changed geometry of the computational domain, which greatly reduces the efficiency of the method. In the past decade, the meshless method
and some other methods have been emerging and found their suitability in analysing such problems.

### 2.3 Finite Difference Method

The FDM consists of transforming the partial derivatives in differential equations over a small interval. The formulae of the FDM are essentially based upon polynomial approximation, which will give exact results when operating on a polynomial of the proper degree. This method can be used to solve any PDE. It is one of the dominant approaches in solving some scientific and engineering problems, e.g., electromagnetic wave simulations and fluid dynamics analysis [20]. The FDM is also very popular when dealing with problems involving both space and transient time period.

Finite difference analysis is one of the oldest methods for numerical solutions of PDEs. The brief history for numerical analysis of PDEs by the FDM can be found in an excellent review paper by Themée in 2001 [35], in which he wrote that the fundamental theory of the finite difference analysis can be traced back to the original theoretical paper by Courant, Friedrichs and Lewy in 1928 [35]. By employing the variational principle for discretization and the emphasis of the importance of mesh-ratio conditions in approximation of time-dependent problems, this paper had a great influence on numerical analysis of PDEs. Themée [35] also reviewed that the FDM was further developed in the 1950s and 1960s to solve time-dependent problems, initial-boundary value problems, and further more the large-scale problems were attempted using the FDM due to the advent of digital computers after World War II.

In 1966, Yee [36] proposed a finite difference scheme for the solutions of initial boundary value problems in electromagnetic field analysis. His scheme ensures the
stability of the FDM in solving three-dimensional (3-D) initial-boundary value problems and reasonable accuracy was achieved. This method is still popular now in dealing with time-varying field problems [19].

In 1992, Sathyanarayana et al. [37] applied the FDM to the determination of the electrostatic field in a multilayered electro-optic device. The Laplace equation governing the problem was solved in a closed area by taking different permittivities in each layer. Hussein and Sebak [38] presented a review of the finite-difference time-domain method and then predicted the radiation patterns of three basic configurations of mobile antennas by the model. The finite difference or finite volume method is also important in the computational fluid dynamics (CFD), especially in the shock wave capture [39, 40].

The FDM has been used with great success for one-dimensional problems with either regular or irregular mesh spacings. However, in two- and three-dimensional analyses, the application of the method is still largely limited to regular meshes, which poses some difficulties when applied to problems with irregular boundaries and shapes. Courant and Varga combined the finite differences with the energy formulation through the use of Green’s formula so that derivatives can be expressed in terms of functions at intermediate nodal points [41, 42]. This technique was applied to a one-dimensional shell analysis in [43, 44] and to two-dimensional shell analyses in [45, 46].

In general, the FDM can be easily implemented for finding the numerical solutions of PDEs. However, the FDM still faces certain difficulties in dealing with irregular geometries in some complex physical problems and the convergence and accuracy of the methods need to be improved to compete with the popular FEM. The major disadvantage of this approach is the difficulty arising from the determination of the second order derivatives; its application to general problems still remains to be explored. The higher-order FDM was developed to address this problem [47, 48].
One of the obvious critical differences between the finite difference and finite element techniques is the ability of the latter to treat irregular domains as stated by Zienkiewicz [49].

2.4 Boundary Element Method

Over the past three decades, the BEM has received much attention from researchers. Nowadays, it is a well established method for engineering analysis and is rapidly gaining more acceptances within the engineering profession. It has become an important technique in the computational solution of a number of physical problems, e.g., soil mechanics, electromagnetic wave scattering, hydraulic, water waves and other half space problems [50-54].

The BEM was first proposed by Brebbia and his associate [55-56]. The BEM is an approach which combines the conventional boundary integral equation method and the discretization technique. The domain is normally assumed to be homogeneous and only nodes along the boundary of the domain are needed in the BEM modelling.

The main characteristic of the method is that it reduces the dimensionality of a problem and the data required to solve the problem [57]. Comparing with the FEM and the FDM which require discretization over the whole computational domain, the BEM discretizes only the boundaries of the domain. Therefore, the number of unknowns involved with the BEM can be greatly reduced. The method is well suited for solving problems with infinite domains such as those frequently encountered in soil mechanics, hydraulics, and stress analysis, etc.

Normann et al. [58] presented a simple boundary element approach to solve the three-dimensional magnetostatic problems in 1985. The study showed that the reduction of calculating time in the mesh generation step remains an important factor
concerning the costs of calculation. For studying electromagnetic fields, the BEM is often used for the analysis of scattering and wave propagation problems. In 1985, Yashiro et al. [59] applied the BEM to solving the magnetostatic wave propagation problems. Bendali et al. [53] employed the BEM in 1999 to solve the electromagnetic scattering problems of the frequency domain related to an impedance boundary condition on an obstacle of arbitrary shape.

The BEM has also found its application in solving soil-structure interaction. Many researchers have employed this method to obtain the responses of structures and soils under various loading conditions [60-62]. The interaction among fluid-soil-structure has also been studied using the BEM [63].

A three-dimensional contact problem was studied by Segond and Tafreshi [64] in 1988 using a technique based on the BEM. The numerical implementation involved the use of linear triangular elements for the representation of the boundary and variables of the bodies in contact.

Calve and Gracia [65] presented a study in 2001 to deal with the sensitivity analysis on the optimum shape design in elasticity using the BEM. In 2005, Yue et al. [66] presented a study on penny-shaped crack problems in two joined transversely isotropic solids by the BEM.

There are also new approaches by using the hybrid methods such as the FEM and BEM to solve some of the engineering problems. In 2006, Bonola and Aviles [54] employed a hybrid boundary and finite element method for gravity walls in which the backfill can be represented by a horizontally layered medium. Degrande et al. [67] presented a finite element-boundary element formulation to predict vibrations in the free field from excitation due to metro trains in tunnels. Specifically, the finite element formulation is used for the tunnel while the boundary element method is used for the soil of the three-dimensional dynamic tunnel-soil interaction problem.
2.5 Ritz Method

In the numerical solution of scientific and engineering problems, there are two major approaches available: (1) variational techniques used in conjunction with energy formulations, and (2) direct solution of the governing differential equations. The typical numerical methods using the direction solution of differential equations are the FDM, the differential quadrature and the collocation method. On the other hand, the FEM and the Ritz or Rayleigh-Ritz method are based on the variational techniques. A more detailed review on the development and applications of the Ritz method will be presented in this section.

2.5.1 General background of Ritz method

The Ritz method, also known as Rayleigh-Ritz method, provides a means to obtain approximate solutions to a differential equation based on minimizing the functional of the equation. The Ritz method is a variational method in which the boundary-value problem is formulated in terms of variational expression, referred to as functional. The minimum of the functional corresponding to approximate solution is then obtained by minimizing the functional with respect to its variables.

In late 1800s, Lord Rayleigh [68, 69] presented a theory to determine the natural frequencies of vibrating structures. Rayleigh was interested in the potential and kinetic energies of the system and, in some cases, attacked the problems from this perspective. In particular, in many cases, he assumed a mode shape, and calculated the corresponding free vibration frequency by equating the potential and the kinetic energies during a vibration cycle. This has generally become known as the Rayleigh method of solution. Its accuracy depends upon how closely the assumed mode shape fits the correct (exact) one [70].
In 1908, Walter Ritz published a paper to present a method by involving multiple terms and minimizing the potential and kinetic energies to determine vibration frequencies of structures [71]. In 1909, Ritz used the method to present novel results for the vibrations of a completely free square plate. These two published papers detailed and demonstrated a straightforward procedure for solving boundary value and eigenvalue problems numerically, to any degree of exactitude desired, also using the energy functional. Ritz extended the Rayleigh method by using a set of admissible trial functions, each of which possesses an independent amplitude coefficient. From this, a more accurate upper bound solution may be obtained via minimizing the energy functional with respect to each of these coefficients. According to Leissa [70], while a first approximation to a vibration frequency may be obtained by the Rayleigh method by using a single admissible function for the mode shape, much better results are typically obtained by using the Ritz method with a series of admissible functions.

In engineering, the Ritz method is a numerical approach widely adopted in computational mechanics due to its simplicity, stability and efficiency in numerical implementation. The classical Ritz method is also widely used in mechanical engineering, electrical engineering and quantum chemistry. Typically, it is used for finding the approximate real resonant frequencies of multi-degree of freedom systems, such as the natural vibration frequencies of a structure in the second or higher modes, the spring mass systems or flywheels on a shaft with varying cross section. It can also be used to determine buckling loads for columns, as well as more complicated problems.

In the Ritz method, the accuracy and convergence rate are highly dependent on the trial functions selected. The frequently used trial functions include the products of eigen-functions of vibrating beams, 2-D orthogonal polynomials, and spline functions.
Although the Ritz method may require less computational effort as compared with the FDM, FEM and BEM, one of the major difficulties is to establish the Ritz trial functions that must satisfy the boundary conditions of the calculation domain. When the domain is irregular and contains multiple mediums, the Ritz method is not convenient to use.

2.5.2 Application of Ritz method for analysis of solid mechanics problems, in particular plate analysis

The application of the Ritz method to solve the PDE that governs the plate vibration problems can be dated back to early 1960s. There are hundreds of published papers dealing with solid mechanics problems such as vibration and buckling of columns, plates and shells by employing the Ritz method. The excellent monographs on vibration of plates and shells by Leissa [72, 73] in 1969 and 1973 summarized the previous studies in these areas and many of the past researches were based on the Ritz methods.

Since 1970, the Ritz method gained more popularity in the analysis of plates, especially in solving the buckling and vibration of plates. Simons and Leissa [74] employed the Ritz method to analyse the transverse free vibration of plates by using mode shapes which are the sum of products of eigen-functions for vibrating beams. Bassily and Dickinson [75] extended the solutions for the cantilever plate in Simons and Liessa’s work to a more general solution. In their study, a series, composed of multiplications of beam vibration modes, was used to derive the frequency equations for plates with various combinations of boundary conditions and subject to any arbitrarily prescribed in-plane direct and shear stress patterns. Greif and Mittendorf [76] introduced a method for the vibration analysis of a wide class of beam, plate and shell problems including the effects of variable geometry and material properties. Filipich et al. [77] employed the Rayleigh-Ritz method to evaluate the natural frequencies of a stepped rectangular plate with elastically restrained edges. Kajita
and Naruoka [78] studied the free vibration of skew plates by the Rayleigh-Ritz method with B-spline functions as Ritz trial functions.

By the 1980s, the numerical solution of scientific and engineering problems became more attainable and convenient due to the introduction of personal computer technology. The Ritz method was employed to analyse plate problems by many researchers in this period of time as the implementation of the method is relatively easier and sometimes it is more efficient than the FEM and FDM. Tielking [79] employed the Ritz method to calculate the deformation stress of an isotropic, variable thickness, annular plate based on the potential energy formulation of von Karman plate theory. Dawe and Roufaeil [80] applied the Rayleigh-Ritz method to predict the natural frequencies of flexural vibration of isotropic plates when the effects of transverse shear and rotary inertia were taken into account. The Mindlin first-order shear deformable plate theory was employed in their study. Mizusawa and Kajita [81] applied the Rayleigh-Ritz method with B-spline functions to analyse the elastic limit load of skew plates which are commonly used as bridge decks and floors. They also studied the vibration and buckling of skew plates with various complicating effect by the Ritz method [82, 83]. In 1986, Baharlou and Leissa [84] developed a method to find frequencies and buckling loads for all the possible combinations of boundary conditions that can exist for generally laminated rectangular plates. Within the limitations of the classical linear theory, the Ritz method was used with modified polynomials as displacement functions to satisfy the geometric boundary conditions.

Liew and his associates proposed a p-version Ritz method to study the buckling and vibration of plates [85, 86]. In the trial function, a basic function which consists of the product of the plate boundary equations is multiplied to an orthogonal polynomial series. The geometric boundary conditions of the plate can be satisfied automatically. Their method has expanded the application of the Ritz method to plates of arbitrary shape and different combination of boundary conditions [87-89]. Liew and his colleagues has also conducted a series of studies on the 3-D vibration
analysis of rectangular, skew trapezoidal and skew plates using the Ritz method in conjunction with the orthogonal polynomial functions [90-92].

Leissa and Martin [93] analysed the structural behaviour of composite plates having nonuniformly spaced fibres. They were the first to treat the free vibration and buckling problems for composite plates having variable fibre spacing. A plane elasticity problem was solved to determine the in-plane stresses caused by the applied boundary loading, and these stresses became inputs to the vibration and buckling problems in their analysis. Both vibration and buckling of the composite plates were studied by the Ritz method. In 1991, Qatu and Leissa [94] studied the free vibration of cantilevered laminated composite shallow shells. Based on the thin shell theory, the Ritz method with algebraic polynomial displacement functions was employed to obtain vibration frequencies of thin cantilevered laminated plates and shallow shells having rectangular planforms.

McGee, Leissa and their associates studied the 3-D vibration of thick plates by the Ritz method. Leissa and Zhang [95] investigated the 3-D free vibration of cantilevered parallelepiped using the Ritz method with a set of simple algebraic polynomial being the Ritz trial functions. McGee and Leissa [96] determined the natural frequencies of skewed cantilevered thick plates. The work was the first known 3-D study of thick skew plates. The numerical studies revealed interesting trends concerning the variation of frequencies with increasing skew angle. McGee and Giaimo [97] presented the first-known 3-D vibration solutions for cantilevered right-angled triangular plates with varying thickness. In 1998, Kang and Leissa [98] applied the Ritz method in a three-dimensional analysis to obtain accurate frequencies for thick, linearly tapered, annular plates. Kang and Leissa [99] also analysed 3-D vibrations of thick spherical shell segments with variable thickness.

In 1996, Zhou [100] combined the sine series and polynomials as the basis functions in the Ritz method to solve the transverse vibration of thin elastic plates. He believed
that this method was more effective in solving these problems than some of the popular numerical methods (including the general FEM). In 1999, Cheung and Zhou [101] investigated the free vibrations of a wide range of tapered rectangular plates with an arbitrary number of intermediate line supports in one or two directions. Recently, Zhou and his associates [102-105] studied 3-D vibration problems for thick circular, triangular and rectangular plates with various complications. They employed the Ritz method with Chebyshev polynomials as the trial functions in their study.

In 2002, Wang and Reddy [106] investigated the problems of determining accurate stress resultants when using the Ritz method for corner supported rectangular plates under transverse uniformly distributed load. They proposed a remedy to address the problems encountered in the Ritz method to obtain accurate stress resultants in plates. Su and Xiang [107] presented a non-discrete approach for the vibration and buckling analysis of rectangular Kirchhoff plates with mixed edge support conditions. Romero et al. [108] studied a vibrating plate experimentally and analytically. Their analytical approach was based on the Rayleigh-Ritz method and on the use of nonorthogonal right triangular co-ordinates.

In 2004, Hu et al. [109] studied the vibration of angle-ply laminated plates with twist by the Ritz procedure. The Ritz trial functions were based on the normalized characteristic orthogonal polynomials generated by the Gram–Schmidt process. Liew and Zhao et al. [110-112] developed the mesh-free $kp$-Ritz method for the analysis of the free vibration of laminated two-side simply-supported cylindrical panels, postbuckling analysis of laminated composite plates, and conical panels. Recently, Kurpa, et al. [113] proposed an effective method to analyse the free vibration of arbitrary platform shallow shells using the Ritz method associated with the R-function.
2.5.3 Application of Ritz method for analysis of electromagnetic field problems

Although the Ritz method is less frequently used for electromagnetic field analysis than for solid mechanics, there has still been significant amount of studies in this area, especially in the analysis of waveguide problems.

In 1957, Vailancourt and Collin [114] employed the Rayleigh-Ritz method to obtain the eigenfunctions and eigenvalues of an inhomogeneously filled rectangular waveguide. Collin [115] also presented a work on the analysis of slotted dielectric interfaces based on similar methods.

Bulley [116] developed a computer program EHPOL to calculate the waveguide eigenvalues and eigenfunctions of arbitrarily shaped waveguides in 1970, based on the Ritz method with polynomial functions being used as the Ritz trial functions. Gish and Graham [117] studied the characteristic impedance and phase velocity of a dielectric-supported air strip transmission line with side walls. The Rayleigh-Ritz method was applied in the calculation of the characteristic impedance and phase velocity of the strip transmission line in their study. In 1977, Balaban [118] obtained the capacitance coefficients of multiconductor transmission lines on a planar surface using the Ritz method. A piece-wise polynomial function over a rectangular region was used as the Ritz trial function in the analysis.

In 1984, Kuttle [119] presented a new method for calculating the TE and TM cutoff frequencies of uniform waveguides with lunar or eccentric annular cross section. The method combined the intermediate method for lower bounds with the Rayleigh-Ritz method for upper bounds of the cutoff frequencies. In 1990, Fatić [120] developed a new variational principle, suitable for the application to dissipative systems, by considering the Lagrangian functions which depend on a set of generalized coordinates and their time-derivatives up to the second order. Young [121] proposed a modified Ritz method for the analysis of a general full-wave frequency-dependent waveguide problem. The method was a combination of the
classical Rayleigh-Ritz method and an additional optimization process. It was found that this method can be used to obtain accurate propagation constant with high computational efficiency. Lee et al. [122] studied the changes in the frequencies or velocities of guided EM waves in infinite and anisotropic dielectric plates affected by the uniform and thickness-dependent stress fields. They obtained the changes of frequencies as a function of wave number using the variational techniques and the Rayleigh-Ritz method.

In 2002, Krupka et al. [123] used the mode-matching and Rayleigh-Ritz methods for complex permittivity calculations using a TE\textsubscript{01δ} mode dielectric resonator to determine lower and upper bounds of the permittivity. The results were obtained by employing 126 basis functions with the Rayleigh-Ritz method and 10 basis functions with the mode-matching method. In 2003, Krupka et al. [124] employed the Garlerkin-Rayleigh-Ritz method to analyse the 3-D distribution of electromagnetic fields in multilateral dielectric resonators. Mahanfar et al. [125] presented a study on rectangular waveguides with curved corners. The Ritz method associated with the super-quadric function was employed to obtain the waveguide eigenvalues for the rectangular waveguides.

In 2004, Tadjalli et al. [126] employed the Ritz method to calculate the modes and resonant frequencies of elliptical cylinder dielectric resonator. Similar to Liew et al. [85, 86] in analysing vibration of plates, Tadjalli et al. [126] used the boundary equation of the elliptical cylinder dielectric resonator to construct the Ritz trial function that satisfies the geometric boundary conditions of the problem.

## 2.6 Meshless Method

The meshless method is now proven to be a robust numerical method for the analysis of electromagnetic field problems and solid mechanics problems. The meshless
method does not require the mesh generation and adaptive mesh updating such as the ones in the finite element analysis. It is well fit for many electromagnetic field and solid mechanics problems with large deformations, crack propagation and moving conductors.

In the meshless method, the problem domain is discretized by a set of scattered nodes and element connectivity among the nodes is not required. There are different ways to establish the shape functions. According to Belytschko et al. [127], in and before 1996, there existed three types of the models used in the meshless methods: (a) kernel method, (b) moving least square technique, and (c) partition of unity. In 1996, Belytschko et al. [127] compared the features of the three meshless models, discussed the issue of functions with discontinuities and discontinuous derivatives, and highlighted the approaches for implementation of essential boundary conditions.

The meshless method has received considerable research attentions from engineering and science disciplines, which has led to the many publications in distinct areas such as in solid mechanics and electromagnetic field analysis. One of the major reasons of the popularity is its versatility and flexibility for solving different engineering problems. For instance, in the simulation of manufacturing processes such as extrusion and moulding, it is vital to predict the behaviour of the object with extremely large deformation. In simulations of failure processes, the propagation of cracks with arbitrary and complex paths needs to be determined. The conventional computational methods such as the FEM, FDM or finite volume method are not suitable for dealing with such problems. The underlying structure of these methods with a predetermined mesh limits their applications to these problems. On the other hand, the meshless method is ideal to analyse such problems as the reliance on the predetermined mesh required by other methods is removed in the meshless method.

Belytschko and his associates were among the pioneers in developing the meshless method for the analysis of solid mechanics problems. According to Belytschko et al.
[127]. Nayroles et al. [128] were evidently the first to use moving least square approximations in a Galerkin method and called their approach the diffuse element method. In 1994, Belytschko et al. [129, 130] refined and modified the method and called their method the element free Galerkin (EFG) method. The method was based on the use of moving least-square interpolators in conjunction with the Galerkin method. The new method was applied to solve PDEs that require only nodal data and a description of the geometry. Using this method, Belytschko et al. [129] analysed elasticity of solids and heat conduction problems. The implementation of the EFG method for the problems of fracture and static crack growth was presented in [130]. Their study showed that accurate stress intensity factors could be obtained without any enrichment of the displacement field by a near-crack-tip singularity and that crack growth can be easily modelled since it requires hardly any remeshing. They later further developed and applied this method to solve large deformation [131], fracture and crack propagation [132-133], wave propagation [134] and plate and shell problems [135, 136].

Following Belyschko and his associates’ pioneering work, many researchers applied the meshless method to study solid mechanics problems. Babuska and Melenk [137] proposed the partition of unity method, another version of the meshless method, to solve various scientific and engineering problems. Chen et al. [138] took advantage of the meshless method to analyse metal forming process. Bonet and Kulasegaram [139] used the smooth particle hydrodynamics method to simulate the metal forming process.

In 1998, Atluri and Zhu [140] proposed a new meshless local Petrov-Galerkin method in the analysis of various computational mechanics problems. The moving least square technique was utilised in establishing the shape function in their approach. Liu and Gu [140, 142] proposed a local point interpolation method for the stress analysis of 2-D solids. In 2001, Gu and Liu [143] coupled the element free Galerkin method and the boundary element method to improve the solution
efficiency. Their method overcame two difficulties of the conventional meshless method: (a) the EFG shape functions along the combination boundary lack the Kronecker delta function property, and (b) the equivalent boundary element stiffness matrix is asymmetric.

Liew et al. [144] developed a reproducing kernel particle approach for the large deformation analysis of structures. The penalty technique was employed to enforce the geometric boundary conditions of structures. They later applied this method to analyse stresses in human proximal femur [145], elasto-plastic analysis of solids [146] and buckling of folded plates [147].

Rossi [148] investigated the $h$-adaptive Modified Element-Free Galerkin method in 2005. He proposed an error estimator based on a recovery by equilibrium of nodal patches where a recovered stress field was obtained by a moving least square approximation. In 2006, Belinha and Dinism [149] extended the EFG method with nodal direct integration to analyse anisotropic plates and laminates based on the Mindlin laminated plate theory. An alternative integration method avoiding the consideration of background cells was proposed in their study.

The meshless method has also been applied to analyse electromagnetic field problems in recent years. In 1998, Cingoski et al. [150] introduced the element-free Galerkin method for electromagnetic field computation in one- and two- dimensional space. Viana and Mesquita [151] applied the meshless moving least square reproducing kernel method to the solution of electromagnetic problems. Two-dimensional static problems were studied in this work.

Kim and Kim [152] presented the methodology of a mesh-free point collocation method and its application to the electromagnetic field computation in 2004. A point collocation scheme based on the moving least square technique was employed. Ho et al. [153] proposed a meshless method based on collocation with radial basis
functions and wavelets. The bridging scales were employed in their study to reserve the mathematical properties of the entire bases in terms of consistency and linear independence. Xuan et al [154] applied the EFG method for solving pulsed eddy-current problems.

In 2005, Zhao et al. [155] applied the element-free method to compute the cutoff wavenumbers of waveguides partially filled with dielectric medium. Zhang, et al. [156] presented a study in dealing with the problems on numerical oscillations of the solution in the element-free Galerkin method when it uses high order polynomial basis for electromagnetic problems. Zhou and Zheng [157] proposed a novel moving least square Ritz method which essentially is a meshless numerical method to study 2-D electromagnetic field problems. Chen et al. [158] employed the element-free Galerkin method to study giant magnetostrictive thin films. They believed that the meshless method can overcome the numerical difficulty facing the FEM when a large distortion of the film occurs.

2.7 Other Emerging Methods

2.7.1 Differential Quadrature (Collocation-Interpolation) Method

The differential quadrature method (DQM) was introduced by Bellman and his associates [159-161] in the early 1970s for solving linear and nonlinear PDEs. The differential quadrature approximates the partial derivative of a function with respect to a space variable at a given discrete point as a weighted linear sum of the function values at all discrete points. This is in contrast to the standard FDM in which a solution value at a point is a function of values at adjacent points only [162]. The DQM discretizes the original continuous model (and problem) into a discrete (in space) model, with a finite number of degrees of freedom, while the initial-boundary
value problem is transformed into an initial-value problem for ordinary differential equations [163].

The DQM is a rather efficient numerical method for rapid solution of linear and nonlinear PDEs. It has been applied successfully to solve a wide range of problems such as nonlinear diffusion [164] and system identification [165]. In 1984, Civan and Sliepcevich [166] applied this method and elaborated upon further to solve multidimensional problems.

Since 1988, Bert and his associates have successfully applied this method in the solution of solid mechanics problems in a series of excellent papers [162, 167-172]. The problems they studied includes: large deflection analysis, free vibration of isotropic structural components, behaviour of thin, circular, isotropic elastic plates with immovable edges and undergoing large deflections, and buckling and free vibration of beams and rectangular plates under various boundary conditions and with different material properties.

The early DQMs employed the classic polynomials, including the Legendre polynomials, based on collocation technique to approximate the function value and its derivatives [159-162, 167]. These DQMs often encounter severe numerical instability problem when the number of grid points is greater than 13. Shu and Richard [173] proposed a generalized differential quadrature method (GDQM) that employs the Lagrange interpolating functions to approximate a function value and its derivatives. Their method improved the numerical stability of the DQM, and they applied the GDQM to analyse many scientific and engineering problems [174-176].

There are many other researchers who have also contributed in the development of the DQM. Striz et al. [177] presented a hybrid quadrature element method to analyse the free vibration of plates in 1997. The hybrid method combined the advantages of both the DQM and FEM. In 1999, Liew and Teo [178] employed the DQM to
analyse plate problems based on the 3-D elasticity plate model. Liew et al. [179] presented a study for the vibration analysis of moderately thick symmetrically laminated composite plates. Their DQM was based on the moving least square interpolation. In 2001, Wu and Liu [180, 181] presented a generalized differential quadrature rule that they proposed earlier to analyse vibration of circular plate and circular arches. The circular arches are governed by a sixth-order differential equation and constrained by three boundary conditions at each end.

Wang and Wang [182] proposed a new version of the DQM in 2004 to analyse the free vibration of thin sector plates with various sector angles and different combinations of boundary conditions. Recently, Zhang et al. [183] presented an adaptive local DQM to investigate the vibration behaviour of circular cylindrical shells. Instead of using all grid points to approximate a function value and its derivatives at a given point, the local adaptive DQM is able to use a set of predefined number of grid points on each side of the given point. It improves the computational efficiency of the DQM, especially for the analysis of large scale problems.

2.7.2 Discrete singular convolution method
The discrete singular convolution algorithm was proposed by Wei in 2000 [184]. The discrete singular convolution (DSC) method is a potential numerical approach for solving many scientific and engineering problems governed by differential equations. Mathematically, the DSC is the theory of distributions and wavelet analysis. A function value and its derivatives at a given point can be approximated by the function values of several points on each side of the given point. The implementation of the DSC is within the collocation scheme.

Wei et al. [185] applied the DSC algorithm for the vibration analysis of rectangular plates with mixed boundary conditions in 2001. A unified scheme was developed to implement the DSC algorithm for the analysis of plates with arbitrarily mixed boundary conditions and their results showed good agreement with those in the open
Zhao et al. [186] investigated a challenging problem in structural analysis, high frequency vibration of plates, and demonstrated the ability of the DSC algorithm for such problem by providing extremely accurate frequency parameters for plates vibrating in the first 5000 modes.

In 2003, Shao et al. [187] introduced the DSC algorithm for solving scattering and guided wave problems described by time-domain Maxwell’s equations. The study explored the ability of the DSC algorithm to obtain accurate results and demonstrated that the DSC time-domain algorithm was cost-efficient. In 2006, Civalek [188] carried out the vibration analysis of laminated conical and cylindrical shell by the DSC method. The results of the study showed that the convergence of the DSC approach was very good and the results agreed well with those obtained by other researchers.

### 2.8 Conclusions

This chapter reviewed the existing important numerical methods in science and engineering computations. It highlighted some of the advantages and disadvantages of these methods. The literature review showed that numerical computations are indispensable tools that are essential in science and engineering research and practices. With the exponential increase of computing power in the last few years, numerical methods became even more important and had broader applications not only in science and engineering fields but also in finance, medical research, defence and so on.

The literature review also revealed that no single method is perfect and can solve all problems. There is a great need for improving the existing methods and developing new methods to meet the challenges to solve complex problems. This study is
focused on the development of a novel numerical method, the MLS-Ritz method, that will significantly improve the existing Ritz and meshless methods in dealing with solid mechanics and electromagnetic field problems.
CHAPTER 3

MOVING LEAST SQUARE INTERPOLATION

3.1 Introduction

The least square method was developed for curve fitting [189]. As the raw data usually contains noise, even though all the control parameters (the independent variables) are constant, the resultant outcomes which are dependent on the variables are varying. It would be ideal if we could predict one quantity exactly in term of another, but this is seldom possible. Therefore, predicting the trend of the dependent variables is needed. This process is called regression or curve fitting. The estimated equation (matrix) satisfies the raw data. However, for a given set of data, the equation is not usually unique. Thus, a curve with a minimal deviation from all data points is desired. The least square method is developed to find the best-fitting equation. This method involves all the data points in the whole domain. This desirable best-fitting equation can be obtained by the least square method which uses the minimal sum of the deviations squared from a given set of data.

The moving least square method was initially introduced for the same curve fitting purpose. However, to fit a fixed point, a weighted least squares formulation is formed with data points within a given domain of influence. The weighted function has a positive value between 0 and 1. The value of the weighted function decreases as the distance between the fixed point and the data point increases. This value becomes
zero when the data point is outside the domain of influence. Repeating this procedure for all pre-selected fixed points, we can obtain the curve fitting coefficients based on the raw data points. The moving least square method is a powerful tool for data interpolation, smoothing and derivative approximations. The method is considered as the near-best in the sense that the local error is bounded in terms of the error of local best polynomial approximation [190]. Because of this characteristic, the moving least square method is employed recently in conjunction with other numerical method, i.e. the meshless method, to effectively analyse solid mechanics and electromagnetic field problems [127, 191 – 192].

The purpose of this chapter is to introduce the concept of the moving least square method and its mathematical formulation. Section 3.2 describes the moving least square method from the mathematical point of view. As the weighted function is a very important part of the method, a brief review of the weighted functions used for the moving least square method is given in this section. Section 3.3 presents the mathematical formulation of 2-D moving least square method for the interpolation of a 2-D function. Section 3.3 shows the derivatives of moving least square shape function. Section 3.4 presents the procedure for the calculation of the moving least square shape function and its derivatives in 3-D cases. Section 3.5 concludes this chapter.

### 3.2 Moving Least Square Method

The moving least square (MLS) method is built based on the least square method. In order to understand the concept of the moving least square method, it is necessary to start with the method of least squares.
3.2.1 Least square method

The method of least squares is used for curve fitting of a given set of data. It is assumed that the best-fit curve for a given set of data has the minimal sum of the deviations squared (least square error) between the points on the fitted curve and the given set of data. If we have a set of data \( x_i \) (for 2-D case, \( x_i = (x_i, y_i) \), or for 3-D case \( x_i = (x_i, y_i, z_i) \), respectively) which has the number of \( n \) points \( (x_1), (x_2), \ldots, (x_n) \), we wish to obtain a globally defined function \( f(x) \) that approximates the given scalar values \( f_i \) at point \( x_i \). We can assume that the fitting curve function \( f(x) \) has deviations \( d_1, d_1, \ldots, d_n \) from each data point, i.e., \( d_i = f_i - f(x_i) \), \( d_2 = f_2 - f(x_2) \), \ldots, \( d_n = f_n - f(x_n) \). Based on the least square approach, the best fitting curve has the minimum least square error [193]:

\[
\Pi = d_1^2 + d_2^2 + \ldots + d_n^2 = \sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} \| (f_i - f(x_i)) \|^2 = \text{minimum}
\]

(3.1)

Here the \( f \) is taken from the polynomials of total degree and the spatial dimensions, and it can be written as

\[
f(x) = p(x)^T a
\]

(3.2)

where \( p(x) = [p_1(x), \ldots, p_m(x)]^T \) is the \( m \) terms of polynomial basis function or any other set of basis function which forms a complete space, and \( a = [a_1, \ldots, a_m]^T \) is the unknown coefficients. The different types of least squares depend on the form of \( f \) [190, 194]. The common least square types are: the least square line, the least square parabola, the least square polynomials, and the multiple regression least square.

3.2.2 Moving least square method

The moving least square approximation at a point \( x \) is to take the weighted least square form over a pre-defined domain of influence

\[
\Pi = \sum_{i=1}^{N} g(\| x - x_i \|) \| (f_i - f(x)) \|^2 = \text{minimum}
\]

(3.3)
where \( n_d \) is the number of points \( x_i \) within the influence domain of point \( x \), \( g(\|x-x_i\|) \) is the non-negative weight function, and \( \|x-x_i\| \) is the Euclidean distance between points \( x \) and \( x_i \). Note that the coefficients \( a = [a_1, \ldots, a_m]^T \) in Eq. (3.1) associated with the least square method are a set of fixed values determined from the minimizing process of Eq. (3.1). However, the coefficients \( a = [a_1, \ldots, a_m]^T \) associated with the moving least square method are not fixed values, but vary with the change of the interpolating point \( x \). The method to determine the coefficients \( a \) will be discussed in Section 3.3.

### 3.2.3 Moving least square weight functions

In the moving least square fitting, the weight function can take different forms. According to Belytschko [127], there are three commonly used weight functions, which are the exponential, the cubic spline and quartic spline. For a 2-D problem, if

\[
r = \sqrt{(x-x_i)^2+(y-y_i)^2}/d
\]

is the normalized distance between the point \((x,y)\) and the \(i\)th grid point \((x_i,y_i)\), \(d\) is the radius of support (see Figure. 3.1) and \(k\) is an integer which can be adjusted to optimize the MLS fitting, then the weight functions are

- **exponential:**
  
  \[
g_i(r) = \begin{cases} 
  e^{-(r/k)^2} & \text{for } r \leq 1 \\
  0 & \text{for } r > 1 
  \end{cases} \tag{3.4}
\]

- **cubic spline:**
  
  \[
g_i(r) = \begin{cases} 
  \frac{2}{3} - 4r^2 + 4r^3 & \text{for } r \leq \frac{1}{2} \\
  \frac{4}{3} - 4r + 4r^2 + \frac{4}{3}r^3 & \text{for } \frac{1}{2} < r \leq 1 \\
  0 & \text{for } r > 1 
  \end{cases} \tag{3.5}
\]

- **quartic spline:**
  
  \[
g_i(r) = \begin{cases} 
  1 - 6r^2 + 8r^3 - 3r^4 & \text{for } r \leq 1 \\
  0 & \text{for } r > 1 
  \end{cases} \tag{3.6}
\]

The cubic spline used by many researchers for the Meshless Galerkin method [127]. Liew et al. [191] used the Gaussian type with a circular support for the MLS approximation, which takes the form of
Gaussian: \( g_i(r) = \begin{cases} \frac{e^{-(rd/c)^2} - e^{-(d/c)^2}}{1 - e^{-(d/c)^2}} & \text{for } r \leq 1 \\ 0 & \text{for } r > 1 \end{cases} \) (3.7)

where the constant \( c = r/4 \) was used in Liew et al. [191].

We propose the following weight function to be used in this study:

\[
g_i(r) = \begin{cases} (1-r^2)^k & \text{if } r \leq d \\ 0 & \text{if } r > d \end{cases}
\] (3.8)

All the above-mentioned weight functions have similar trend, i.e. the value of the functions approaches 1 when \( r \) approaches zero and the value of the functions approaches zero when \( r \) is close to the value of \( d \). Figure 3.1 shows the typical trends of the weight function Eq. (3.8). The integer \( k \) can be used to control the distribution of the weight function with respect to the distance \( r \).

Figure 3.1 Influence of the value \( k \) on the MLS-Ritz weight function
3.3 Moving Least Square Interpolation Scheme for 2-D Function

3.3.1 2-D MLS shape function
Section 3.2 outlined the basic principle and formulations for the moving least square method. However, the coefficients of $a = [a_1, \ldots, a_m]^T$ need to be determined. Figure 3.2 shows a calculated domain in a Cartesian coordinate system.

![Figure 3.2 Illustration the distribution of the MLS-Ritz points](image)

Assume that at an arbitrary point $(x, y)$ (see Figure. 3.2), the approximate function of the $f(x, y)$ is $f^h(x, y)$. Apply the moving least square method [191, 195], the approximate function $f^h(x, y)$ on the calculated domain can be approximately evaluated as

$$f^h(x, y) = \sum_{i=1}^{m} p_i(x, y)a_i = p^T(x, y)a \quad (3.9)$$
where \( m \) is the number of basis functions which can take various function forms (i.e. polynomial, sinusoidal functions, etc.) that form a complete space. The finite set of function \( p(x, y) = [p_1(x, y) \ p_2(x, y) \ \cdots \ p_m(x, y)]^T \) forms a complete space, and \( a = [a_1 \ a_2 \ \cdots \ a_m]^T \) is the unknown coefficients, respectively. In this study, we propose to use the 2-D complete polynomial as the basis function.

If the degree of the complete 2-D polynomial is taken to be \( P = 2 \), then the basis function is of the form with \( m = 6 \),

\[
p(x, y) = [1 \ x \ y^2 \ xy \ y^2]^T
\]

(3.10)

The unknown coefficients \( a \) can be determined by minimizing the following weighted quadratic form

\[
\Pi(a) = \sum_{i=1}^{n} g_i(r)(f^h(x_i, y_i) - f_i)^2 = \sum_{i=1}^{n} g_i(r)(p^T T (x_i, y_i)a - f_i)^2
\]

(3.11)

where \((x_i, y_i), i = 1, 2, \ldots, n\), are the \( n \) grid points in the neighbourhood of the point \((x, y)\), \( f_i \) is the nominal displacement at point \((x_i, y_i)\), and \( g_i(r) \) is the weight function.

The weight function has been presented in Eq. (3.8) in Section 3.2. The distance \( r \) is the normalized distance between the point \((x, y)\) and the \( i \)th grid point \((x_i, y_i)\),

\[
r = \frac{(x - x_i)^2 + (y - y_i)^2}{d}, \text{ where } d \text{ is the radius of support (see Figure. 3.2) and } k \text{ is an integer which can be adjusted to optimize the MLS fitting.}
\]

Minimizing Eq. (3.11) with respect to the unknown coefficients \( a \), the unknown coefficients can be obtained as follows:

\[
A(x, y)a = B(x, y)f
\]

(3.12)
Chapter 3 Moving Least Square Interpolation

\[ a = A^{-1}Bf \]  \hspace{1cm} (3.13)

where

\[ A(x, y) = \sum_{i=1}^{n} g_i(r) p(x_i, y_i) p^T(x_i, y_i) \]  \hspace{1cm} (3.14)

\[ B(x, y) = [g_1(r) p(x_1, y_1) \quad g_2(r) p(x_2, y_2) \cdots g_n(r) p(x_n, y_n)] \]  \hspace{1cm} (3.15)

\[ f = [f_1 \quad f_2 \cdots f_n]^T \]  \hspace{1cm} (3.16)

Substituting Eq. (3.13) into Eq. (3.9), the approximate function of the \( f(x, y) \) can be expressed in terms of the nominal values of the grid points within the radius of support \( d \) as:

\[ f(x, y) = f^h(x, y) = \sum_{i=1}^{n} R_i(x, y) f_j = R f = f^T R^T \]  \hspace{1cm} (3.17)

\[ R = [R_1(x, y) \quad R_2(x, y) \cdots R_i(x, y) \cdots R_n(x, y)] \]  \hspace{1cm} (3.18)

\[ R_i(x, y) = p^T(x, y) A^{-1}(x, y) g_i(r) p(x_i, y_i) \]  \hspace{1cm} (3.19)

We may also express the function in terms of the nominal values of all grid points in the calculation domain as

\[ f(x, y) = f^h(x, y) = \sum_{i=1}^{N} R_i(x, y) f_j = R f = f^T R^T \]  \hspace{1cm} (3.20)

where \( N \) is the total number of grid points in the calculation domain. The shape function \( R_i(x, y) \) may be evaluated using Eq. (3.19) if the \( i \)th grid point \((x_i, y_i)\) is within the radius of support \( d \) of point \((x, y)\), or \( R_i(x, y) = 0 \) otherwise.
3.3.2 Differentiation of 2-D MLS shape function

The interpolated function \( f(x, y) \) is continuously differentiable if and only if the weight function is continuously differentiable. The first and second derivatives of the function \( f(x, y) \) in Eq. (3.20) with respect to \( x \) and \( y \) can be expressed as

\[
\frac{\partial f(x, y)}{\partial x} = \frac{\partial f^h(x, y)}{\partial x} = \sum_{i=1}^{N} \frac{\partial R_i(x, y)}{\partial x} w_i = R_x f = f^T R_x^T
\]  

(3.21a)

\[
\frac{\partial f(x, y)}{\partial y} = \frac{\partial f^h(x, y)}{\partial y} = \sum_{i=1}^{N} \frac{\partial R_i(x, y)}{\partial y} w_i = R_y f = f^T R_y^T
\]  

(3.21b)

\[
\frac{\partial^2 f(x, y)}{\partial x^2} = \frac{\partial^2 f^h(x, y)}{\partial x^2} = \sum_{i=1}^{N} \frac{\partial^2 R_i(x, y)}{\partial x^2} f_i = R_{xx} f = f^T R_{xx}^T
\]  

(3.21c)

\[
\frac{\partial^2 f(x, y)}{\partial y^2} = \frac{\partial^2 f^h(x, y)}{\partial y^2} = \sum_{i=1}^{N} \frac{\partial^2 R_i(x, y)}{\partial y^2} f_i = R_{yy} f = f^T R_{yy}^T
\]  

(3.21d)

\[
\frac{\partial^2 f(x, y)}{\partial x \partial y} = \frac{\partial^2 f^h(x, y)}{\partial x \partial y} = \sum_{i=1}^{N} \frac{\partial^2 R_i(x, y)}{\partial x \partial y} f_i = R_{xy} f = f^T R_{xy}^T
\]  

(3.21e)

in which

\[
R_x = \begin{bmatrix} \frac{\partial R_1(x, y)}{\partial x} & \frac{\partial R_2(x, y)}{\partial x} & \ldots & \frac{\partial R_i(x, y)}{\partial x} & \ldots & \frac{\partial R_N(x, y)}{\partial x} \end{bmatrix}
\]  

(3.22a)

\[
R_y = \begin{bmatrix} \frac{\partial R_1(x, y)}{\partial y} & \frac{\partial R_2(x, y)}{\partial y} & \ldots & \frac{\partial R_i(x, y)}{\partial y} & \ldots & \frac{\partial R_N(x, y)}{\partial y} \end{bmatrix}
\]  

(3.22b)

\[
R_{xx} = \begin{bmatrix} \frac{\partial^2 R_1(x, y)}{\partial x^2} & \frac{\partial^2 R_2(x, y)}{\partial x^2} & \ldots & \frac{\partial^2 R_i(x, y)}{\partial x^2} & \ldots & \frac{\partial^2 R_N(x, y)}{\partial x^2} \end{bmatrix}
\]  

(3.22c)

\[
R_{yy} = \begin{bmatrix} \frac{\partial^2 R_1(x, y)}{\partial y^2} & \frac{\partial^2 R_2(x, y)}{\partial y^2} & \ldots & \frac{\partial^2 R_i(x, y)}{\partial y^2} & \ldots & \frac{\partial^2 R_N(x, y)}{\partial y^2} \end{bmatrix}
\]  

(3.22d)

\[
R_{xy} = \begin{bmatrix} \frac{\partial^2 R_1(x, y)}{\partial x \partial y} & \frac{\partial^2 R_2(x, y)}{\partial x \partial y} & \ldots & \frac{\partial^2 R_i(x, y)}{\partial x \partial y} & \ldots & \frac{\partial^2 R_N(x, y)}{\partial x \partial y} \end{bmatrix}
\]  

(3.22e)
3.3.3 Evaluation of 2-D MLS shape function and its derivatives

The partial derivatives of the shape functions can be directly calculated by the partial derivative of the function \( R \) from Eq. (3.19). As this computation would involve the inversion of the matrix \( A \), it is a very time consuming process. In this study, we employ a procedure proposed by Belytschko [195] and Liew [191]. The method used by Belytschko and Liew is very effective to reduce the computational cost and avoid the ill-conditioning of the inverse of matrix \( A \).

In order to transform the Eq. (3.19), we can assume a coefficient vector \( \gamma(x, y) \), which can be expressed as:

\[
\gamma^T (x, y) = p^T (x, y) A^{-1} (x, y)
\]

or

\[
A(x, y) \gamma(x, y) = p(x, y)
\]

Substituting Eq. (3.23) into Eq. (3.19), the \( R_i (x, y) \) becomes

\[
R_i (x, y) = p^T (x, y) A^{-1} (x, y) g_i (r) p(x_i, y_i) = \gamma^T (x, y) g_i (r) p(x_i, y_i)
\]

Therefore, the computation of the shape function and its partial derivatives becomes to calculate the coefficients vector \( \gamma(x, y) \) and its derivatives.

The LU decomposition and back-substitution are employed to calculate the coefficients of \( \gamma(x, y) \) as it requires fewer computations than the inversion of \( A(x, y) \). Similarly, any order of the partial derivatives of \( \gamma(x, y) \) can be determined by the same matrix after LU decomposition of \( A(x, y) \) as in Eq. (3.25).

The following example demonstrates the details of the calculation of the derivative and the weight coefficients [195].

Taking the derivative of Eq. (3.24) along the \( I \) direction in Cartesian coordinates, we have
and re-arranging Eq. (3.26), we obtain

\[ A(x, y)\gamma_f(x, y) = p_f(x, y) - A_f(x, y)\gamma(x, y) \] (3.27)

where \( ()_I \) represents the derivative along the \( I \) direction \((I = x \text{ or } y)\). Eq. (3.27) shows that only one back-substitution is required to obtain \( \gamma_f(x, y) \). The higher order derivatives can also be dealt with in the same way. For example, taking the second-order derivative of Eq. (3.24), we have

\[ A(x, y)\gamma_{,IJ}(x, y) = p_{,IJ}(x, y) - A_{,IJ}(x, y)\gamma(x, y) - A_f(x, y)\gamma_{,IJ}(x, y) - A_{,IJ}(x, y)\gamma_f(x, y) \] (3.28)

where \( ()_{IJ} \) represents the second derivative along the \( I \) and \( J \) directions, where \( I = x \) or \( y \) and \( J = x \) or \( y \), respectively. The second-order derivatives of the \( \gamma(x, y) \) can be determined with a back-substitution using the new right hand vector.

The shape functions can then be obtained accordingly using Eq. (3.25). The following shows the first-order and the second-order derivatives of the shape functions

\[ R_{,i}(x, y) = \left( \gamma^T_f(x, y) g_i(r) + \gamma^T(x, y) g_{,i}(r) \right) p(x_i, y_i) \] (3.29)

And

\[ R_{,ij}(x, y) = \left( \gamma^T_{,IJ}(x, y) g_i(r) + \gamma^T_{,IJ}(x, y) g_{,i}(r) + \gamma^T_{,IJ}(x, y) g_{,i}(r) + \gamma^T(x, y) g_{,i}(r) + \gamma^T(x, y) g_{,i}(r) \right) p(x_i, y_i) \] (3.30)

### 3.4 3-D Moving Least Square Formulation

For a set of preselected points \( x_i = (x_i, y_i, z_i) \), where \( i = 1, 2, \ldots, N \), in the calculation domain of a 3-D problem, the moving least square technique is applied to approximate the function \( f \) at an arbitrary point \( x = (x, y, z) \).
The function $f(x, y, z)$ at an arbitrary point on the calculation domain can be approximately evaluated by

$$f^h(x, y, z) = \sum_{i=1}^{m} P_i(x, y, z)a_i = \mathbf{P}^T(x, y, z) \mathbf{a}$$  \hspace{1cm} (3.31)$$

where the $f^h(x, y, z)$ is the approximate value of $f(x, y, z)$, $m$ is the number of the basis functions, $\mathbf{P}(x, y, z) = [p_1(x, y, z) \quad p_2(x, y, z) \cdots \quad p_m(x, y, z)]^T$ is a row matrix containing the $m$ terms of basis functions which can take various function forms (i.e. polynomial, sinusoidal functions, etc.) of a complete space, and $\mathbf{a} = [a_1 \quad a_2 \cdots \quad a_m]^T$ is the unknown coefficients, respectively.

As proposed in Section 3.3, a complete 3-D polynomial is used to form the basis function. With the degree of the complete 3-D polynomial $P = 2$, the basis function has ten terms ($m = 10$)

$$p(x, y, z) = [1 \quad x \quad y \quad z \quad x^2 \quad y^2 \quad z^2 \quad xy \quad xz \quad yz]^T$$  \hspace{1cm} (3.32)$$
The unknown coefficients $\mathbf{a}(x)$ can be obtained by minimizing the following quadratic form:

$$\Pi(\mathbf{a}) = \sum_{i=1}^{n} g_i(r)(f(x_i, y_i, z_i) - f_i)^2 = \sum_{i=1}^{n} g_i(r)(\mathbf{P}^T(x_i, y_i, z_i) \mathbf{a} - f_i)^2$$  \hspace{1cm} (3.33)$$

where $i = 1, 2, \ldots, n$, are the $n$ grid points in the neighbourhood of the point $(x, y, z)$, $f_i$ is the nominal value of the function $f$ at point $(x_i, y_i, z_i)$, and $g_i(r)$ is the weight function used in the MLS fitting. The weight function used for the 3-D study takes the same form as in the 2-D case, except for $r = \sqrt{(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2}/d$.

Minimizing Eq. (3.33) with respect to $\mathbf{a}$, we have

$$\frac{\partial \Pi}{\partial \mathbf{a}} = \sum_{i=1}^{n} g_i(r) 2(\mathbf{P}^T(x_i, y_i, z_i) \mathbf{a} - u_i)\mathbf{P}^T(x_i, y_i, z_i) = 0$$  \hspace{1cm} (3.34)$$
The coefficients $a$ can be determined as
\[
A(x, y, z)a = B(x, y, z)f
\] (3.35)
\[
a = A^{-1}Bf
\] (3.36)
where
\[
A(x, y, z) = \sum_{i=1}^{n} g_i(r)p(x_i, y_i, z_i) p^T(x_j, y_j, z_j)
\] (3.37)
\[
B(x, y, z) = \left[ g_1(r)p(x_1, y_1, z_1) \ g_2(r)p(x_2, y_2, z_2) \ \cdots \ g_n(r)p(x_n, y_n, z_n) \right]
\] (3.38)
\[
f = [f_1 \ f_2 \ \cdots \ f_n]^T
\] (3.39)

### 3.4.1 3-D MLS shape function
Following the same procedure presented in Section 3.3, the function in terms of the nominal displacement values within the domain of influence is
\[
f(x, y, z) \approx f^h(x, y, z)
\]
\[
= \sum_{i=1}^{n} p^T(x, y, z) A^{-1}(x, y, z) \omega_i(x, y, z) p(x, y, z_i) f_i
\] (3.40)
\[
= \sum_{i=1}^{n} R_i(x, y, z) f_i
\]
where $f_i$ is the nominal value of the function and $R_i(x, y, z)$ is the shape function of the $i$th point, and $n$ is the number of points within the neighbourhood of point $(x, y, z)$. We can also express the function in terms of all MLS-grid points on the calculation domain
\[
f(x, y, z) = \sum_{i=1}^{N} R_i(x, y, z) f_i = Rf = f^T R^T
\] (3.41)
\[
R = \left[ R_1(x, y, z) \ R_2(x, y, z) \ \cdots \ R_i(x, y, z) \ \cdots \ R_n(x, y, z) \right]
\] (3.42)
\[
R_i(x, y, z) = p^T(x, y, z) A^{-1}(x, y, z) g_i(r)p(x_j, y_j, z_j)
\] (3.43)
where \( N \) is the total number of MLS grid points in the calculation domain, and the
shape function can be evaluated by

\[
\begin{aligned}
R_i(x, y, z) &= \mathbf{P}^T(x, y, z)\mathbf{A}^{-1}(x, y, z)g_i(r)\mathbf{P}(x_i, y_i, z_i), & \text{if } r \leq d \\
R_i(x, y, z) &= 0, & \text{if } r > d
\end{aligned}
\]

(3.44)

### 3.4.2 Differentiation of 3-D MLS shape function

The first derivatives of the function \( f(x, y, z) \) with respect to \( x, y \) and \( z \) can be
expressed as

\[
\frac{\partial f}{\partial x} = \frac{\partial f^h}{\partial x} = \sum_{i=1}^{N} \frac{\partial R_i(x, y, z)}{\partial x} f_i = \mathbf{R}_x^T \mathbf{f} = \mathbf{f}^T \mathbf{R}_x^T
\]

(3.45a)

\[
\frac{\partial f}{\partial y} = \frac{\partial f^h}{\partial y} = \sum_{i=1}^{N} \frac{\partial R_i(x, y, z)}{\partial y} f_i = \mathbf{R}_y^T \mathbf{f} = \mathbf{f}^T \mathbf{R}_y^T
\]

(3.45b)

\[
\frac{\partial f}{\partial z} = \frac{\partial f^h}{\partial z} = \sum_{i=1}^{N} \frac{\partial R_i(x, y, z)}{\partial z} f_i = \mathbf{R}_z^T \mathbf{f} = \mathbf{f}^T \mathbf{R}_z^T
\]

(3.45c)

The higher order derivatives of the function \( f(x, y, z) \) have not been encountered
in the 3-D applications in this thesis. However, it can be readily derived in the same
manner as for the 2-D cases in Section 3.3.2.

### 3.4.3 Evaluation of 3-D MLS shape function and its derivatives

Employing the same procedure as in Section 3.3.3, the 3-D MLS shape function and
its derivatives can be evaluated as follows.

First, we assume

\[
\mathbf{\gamma}^T(x, y, z) = \mathbf{p}^T(x, y, z)\mathbf{A}^{-1}(x, y, z)
\]

(3.46)

and from Eq. (3.4-17) we can obtain

\[
\mathbf{A}(x, y, z) \mathbf{\gamma}(x, y, z) = \mathbf{p}(x, y, z)
\]

(3.47)
Then substituting Eq. (3.47) into Eq. (3.43), we can express the $i$-th term of the shape function $R_i(x, y, z)$ as

$$R_i(x, y, z) = p^T (x, y, z) A^{-1}(x, y, z) g_i(r) p(x_i, y_i, z_i)$$

$$= \gamma^T (x, y, z) g_i(r) p(x_i, y_i, z_i)$$

(3.48)

in which $\gamma(x, y, z)$ can be obtained from Eq. (3.47) using the LU decomposition of $A(x, y, z)$ and the back-substitution procedures.

The first derivative of $R_i(x, y, z)$ can be obtained by the following expression

$$R_{i,I}(x, y, z) = \left( \gamma_{i,I}^T (x, y, z) g_i(r) + \gamma^T (x, y, z) g_{i,I}(r) \right) p(x_i, y_i, z_i)$$

(3.49)

where $(,)$ represents the derivative along the $I$ direction ($I = x, y$ or $z$) and $\gamma_{i,I}(x, y, z)$ can be obtained using the LU decomposition $A(x, y, z)$ and back-substitution from the following equation

$$A(x, y, z)\gamma_{i,I}(x, y, z) = p_{i,I}(x, y, z) - A_{i,I}(x, y, z)\gamma(x, y, z)$$

(3.50)

### 3.5 Conclusions

In this chapter, a brief introduction of the moving least square method has been presented. The moving least square data interpolation technique has been utilized to establish the approximate value $f^h(x)$ of the function $f(x)$ in the 2-D and 3-D cases. The details of the computation for the shape function and its derivatives have been presented. The moving least square interpolation in conjunction with the Ritz method will be used in the subsequent chapters to study the vibration of plates and electromagnetic field problems.
CHAPTER 4

APPLICATION OF MLS-RITZ METHOD FOR VIBRATION ANALYSIS OF RECTANGULAR AND TRIANGULAR PLATES

4.1 Introduction

Plate structures are one of the most important types of structures used in civil, mechanical, marine and aerospace engineering. Vibration of plates has been studied extensively since 1787 [27, 89, 196-199] due to its importance in the design of plate structures and many of the important studies in this field were documented in Leissa’s monograph [27] and a series of reviews [200-205].

Various analytical and numerical methods have been developed to investigate the vibration behaviour of plates, ranging from the superposition method [206-208], Levy approach [199, 209-210], point collocation method [211], finite difference method [212], differential quadrature (DQ) method [213], Ritz method [89], meshless method [27] to the finite strip method and the finite element (FE) method [214-215]. Although analytical methods are important to give an insightful understanding of the vibration behaviour and to provide benchmark frequencies of plates, numerical methods are preferred in the vibration analysis of plates due to the fact that most of the plate vibration problems do not admit analytical solutions. While the FE method is still the
dominant numerical method in this field, many alternative methods such as the finite strip method, Ritz method and DQ method are developed to improve the efficiency and accuracy of vibration analysis of plates. Cheung [214] proposed the finite strip method for plate analysis which is proven to be highly efficient to deal with plates of regular shapes. The p-Ritz method developed by Liew and his associates [89] in the past decade has made a significant impact in the vibration analysis of plates and shells. Their approach is able to enforce the geometric boundary conditions of plates automatically and is able to deal with plates of various shapes and different internal line supports. The DQ method proposed by Bellman etc. [160] starts to make its impact in the area of plate analysis and a large number of publications can be found in the open literature [176, 213]. Recently, the discrete singular convolution (DSC) method developed by Wei and his associates [216] showed great potential in the analysis of plates, especially in the high frequency analysis of plates [186].

The moving least square (MLS) technique was originally used for data fitting [127]. In recent years, researchers have applied the MLS technique in the analysis of solid mechanics problems by developing the meshless method or element free Garlerkin method [127]. The MLS technique is employed to establish the shape functions in the numerical analysis process. The MLS technique was also applied in conjunction with the DQ approach to analyse the bending and buckling of plates [217] and electromagnetic field problems [218].

This chapter presents an application of the MLS-Ritz method for the free vibration analysis of classical thin plates. The classical thin plate (Kirchhoff plate) theory is employed in this study. The proposed method utilizes the strength of the moving least square approach to define the Ritz trial function for the transverse displacement of the plates. A set of points is pre-selected on the calculation domain of a plate that forms the basis for the MLS-Ritz trial function. The edge support conditions of the plate are satisfied by forcing the boundary points to meet the geometric boundary conditions of
the plate via a point substitution technique. Virtual points (points outside the plate domain) are introduced for clamped edges to improve the convergence and accuracy of the calculations. Square and right-angled isosceles triangular plates of various combinations of edge support conditions are selected to examine the validity and accuracy of the MLS-Ritz method. Extensive convergence studies are carried out to investigate the influence of the MLS mesh size, the MLS support radius, the number of Gaussian integration points and the shape of the MLS weight function on the proposed method. Comparing to the existing Ritz methods, the MLS-Ritz method is highly stable and accurate and is extremely flexible for dealing with plates of arbitrary shapes and boundary conditions.

The layout of this chapter is as follows. Section 4.2 briefly introduces the Kirchhoff plate theory, including the hypothesis, the energy functional and the boundary conditions. Section 4.3 details the mathematical modeling of the 2-D plates and a rectangular plate is used to illustrate the process to implement the MLS-Ritz method. Section 4.4 shows the process of the enforcement of the boundary conditions. Section 4.5 derives the eigenvalue equation that governs the vibration of the plates. Section 4.6 discusses the vibration results of the MLS-Ritz analysis, including the convergence and comparison studies of square and right-angled triangular plates. And finally Section 4.7 concludes this chapter.

### 4.2 Kirchhoff Plate Theory

A plate is a flat structural component with one of its dimension (thickness) being far less than the other two dimensions (length and width). Plates can be classified to be thin plates or thick plates, depending on the thickness to length ratio of the plates. When the thickness to length ratio of a plate is less than 1/20, the Kirchhoff plate (thin) theory can be applied to obtain accurate results for the plate [27]. However, the
Kirchhoff plate theory will over-predict the vibration frequencies or buckling load of a plate and underestimate the deflection of a plate in bending if the plate thickness to length ratio is greater than 1/20. It is caused by the negligence of the transverse shear deformation and the rotary inertia of the thick plate in the Kirchhoff plate theory. In such cases, the Mindlin first-order shear deformable plate theory [219], the Reddy high-order plate theory [220] or the 3-D elasticity theory can be used to obtain more accurate results for thick plates.

Plates can also be classified by the types of materials used to make the plates, such as isotropic linear elastic plates, orthotropic plates, composite laminated plates and plates made of functionally graded material and nonlinear material, etc. We limit the scope of this study on plates made of isotropic linear elastic material. In this chapter, the Kirchhoff plate theory is employed to analyse the vibration of thin plates. This section will present the Kirchhoff plate theory for vibration of plates. The general boundary conditions of the thin plates will be discussed briefly.

4.2.1 The Kirchhoff hypothesis

Figure 4.1 (a) shows a typical flat plate and a cross section of the plate before deformation. The Kirchhoff hypothesis holds [221]:

1. Straight lines perpendicular to the midsurface (i.e., transverse normals) before deformation remain straight after deformation.

2. The transverse normals do not experience elongation (i.e., they are inextensible).

3. The transverse normals rotate such that they remain perpendicular to the midsurface after deformation.
The first two assumptions imply that the transverse displacement is independent of the transverse (or thickness) coordinate and the transverse normal strain $\varepsilon_{zz}$ is zero. The third assumption results in zero transverse shear strains $\varepsilon_{xz} = 0$, $\varepsilon_{yz} = 0$. Figure 4.1 (b) and (c) show the cross-sectional configuration of the plate before and after deformation.
(a) Dimension and coordinate system for rectangular plate

(b) Un-deformed sideview of the plate

(c) Deformed sideview of the plate

Figure 4.1 A rectangular plate and the un-deformed and deformed sideviews of the plate
4.2.2 Plate energy functional

Figure 4.1 (a) shows a rectangular plate in a Cartesian coordinate system and of thickness $h$, length $a$ and width $b$. The plate is made of isotropic linear elastic material with Young’s modulus $E$, Poisson’s ratio $\nu$, the shear modulus $G = E/[2(1 + \nu)]$ and the mass density $\rho$, respectively.

Based on the Kirchhoff plate theory, the displacements on an arbitrary point $(x, y, z)$ of the plate can be expressed as follows [27]:

\[
U(x, y, z) = u(x, y) - z \frac{\partial w(x, y)}{\partial x} \tag{4.1}
\]

\[
V(x, y, z) = v(x, y) - z \frac{\partial w(x, y)}{\partial y} \tag{4.2}
\]

\[
W(x, y, z) = w(x, y) \tag{4.3}
\]

where $U(x, y, z)$, $V(x, y, z)$ and $W(x, y, z)$ are the displacements of the plate along the $x$, $y$ and $z$ directions at an arbitrary point $(x, y, z)$, respectively, $u(x, y)$ and $v(x, y)$ are the in-plane displacements at the midsurface and $w(x, y)$ is the transverse displacement at the midsurface. It is noted that $u(x, y)$ and $v(x, y)$ can be decoupled from the equations for the free vibration analysis of isotropic plates.

The strain and displacements have the following relationships [27]:

\[
\varepsilon_x = \frac{\partial U}{\partial x} \tag{4.4}
\]

\[
\varepsilon_y = \frac{\partial V}{\partial y} \tag{4.5}
\]

\[
\gamma_{xy} = \frac{\partial V}{\partial x} + \frac{\partial U}{\partial y} \tag{4.6}
\]

where $\varepsilon_x$ and $\varepsilon_y$ are the normal strains along the $x$ and $y$ directions, and $\gamma_{xy}$ is the in-plane shear strain in the plate. Substituting Eqs. (4.1), (4.2) and (4.3) into Eqs.
(4.4), (4.5) and (4.6) and neglecting \(u(x, y)\) and \(v(x, y)\), we have the relationship of the strains and the transverse displacement as follows:

\[
\varepsilon_x = -z \frac{\partial^2 w}{\partial x^2}
\]

\[
\varepsilon_y = -z \frac{\partial^2 w}{\partial y^2}
\]

\[
\gamma_{xy} = -2z \frac{\partial^2 w}{\partial x \partial y}
\]

The general 3-D strain and stress relationship for isotropic linear elastic material may be written in the matrix form as:

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z \\
\gamma_{xy} \\
\gamma_{yz} \\
\gamma_{zx}
\end{bmatrix} = \frac{1}{E}
\begin{bmatrix}
1 & -\nu & -\nu & 0 & 0 & 0 \\
-\nu & 1 & -\nu & 0 & 0 & 0 \\
-\nu & -\nu & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & E/G & 0 & 0 \\
0 & 0 & 0 & 0 & E/G & 0 \\
0 & 0 & 0 & 0 & 0 & E/G
\end{bmatrix}
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{xy} \\
\tau_{yz} \\
\tau_{zx}
\end{bmatrix}
\]

(4.10)

This relationship can also be expressed as stresses in term of strains:

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{xy} \\
\tau_{yz} \\
\tau_{zx}
\end{bmatrix} = c
\begin{bmatrix}
1-\nu & \nu & \nu & 0 & 0 & 0 \\
\nu & 1-\nu & -\nu & 0 & 0 & 0 \\
\nu & -\nu & 1-\nu & 0 & 0 & 0 \\
0 & 0 & 0 & G/c & 0 & 0 \\
0 & 0 & 0 & 0 & G/c & 0 \\
0 & 0 & 0 & 0 & 0 & G/c
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z \\
\gamma_{xy} \\
\gamma_{yz} \\
\gamma_{zx}
\end{bmatrix}
\]

(4.11)

in which \(\varepsilon_z\) is the normal strain in the \(z\) direction, \(\gamma_{yz}\) and \(\gamma_{zx}\) are the transverse shear strains, \(\sigma_x\), \(\sigma_y\) and \(\sigma_z\) are the normal stresses in the \(x\), \(y\) and \(z\) directions, \(\tau_{xy}\), \(\tau_{yz}\) and \(\tau_{zx}\) are the in-plane and transverse shear stresses, respectively, and the constant \(c\) is given by

\[
c = \frac{E}{(1+\nu)(1-2\nu)}
\]

(4.12)
In the case of thin plate analysis, the stresses $\sigma_z$, $\tau_{yz}$, and $\tau_{zx}$ are assumed to be small relative to the in-plane stresses (plane stress problem) and can be neglected in the analysis. The stress and strain relationship for a plane stress problem can be derived as follows:

$$
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} =
\begin{bmatrix}
1 & v & 0 \\
v & 1 & 0 \\
0 & 0 & 1-\nu
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}
$$

(4.13)

The in-plane bending moments are obtained by integrating the in-plane stresses over the plate thickness. In the case of an isotropic plate these integrals are

$$
M_x = \int_{-h/2}^{h/2} \sigma_x z \, dz
$$

(4.14)

$$
M_y = \int_{-h/2}^{h/2} \sigma_y z \, dz
$$

(4.15)

$$
M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z \, dz
$$

(4.16)

The bending moments can be expressed in terms of transverse displacement $w$ as follows:

$$
M_x = -D \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right)
$$

(4.17)

$$
M_y = -D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right)
$$

(4.18)

$$
M_{xy} = -D(1-\nu) \frac{\partial^2 w}{\partial x \partial y}
$$

(4.19)

The shear forces can be derived as [27]:

$$
Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}
$$

(4.20)

$$
Q_y = \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x}
$$

(4.21)

where $D$ is the flexural rigidity of the plate and is given by
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MLS-Ritz Method for Rectangular and Triangular Plates

\[ D = \frac{Eh^3}{12(1-\nu^2)} \]  (4.22)

As the MLS-Ritz method will be applied to analyse the vibration of plates, the total potential energy functional of the plates will be derived first. The strain energy stored in a deformed plate is given by

\[ S = \frac{1}{2} \int_\Omega \left( \sigma_{xx} \varepsilon_{xx} + \sigma_{yy} \varepsilon_{yy} + \tau_{xy} \gamma_{xy} \right) d\Omega \]  (4.23)

where \( \Omega \) is the volume of the plate. Substituting Eqs. (4.4), (4.5), (4.6) and (4.13) into Eq. (4.23) and evaluating the integration over the \( z \) direction, we obtain

\[ S = \frac{D}{2} \int_A \left\{ \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\nu) \left[ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left( \frac{\partial^2 w}{\partial x\partial y} \right)^2 \right] \right\} dA \]  (4.24)

where \( A \) is the area of the plate. The kinetic energy due to harmonic vibration of the plate can be expressed as [27]

\[ T = \frac{1}{2} \rho h \omega^2 \int_A w^2 dA \]  (4.25)

where \( \omega \) is the circular frequency of the plate and it will be determined by applying the MLS-Ritz method in the subsequent sections.

Therefore, the total potential energy functional (the Lagrangian) of a thin plate can be expressed as

\[ F = S - T = \frac{D}{2} \int_A \left\{ \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\nu) \left[ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left( \frac{\partial^2 w}{\partial x\partial y} \right)^2 \right] \right\} dA \]

\[- \frac{1}{2} \rho h \omega^2 \int_A w^2 dA \]  (4.26)
4.2.3 Boundary conditions

There are a few typical boundary conditions in thin plate analysis, namely free, simply supported and clamped edge support conditions. The geometric and natural boundary conditions for the three support conditions can be expressed as follows:

**A simply supported edge:**

\[ w = 0 \quad \text{(geometric)} \]  \hspace{1cm} (4.27)

\[ M_s = -D \left( \frac{\partial^2 w}{\partial s^2} + \nu \frac{\partial^2 w}{\partial t^2} \right) = 0 \quad \text{(natural)} \]  \hspace{1cm} (4.28)

**A clamped edge:**

\[ w = 0 \quad \text{(geometric)} \]  \hspace{1cm} (4.29)

\[ \frac{\partial w}{\partial s} = 0 \quad \text{(geometric)} \]  \hspace{1cm} (4.30)

**A free edge:**

\[ M_s = -D \left( \frac{\partial^2 w}{\partial s^2} + \nu \frac{\partial^2 w}{\partial t^2} \right) = 0 \quad \text{(natural)} \]  \hspace{1cm} (4.31)

\[ V_s = Q_s + \frac{\partial M_s}{\partial t} = -D \left[ \frac{\partial^3 w}{\partial s^3} + (2 - \nu) \frac{\partial^3 w}{\partial s \partial t^2} \right] = 0 \quad \text{(natural)} \]  \hspace{1cm} (4.32)

where \( s \) and \( t \) are the normal and tangential directions to the edge and \( V_s \) is the effective shear force at the free edge as given in [27].

In this study, the Ritz method will be employed. Therefore only geometric boundary conditions need to be implemented.
4.3 MLS-Ritz Modelling of 2-D Kirchhoff Plates

The MLS-Ritz method is applied to analyse the vibration of 2-D Kirchhoff plates. A rectangular plate shown in Figure 4.2 is used to illustrate the process for the implementation of the MLS-Ritz method. The Ritz trial function is first established through the moving least square (MLS) technique. A number of pre-determined points are selected on the calculation domain of the plate (see Figure 4.2). The distribution of the points can be regular or irregular, depending on the requirement of the problem at hand. For convenience and simplicity, the uniformly distributed grid points are used in this chapter.

Based on the MLS interpolation scheme presented in Chapter 3, the transverse displacement $w(x, y)$ at an arbitrary point $(x, y)$ (see Figure 4.2) on the plate domain can be approximately expressed as

$$w(x, y) \approx w^h(x, y) = \sum_{i=1}^{N} R_i(x, y)w_i = Rw = w^T R^T \quad (4.33)$$

Figure. 4.2 Dimensions and coordinate system for a rectangular plate
where $N$ is the total number of MLS-Ritz grid points in the calculation domain, $w_i$ is the nominal displacement value at grid point $i$, and $R_i(x, y)$ is the MLS shape function, which has been derived in Eq. (3.19) in Section 3.3. 

$$\mathbf{R} = \left[ R_1(x, y) \ R_2(x, y) \ \cdots \ R_N(x, y) \right]$$ 

is a row matrix containing the shape functions, and 

$$\mathbf{w} = \left[ w_1 \ w_2 \ \cdots \ w_N \right]^T$$ 

is a column matrix with the nominal displacement values at the grid points, respectively. Substituting Eq. (4.33) into Eq. (4.26), the total potential energy functional of the plate can be expressed as follows

$$F = \frac{1}{2} \mathbf{w}^T (\mathbf{K} - \omega^2 \mathbf{M}) \mathbf{w}$$ (4.34)

where the stiffness matrix $\mathbf{K}$ and the mass matrix $\mathbf{M}$ have the dimension of $N \times N$ and are given respectively by

$$\mathbf{K} = D \int_A \left[ \mathbf{R}^T_{xx} \mathbf{R}_{xx} + \mathbf{R}^T_{yy} \mathbf{R}_{yy} + \nu \mathbf{R}^T_{xy} \mathbf{R}_{xy} + \nu \mathbf{R}^T_{yx} \mathbf{R}_{yx} + 2(1-\nu) \mathbf{R}^T_{xx} \mathbf{R}_{yy} \right] dA$$ (4.35)

$$\mathbf{M} = \rho h \int_A \mathbf{R}^T \mathbf{R} dA$$ (4.36)

in which the subscripts $x$ or $y$ of $\mathbf{R}$ denote the partial derivative with respect to the $x$ and $y$ directions, respectively.

### 4.4 Implementation of Boundary Conditions

A point substitution approach is proposed in conjunction with the MLS interpolation scheme to impose the geometric boundary conditions of the plate. Figure 4.3 shows a typical rectangular plate with free, simply supported and clamped edges. A set of pre-selected uniform grid points are assigned on the calculation domain. Grid points are located on the plate boundaries, inner domain as well as outer domain (virtual points) for clamped edges. Note that the number of grid points on the outer domain is equal to the number of grid points on the clamped edges as shown in Figure 4.3.
We can group the nominal displacements on the grid points of the plate into two categories \( w_B \) and \( w_I \), i.e.
\[
\mathbf{w} = \begin{bmatrix} \mathbf{w}_B \\ \mathbf{w}_I \end{bmatrix}
\]
(4.37)

where \( \mathbf{w}_B \) contains all nominal displacements of the points on the simply supported and clamped edges and on the outer domain, and \( \mathbf{w}_I \) contains all nominal displacements of the points on the free edge/s and the inner domain, respectively. The geometric boundary condition for a grid point \((x_j, y_j)\) on a simply supported edge is given by

![Figure 4.3 MLS-Ritz grid point arrangement for a rectangular plate with free, simply supported and clamped edges](image-url)
Chapter 4  MLS-Ritz Method for Rectangular and Triangular Plates

\[ w(x_j, y_j) \approx w^h(x_j, y_j) = \sum_{i=1}^{N} R_i(x_j, y_j) w_i = 0 \]

(4.38)

And if the point is on a clamped edge, the geometric boundary conditions are

\[ w(x_j, y_j) = w^h(x_j, y_j) = \sum_{i=1}^{N} R_i(x_j, y_j) w_i = 0 \]

(4.39)

\[ \frac{\partial w(x_j, y_j)}{\partial s} = \frac{\partial w^h(x_j, y_j)}{\partial s} = \sum_{i=1}^{N} \frac{\partial R_i(x_j, y_j)}{\partial s} w_i = 0 \]

(4.40)

where \( s \) denotes the normal direction to the clamped edge/s. Applying boundary conditions for all grid points on the simply supported and clamped edges, we can obtain a system of linear equations given by

\[
\begin{bmatrix} Q & S \\ \end{bmatrix} \begin{bmatrix} w_B \\ w_I \end{bmatrix} = 0
\]

(4.41)

The nominal displacements of the grid points on the simply supported and clamped edges and on the outer domain can be expressed as

\[ w_B = -Q^{-1}Sw_I \]

(4.42)

The nominal displacements for all grid points can then be expressed in terms of \( w_I \) as follows:

\[ w = \begin{bmatrix} w_B \\ w_I \end{bmatrix} = \begin{bmatrix} -Q^{-1}S \\ I \end{bmatrix} w_I = Tw, \]

(4.43)

Note that the geometric boundary conditions of the plate are effectively enforced by applying the above point substitution approach to the plate.
4.5 Eigenvalue Equation

Substituting Eq. (4.43) into Eq. (4.35), the total potential energy functional can be expressed as

$$ F = \frac{1}{2} w_i^T (\bar{K} - \omega^2 \bar{M}) w_i $$

(4.44)

where

$$ \bar{K} = T^T K T $$

(4.45)

$$ \bar{M} = T^T M T $$

(4.46)

Applying the Ritz procedure by minimizing the Eq. (4.44) with respect to $w_i$, we have

$$ (\bar{K} - \omega^2 \bar{M}) w_i = 0 $$

(4.47)

The vibration frequency $\omega$ can be determined by solving the generalized eigenvalue equation defined by Eq. (4.47).

4.6 Results and Discussion

The proposed MLS-Ritz method is examined in this section for its validity and accuracy. The natural frequencies of several selected square and right-angled isosceles triangular plates are obtained. The Poisson ratio $\nu$ is set to be 0.3 and the non-dimensional frequency parameter is defined as $\lambda = (\omega a^2 / \pi^2) \sqrt{\rho h / D}$, where $a$ is the length of the plates.

Convergence studies must be carried out to verify the validity of the MLS-Ritz method. There are several parameters that can vary in the proposed method. Firstly, the basis function used in Eq. (3.9) can be any finite basis function of a complete space.
We propose to use the 2-D complete polynomial as the basis function in this study. If the degree $P$ of the polynomial is zero, then $p(x, y) = 1$. If $P = 2$, then $p(x, y) = [1, x, y, x^2, xy, y^2]^T$. Integration needs to be carried out for Eqs. (4.35) and (4.36) over the plate domain. The influence of the integer number $k$ in the weight function in Eq. (3.8) on the accuracy of the method will be examined. Finally, the number of MLS-Ritz grid points and the effective range of the radius of support $d$ need to be evaluated. The accuracy of the MLS-Ritz method is verified against the known benchmark results.

4.6.1. Square plates

For a square plate, the plate domain is divided into four equal segments as shown in Figure. 4.4 and the Gaussian quadrature is employed to evaluate Eqs. (4.35) and (4.36). The number of sufficient Gaussian points will be determined for carrying out the integration.

A simply supported square plate is first considered in the convergence study. To study the influence of the degree $P$ of the 2-D complete polynomial basis function $p(x, y)$,
the number of Gaussian points in each segment is set to be $20 \times 20$, the value of $k$ in the weight function is fixed at 10, a set of $15 \times 15$ MLS-Ritz grid points is used and the radius of support $d$ is equal to $0.5a$, where $a$ is the length of the square plate. Table 4.1 shows the variation of the first 6 frequency parameters for the simply supported square plate as the degree $P$ of the 2-D polynomial changes from 0 to 6. We observe that the frequency parameters in general decrease as the degree $P$ of the polynomial basis function increases. Good convergence is achieved even with $P = 2$. The frequency parameters of the 5th and 6th modes with $P = 6$ are 9.9999 which is slightly lower than the exact value of 10. It may be caused by numerical roundoff error in the calculation. We also note that the computational time of the MLS-Ritz method is increased rapidly when the degree $P$ of the 2-D polynomial basis function increases. This is because the number of terms used in the approximation of $w(x, y)$ in Eq. (3.9) increases significantly as $P$ increases. Note that the CPU time in Table 4.1 is recorded on a PC with an Intel Celeron 3.2GHz processor and 1.5GB of RAM.

Table 4.1 Variation of frequency parameters $\lambda$ versus the value of $P$ for a simply supported square plate with $k = 10$, $d = 0.5a$, $20 \times 20$ Gaussian points and $15 \times 15$ MLS-Ritz grid points

<table>
<thead>
<tr>
<th>$P$</th>
<th>CPU Time (in sec)</th>
<th>Mode Sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>7.23</td>
<td>2.0001</td>
</tr>
<tr>
<td>1</td>
<td>7.56</td>
<td>2.0000</td>
</tr>
<tr>
<td>2</td>
<td>8.63</td>
<td>2.0000</td>
</tr>
<tr>
<td>3</td>
<td>10.70</td>
<td>2.0000</td>
</tr>
<tr>
<td>4</td>
<td>14.84</td>
<td>2.0000</td>
</tr>
<tr>
<td>5</td>
<td>19.56</td>
<td>2.0000</td>
</tr>
<tr>
<td>6</td>
<td>27.49</td>
<td>2.0000</td>
</tr>
</tbody>
</table>
Table 4.2 shows the same convergence study for a clamped square plate. It is observed that for the first four vibration modes, the frequency parameters converge well as the degree of the 2-D polynomial basis function \( P = 1 \). Further increase in the value of \( P \), we observe that the frequency parameters for the 4\(^{th}\) to 6\(^{th}\) modes oscillate slightly. In general, \( P = 2 \) is sufficient to provide converged results.

Table 4.2 Variation of frequency parameters \( \lambda \) versus the value of \( P \) for a clamped square plate with \( k = 10, d = 0.5a \), 20 \( \times \) 20 Gaussian points and 15\( \times \)15 MLS-Ritz grid points

<table>
<thead>
<tr>
<th>Mode Sequence</th>
<th>P=0</th>
<th>P=1</th>
<th>P=2</th>
<th>P=3</th>
<th>P=4</th>
<th>P=5</th>
<th>P=6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.6461</td>
<td>7.4377</td>
<td>7.4377</td>
<td>10.9704</td>
<td>13.3322</td>
<td>13.3964</td>
<td></td>
</tr>
</tbody>
</table>

Tables 4.3 and 4.4 examine the variation of the frequency parameters of a simply supported square plate and a clamped square plate with respect to the size of the MLS-Ritz grid points. While the MLS-Ritz grid point size varies from 5\( \times \)5 to 19\( \times \)19, the values of \( P = 2 \), \( k = 10 \) and \( d = 0.5a \) and 20\( \times \)20 Gaussian points are employed in the calculation. The frequency parameters for both simply supported and clamped square plates decrease monotonically as the MLS-Ritz grid point size increases. Excellent convergence is achieved for all cases in Tables 4.3 and 4.4 when the MLS-Ritz grid size reaches 15\( \times \)15 and more.
Table 4.3 Convergence of frequency parameters $\lambda$ against the MLS-Ritz grid point size for a simply supported square plate with $P = 2$, $k = 10$, $d = 0.5a$ and $20 \times 20$ Gaussian points

<table>
<thead>
<tr>
<th>MLS Grid Points</th>
<th>Mode Sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>5×5</td>
<td>6061 53051 55369 72450 226535 3715952</td>
</tr>
<tr>
<td>7×7</td>
<td>2.0210 5.2818 5.2818 8.2356 11.0872 11.0916</td>
</tr>
<tr>
<td>9×9</td>
<td>2.0019 5.0357 5.0357 8.0330 10.2036 10.2039</td>
</tr>
<tr>
<td>11×11</td>
<td>2.0000 5.0012 5.0012 8.0022 10.0041 10.0044</td>
</tr>
<tr>
<td>13×13</td>
<td>2.0000 5.0001 5.0001 8.0000 10.0005 10.0006</td>
</tr>
<tr>
<td>15×15</td>
<td>2.0000 5.0001 5.0001 8.0000 10.0003 10.0003</td>
</tr>
<tr>
<td>17×17</td>
<td>2.0000 5.0000 5.0000 8.0000 10.0002 10.0002</td>
</tr>
<tr>
<td>19×19</td>
<td>2.0000 5.0000 5.0000 8.0000 10.0000 10.0000</td>
</tr>
</tbody>
</table>

Table 4.4 Convergence of frequency parameters $\lambda$ against the MLS-Ritz grid point size for a clamped square plate with $P = 2$, $k = 10$, $d = 0.5a$ and $20 \times 20$ Gaussian points

<table>
<thead>
<tr>
<th>MLS Grid Points</th>
<th>Mode Sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>5×5</td>
<td>8.4900 20.2614 20.2631 28.5606 39.0445 40.8206</td>
</tr>
<tr>
<td>7×7</td>
<td>2.4228 5.8519 5.8519 8.8486 12.0883 12.0965</td>
</tr>
<tr>
<td>9×9</td>
<td>3.6521 7.4610 7.4610 11.0214 13.4037 13.4589</td>
</tr>
<tr>
<td>17×17</td>
<td>3.6461 7.4364 7.4364 10.9647 13.3320 13.3953</td>
</tr>
</tbody>
</table>
Now the number of Gaussian points required for generating accurate results by the MLS-Ritz method is examined. Tables 4.5 and 4.6 show the convergence pattern of the frequency parameters for a simply supported square plate and a clamped square plate against the Gaussian points used in each of the four segments in the plate domain, respectively. The values of $k = 10$, $d = 0.5a$ and $P = 2$ and $15 \times 15$ MLS-Ritz grid points are used in the computation. We observe that the number of Gaussian points affects the frequency parameters significantly. When the number of Gaussian points is small ($\leq 5 \times 5$), the frequency parameters are erroneous due to the poor accuracy of the Gaussian integration. The frequency parameters oscillate around the converged values as the number of Gaussian points increases from $6 \times 6$ to $13 \times 13$ for the simply supported square plate and to $14 \times 14$ for the clamped square plate, respectively. The frequency parameters are stabilized when the number of Gaussian points increases further. We observe that $20 \times 20$ Gaussian points for each integration segment are sufficient to provide accurate integration results used in the MLS-Ritz method for square plates. Obviously, increasing the number of Gaussian points will increase the computational time in applying the MLS-Ritz method.

The effect of the values of the integer $k$ on the weight function has been depicted in Figure 3.1 in Section 3.2. It is evident that as the value of $k$ increases, the weighted influence in the MLS technique is more concentrated around the region near the fitting point $(x, y)$. The influence of $k$ on the accuracy of the proposed MLS-Ritz method is examined next.
Chapter 4  MLS-Ritz Method for Rectangular and Triangular Plates

Table 4.5 Variation of frequency parameters $\lambda$ versus the number of Gaussian points for a simply supported square plate with $k = 10$, $d = 0.5a$, $P = 2$ and $15 \times 15$ MLS-Ritz grid points

<table>
<thead>
<tr>
<th>Gaussian Points</th>
<th>Mode Sequence</th>
<th>Mode Sequence</th>
<th>Mode Sequence</th>
<th>Mode Sequence</th>
<th>Mode Sequence</th>
<th>Mode Sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>3×3</td>
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<td>2.82E-06</td>
<td>5.42E-06</td>
<td>6.14E-06</td>
<td>1.82E-05</td>
<td>5.54E-05</td>
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<tr>
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<td>0.4928</td>
<td>2.2243</td>
<td>2.5920</td>
<td>4.1820</td>
<td>4.3016</td>
</tr>
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<td>4.1372</td>
<td>4.1372</td>
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</tr>
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<td>4.9583</td>
<td>4.9583</td>
<td>7.9836</td>
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<tr>
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<td>1.9997</td>
<td>4.9983</td>
<td>4.9983</td>
<td>7.9978</td>
<td>10.0104</td>
<td>10.0106</td>
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<tr>
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<td>5.0008</td>
<td>8.0009</td>
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<td>5.0001</td>
<td>8.0000</td>
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<td>10.0003</td>
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<tr>
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<td>5.0001</td>
<td>8.0000</td>
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<td>5.0001</td>
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<td>10.0003</td>
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<td>10.0003</td>
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<td>8.0000</td>
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</table>
Table 4.6 Variation of frequency parameters $\lambda$ versus the number of Gaussian points for a clamped square plate with $k = 10$, $d = 0.5a$, $P = 2$ and 15×15 MLS-Ritz grid points

<table>
<thead>
<tr>
<th>Gaussian Points</th>
<th>Mode Sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>3×3</td>
<td>1.15E-05</td>
</tr>
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</tr>
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<tr>
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</table>

Tables 4.7 and 4.8 present the variation of frequency parameters of a simply supported square plate and a clamped square plate against various values of $k$ in the weight function, respectively. The values of $P = 2$ and $d = 0.5a$, 20×20 Gaussian points and 15×15 MLS-Ritz grid points are used in this calculation. We observe that when $k = 1$, the frequency parameters are poorly converged. Further increasing the values of $k$, the frequency parameters oscillate about the converged values for both the simply support
and the clamped square plates, respectively. In general, the MLS-Ritz method can generate accurate frequency parameters when \( k \geq 10 \) and the increase of the \( k \) values has a negligible impact on the computational efficiency of the MLS-Ritz method.

Table 4.7 Variation of frequency parameters \( \lambda \) versus the value of \( k \) for a simply supported square plate with \( P = 2, \ d = 0.5a, \ 20 \times 20 \) Gaussian points and \( 15 \times 15 \) MLS-Ritz grid points

<table>
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<th>( k )</th>
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<th>5</th>
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<td>5.0823</td>
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<td>5.0024</td>
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<td>10.0390</td>
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</tr>
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<td>10.0019</td>
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<td>10.0039</td>
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<td>10.0006</td>
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<td>5.0000</td>
<td>8.0000</td>
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<td>10.0003</td>
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<td>5.0000</td>
<td>5.0000</td>
<td>8.0000</td>
<td>10.0005</td>
<td>10.0005</td>
</tr>
<tr>
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<td></td>
<td>2.0000</td>
<td>5.0000</td>
<td>5.0000</td>
<td>8.0000</td>
<td>10.0002</td>
<td>10.0002</td>
</tr>
<tr>
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<td>5.0000</td>
<td>5.0000</td>
<td>8.0000</td>
<td>10.0000</td>
<td>10.0001</td>
</tr>
<tr>
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<td></td>
<td>2.0000</td>
<td>5.0000</td>
<td>5.0000</td>
<td>8.0000</td>
<td>10.0001</td>
<td>10.0001</td>
</tr>
<tr>
<td>15</td>
<td></td>
<td>2.0000</td>
<td>5.0000</td>
<td>5.0000</td>
<td>8.0000</td>
<td>10.0001</td>
<td>10.0001</td>
</tr>
</tbody>
</table>
Table 4.8 Variation of frequency parameters $\lambda$ versus the value of $k$ for a clamped square plate with $P = 2$, $d = 0.5a$, $20 \times 20$ Gaussian points and $15 \times 15$ MLS-Ritz grid points

<table>
<thead>
<tr>
<th>$k$</th>
<th>Mode Sequence</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
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<td>2.2356</td>
<td>5.0639</td>
<td>5.0709</td>
<td>11.2470</td>
<td>12.4406</td>
</tr>
</tbody>
</table>

The radius of support $d$ of the MLS scheme also plays an important role in obtaining accurate MLS-Ritz results. Tables 4.9 and 4.10 show the influence of the radius of support $d$ on the frequency parameters of a simply supported square plate and a clamped square plate, respectively. We use $P = 2$, $k = 10$, $20 \times 20$ Gaussian points and $15 \times 15$ MLS-Ritz grid points in this calculation. While a small radius of support $d$ gives erroneous frequency parameters, a very large $d$ value may lead to missing modes as shown in the case for clamped square plate. The optimal range of the radius of support is between $0.4a$ to $0.6a$ in the considered cases. It is noted that increasing the radius of support $d$ will increase the computational time of the MLS-Ritz method as more neighbourhood grid points are involved in the MLS fitting process (see Table...
4.9). Note that the CPU time in Table 4.9 is recorded on a PC with an Intel Celeron 3.2GHz processor and 1.5GB of RAM.

Table 4.9 Convergence of frequency parameters $\lambda$ against the radius of support $d$ for a simply supported square plate with $P = 2$, $k = 10$, $20 \times 20$ Gaussian points and $15 \times 15$ MLS-Ritz grid points

<table>
<thead>
<tr>
<th>$d/a$</th>
<th>CPU Time (in sec)</th>
<th>Mode Sequence</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
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<td>1.62</td>
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<td>52.6090</td>
<td>57.7458</td>
<td>61.4896</td>
<td>66.4244</td>
</tr>
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<td>5.3068</td>
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<td>5.0144</td>
<td>8.0113</td>
<td>10.0787</td>
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<td>5.0040</td>
<td>8.0039</td>
<td>10.0254</td>
</tr>
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<td>5.0003</td>
<td>5.0003</td>
<td>8.0006</td>
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</tr>
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Table 4.10 Convergence of frequency parameters $\lambda$ against the radius of support $d$ for a clamped square plate with $P = 2$, $k = 10$, $20 \times 20$ Gaussian points and $15 \times 15$ MLS-Ritz grid points

<table>
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<tr>
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</tbody>
</table>

The accuracy of the MLS-Ritz method can be verified against available benchmark solutions [197] in the literature. Table 4.11 presents the frequency parameters for square plates of various boundary conditions obtained by applying the MLS-Ritz method and by Leissa [197]. A four-letter symbol is used to denote the boundary conditions of a square plate. For example, an $SFCS$ plate has a simply supported left edge, a free bottom edge, a clamped right edge and a simply supported top edge, respectively. The MLS-Ritz results are based on $P = 2$, $k = 10$, $d = 0.5a$, $20 \times 20$ Gaussian points and $15 \times 15$ MLS-Ritz grid points on the boundaries and inner domain of the plate, respectively. We observe that the MLS-Ritz results are in excellent agreement with the exact solutions for $SSSS$, $SCSC$, $SFSF$ and $SSFS$ plates in [197] and
the approximate results for the $CCCC$ plate [197]. The comparison study confirms the high accuracy that the MLS-Ritz method can achieve.

Table 4.11 Comparison study of frequency parameters $\lambda$ for square plates with $P = 2$, $k = 10$, $d = 0.5a$, $20 \times 20$ Gaussian points and $15 \times 15$ MLS-Ritz grid points

<table>
<thead>
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<td></td>
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<tr>
<td>$SSSS$</td>
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</tr>
<tr>
<td>[197]</td>
<td></td>
<td>2.00000</td>
</tr>
<tr>
<td>$SCSC$</td>
<td>Present</td>
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</tr>
<tr>
<td>$SFSF$</td>
<td>Present</td>
<td>0.9759</td>
</tr>
<tr>
<td>[197]</td>
<td></td>
<td>0.97586</td>
</tr>
<tr>
<td>$SSFS$</td>
<td>Present</td>
<td>1.1839</td>
</tr>
</tbody>
</table>

The first six vibration mode contour shapes of an $SCSC$ plate and an $SSFS$ plate are presented in Figures 4.5 and 4.6, respectively. We observe that the MLS-Ritz method can predict the correct vibration mode shapes for the considered cases.
Figure 4.5 Mode contour shapes of the first 6 modes for an SCSC square plate
Figure 4.6 Mode contour shapes of the first 6 modes for an SSFS square plate
4.6.2 Right-angled isosceles triangular plates

The same procedure used in the analysis of rectangular plates can be employed for the vibration study of triangular plates. The validity and accuracy of the MLS-Ritz method is further examined through the vibration analysis of several selected right-angled isosceles triangular plates. The MLS-Ritz grid points are uniformly distributed over the triangular plate domain. For example, Figure 4.7 shows the distribution of a $5 \times 5$ MLS-Ritz grid points for a CSF plate where $C$ denotes the clamped left edge, $S$ the simply supported bottom edge and $F$ the free inclining edge, respectively. The notation $5 \times 5$ denotes that there are 5 MLS-Ritz grid points on the left edge and 5 on the bottom edge of the plate, respectively. Note that virtual points (points outside the plate domain) are used to enforce the clamped boundary conditions on the left edge of the plate.

Figure 4.7 Distribution of a $5 \times 5$ MLS-Ritz grid points in the analysis
Chapter 4  
MLS-Ritz Method for Rectangular and Triangular Plates

The Gaussian quadrature is carried out on the triangular plate domain with the following scheme to determine the number of Gaussian points used in the integration. First, we select the largest dimension in the $x$ direction on the plate domain (in this case, the length of the bottom edge of the plate) and determine the number of Gaussian points $N_g$ to be used. Then, we draw vertical lines through the $x$ coordinates of the $N_g$ Gaussian points. The number of Gaussian points used on a vertical line is determined by the ratio between the length of the vertical line in the triangular plate domain and the largest $x$ dimension of the triangular plate. We found that when $N_g = 40$, the Gaussian quadrature provides accurate integration results used in the MLS-Ritz method for the vibration analysis of right-angled isosceles triangular plates.

Tables 4.12 and 4.13 show the influence of the degree of the 2-D polynomial basis function $P$ on the frequency parameters of a simply supported triangular plate and a clamped triangular plate, respectively. The values of $k = 15$, $d = 0.5a$, $N_g = 40$ and $21 \times 21$ MLS-Ritz grid points are used in the calculation. It is observed that for the clamped plate the increase of the value of $P$ shows no significant impact on the frequency parameters. However, for the simply supported plate, the best range of the $P$ values is between 2 to 4. To strike a balance of efficiency and accuracy, we choose to use $P = 3$ for all subsequent calculations. The poor results with $P = 6$ in Table 4.12 may be caused by insufficient number of Gaussian points used because the higher the polynomial degree is, the more Gaussian points are needed for accurate results.
Table 4.12 Variation of frequency parameters $\lambda$ versus the value of $P$ for a simply supported right-angled isosceles triangular plate with $k = 15$, $d = 0.5a$, $N_g = 40$ and 21×21 MLS-Ritz grid points

<table>
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<th>4</th>
<th>5</th>
<th>6</th>
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<tbody>
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<td>17.0011</td>
<td>20.0015</td>
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</tr>
<tr>
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<td>2</td>
<td>5.0000</td>
<td>10.0001</td>
<td>13.0001</td>
<td>17.0007</td>
<td>20.0009</td>
<td>25.0010</td>
</tr>
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<td>10.0000</td>
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<td>25.0009</td>
</tr>
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<tr>
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Table 4.13 Variation of frequency parameters $\lambda$ versus the value of $P$ for a clamped right-angled isosceles triangular plate with $k = 10$, $d = 0.5a$, $N_g = 40$ and 21×21 MLS-Ritz grid points

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<td>15.9871</td>
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</tr>
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</table>

The MLS-Ritz grid point size varies from 7×7 to 23×23 while the other parameters of the MLS-Ritz method are kept constants, i.e. $P = 3$, $k = 15$, $d = 0.5a$ and $N_g = 40$ for the two considered triangular plates (see Tables 4.14 and 4.15). The frequency
parameters of the plates are well converged as the MLS-Ritz grid point size reaches 15×15 and above. To ensure the accuracy of the results presented in this study, we choose to use MLS-Ritz grid point size 21×21 in all calculations for the right-angled isosceles triangular plates.

Table 4.16 shows the variation of the frequency parameters of a clamped right-angled isosceles triangular plate versus the value of $k$ in the MLS weight function. We observe that a wide range of $k$ values can be used in the MLS-Ritz method to generate accurate vibration frequencies. In general, a larger value of $k$ leads to better converged results. We choose to use $k = 15$ in this study for the vibration analysis of right-angled isosceles triangular plates.
Table 4.14 Convergence of frequency parameters $\hat{\lambda}$ against the MLS-Ritz grid point size for a simply supported right-angled isosceles triangular plate with $P = 3$, $k = 15$, $d = 0.5a$ and $N_g = 40$

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</tr>
<tr>
<td>17×17</td>
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<tr>
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</tr>
<tr>
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<tr>
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Table 4.15 Convergence of frequency parameters $\hat{\lambda}$ against the MLS-Ritz grid point size for a clamped right-angled isosceles triangular plate with $P = 3$, $k = 15$, $d = 0.5a$ and $N_g = 40$

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Table 4.16 Variation of frequency parameters $\lambda$ versus the value of $k$ for a clamped right-angled isosceles triangular plate with $P = 3$, $d = 0.5a$, $N_g = 40$ and $21 \times 21$ MLS-Ritz grid points

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<tr>
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</table>

The influence of the radius of support $d$ of the MLS scheme on the frequency parameters of triangular plates is examined. Tables 4.17 and 4.18 present the variation of the frequency parameters of a simply supported and a clamped right-angled isosceles triangular plate against the radius of support $d$, respectively. We use $P = 3$, $k = 15$, $N_g = 40$ and $21 \times 21$ MLS-Ritz grid points in this calculation. Similar to its square plate counterpart, a small radius of support $d$ gives erroneous frequency parameters and a very large $d$ value may lead to erroneous/missing modes as shown in the case for clamped triangular plate (see Table 4.18). Again, the optimal range of the
radius of support $d$ is observed to be between $0.4a$ to $0.6a$ in the considered triangular plate cases. We choose $d = 0.5a$ in the calculations.

Table 4.17 Convergence of frequency parameters $\lambda$ against the radius of support $d$ for a simply supported right-angled isosceles triangular plate with $P = 3$, $k = 15$, $N_g = 40$ and $21 \times 21$ MLS-Ritz grid points

<table>
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Table 4.18 Convergence of frequency parameters $\lambda$ against the radius of support $d$ for a clamped right-angled isosceles triangular plate with $P = 3$, $k = 15$, $N_g = 40$ and $21 \times 21$ MLS-Ritz grid points

<table>
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<td>15.9869</td>
<td>19.7340</td>
<td>24.6004</td>
<td>28.1337</td>
<td>34.0206</td>
</tr>
<tr>
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<td>9.5028</td>
<td>15.9871</td>
<td>19.7338</td>
<td>24.6003</td>
<td>28.1339</td>
<td>34.0199</td>
</tr>
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<td>34.0198</td>
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<td>15.9870</td>
<td>17.3138</td>
<td>19.7334</td>
<td>24.6010</td>
<td>28.1358</td>
</tr>
</tbody>
</table>
To confirm the correctness of the MLS-Ritz method for the vibration analysis of right-angled isosceles triangular plates, a comparison study is carried out against solutions obtained by other researchers [206-208, 222-226] and the results for several selected triangular plates are presented in Table 4.19. Note that the study by Kitipornchai et al. [222] is based on the Mindlin shear deformable plate theory with small plate thickness to length ratio to simulate thin plates.

For a simply supported right-angled isosceles triangular plate (SSS plate), it is possible to obtain exact vibration solutions by using a simply supported square plate with appropriate modes of vibration [27]. For example, the exact frequency parameter of the first mode of the SSS triangular plate is 5 which is the same as the one for the second mode of the simply supported square plate. We observe that comparing with other numerical and analytical methods (see Table 4.19), the MLS-Ritz method generates very accurate frequency parameters for SSS right-angled isosceles triangular plate.

For a fully clamped right-angled isosceles triangular plate, the MLS-Ritz results are again in very close agreement with the ones by Gorman [206], Kim and Dickinson [225] and Kitipornchai et al. [222]. As there are no known exact solutions available for this case, the convergence study given in Table 4.15 together with the available solutions in the open literature is very useful to verify the accuracy of the current results.

Four other cases (CCF, FFC, CSF and CCS plates) are also compared with known solutions in the open literature (see Table 4.19). They all confirm the high accuracy of the MLS-Ritz method in the vibration analysis of right-angled isosceles triangular plates.
Table 4.19 Comparison study of frequency parameters $\lambda$ for right-angled isosceles triangular plates with $P = 3$, $k = 15$, $d = 0.5a$, $N_z = 40$ and $21 \times 21$ MLS-Ritz grid points

<table>
<thead>
<tr>
<th>Cases</th>
<th>Sources</th>
<th>Mode Sequence</th>
</tr>
</thead>
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<tr>
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<tr>
<td>SSS</td>
<td>Present</td>
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</tr>
<tr>
<td></td>
<td>[224]</td>
<td>4.993</td>
</tr>
<tr>
<td></td>
<td>[206]</td>
<td>5.00</td>
</tr>
<tr>
<td></td>
<td>[225]</td>
<td>5.00</td>
</tr>
<tr>
<td></td>
<td>[226]</td>
<td>4.999</td>
</tr>
<tr>
<td></td>
<td>[222]</td>
<td>5.001</td>
</tr>
<tr>
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<td>Present</td>
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</tr>
<tr>
<td></td>
<td>[223]</td>
<td>9.48</td>
</tr>
<tr>
<td></td>
<td>[206]</td>
<td>9.510</td>
</tr>
<tr>
<td></td>
<td>[208]</td>
<td>2.927</td>
</tr>
<tr>
<td></td>
<td>[226]</td>
<td>7.436</td>
</tr>
<tr>
<td></td>
<td>[222]</td>
<td>7.436</td>
</tr>
</tbody>
</table>
The first six vibration mode contour shapes of a simply supported right-angled isosceles triangular plate and an \textit{FFC} right-angled isosceles triangular plate are presented in Figures 4.8 and 4.9, respectively. We observe that the MLS-Ritz method can predict the correct vibration mode shapes for the considered cases. Note that the dotted contour lines in Figures 4.8 and 4.9 represent the nodal lines of the vibration modes.
Figure 4.8 Mode shapes for the first 6 modes of a simply supported right-angled isosceles triangular plate
Figure 4.9 Mode shapes for the first 6 modes of an FFC right-angled isosceles triangular plate
4.7 Conclusions

This chapter has applied the novel numerical method, the MLS-Ritz method, for the vibration analysis of thin plates. The moving least square (MLS) technique has been employed to establish the Ritz trial function for the transverse displacement of a plate. A point substitution technique has been proposed to enforce the geometric boundary conditions of the plate. Convergence studies have been carried out to determine the size of the MLS-Ritz grid points, the number of Gaussian points, the degree of 2-D polynomial basis function and the MLS radius of support that are required to produce accurate MLS-Ritz vibration results. Comparison studies have proven that the proposed method is highly accurate for predicting the vibration frequencies of thin plates.

It is noted that the pattern of convergence of the frequency parameters against the polynomial degree $P$, the radius of support $d$, the MLS grid point size and even the number of Gaussian points does not show the monotonically decreasing nature as seen in many versions of Ritz method. The frequency parameters oscillate slightly when increasing these controlling parameters. This phenomenon was shown in the convergence results by other researchers using the MLS technique [191] and will be seen in later chapters when the MLS-Ritz method is applied to solve other problems. It is also observed that these controlling parameters have their own “optimal range” for the convergence of the frequency parameters.
CHAPTER 5

APPLICATION OF MLS-RITZ METHOD FOR VIBRATION ANALYSIS OF SKEW PLATES

5.1 Introduction

The study of buckling and vibration of skew plates dated back to the early 1950s when there was a need to investigate the mechanical properties of the then new swept-wing aircraft concept [227]. Skew plates are also widely employed in other practical structures, such as skew bridge decks, skew floor slabs, vehicle bodies and ship decks. Leissa [72] pointed out that no exact solutions exist for the vibration of skew plates and approximate numerical methods must be used to obtained solutions for such plates. A series of early investigations on the vibration behaviour of skew plates [224, 228-230] were reviewed in Leissa’s monograph [72]. Among these early studies, the Rayleigh-Ritz method was one of the most frequently used methods in analysing the vibration of skew plates [231, 228-229]. Other methods used are Trefftz method [232], the perturbation method [233] and the point matching method [224].

The study on the vibration analysis of skew plates has attracted much attention since 1970s. Sathyamoorthy and Pandalai [234] employed the Berger approximation to study the relationship between the period and amplitude of skew plates based on an assumed mode shape. Mizusawa et al. [78] studied the free vibration of skew plates
by the Rayleigh-Ritz method with B-spline functions as the Ritz trial functions to solve vibration of skew plates with arbitrary boundary conditions. It was found that, in general, the convergence of the vibration frequencies became less satisfactory with the increase in skew angle of the plates. Mizusawa et al. [82] proposed a modified Rayleigh-Ritz method to analyse skew plates. Both geometric and natural boundary conditions were satisfied by using the Lagrange multiplier technique. Mizusawa and Kajita [235] also employed the spline strip method to investigate the vibration and buckling of skew plates with edges elastically restrained in rotation. A reduction method was proposed by Sakata [236] with a few approximation formulae for numerically estimating the natural frequency of simply supported isotropic and orthotropic skew plates. Gorman [237] studied the vibration of simply supported and clamped rhombic plates using the superposition method. Bardell [238] proposed a hierarchical finite element method to determine the natural frequencies and modes of flat, isotropic skew plates. The free edges and point supports were considered in his study.

Liew and Lam [239] employed the Rayleigh-Ritz method with 2-D orthogonal plate functions as the Ritz trial function to study free vibration of skew plates. Rhombic plates with various combinations of edge support conditions were considered and good convergence and accuracy were demonstrated in their study. Liew and Wang [240] developed the pb-2 Ritz method to study the vibration of skew plates with different edge conditions, skew angles, aspect ratios and internal line supports. A comprehensive literature survey on the vibration of thin skew plates was presented in the paper. The pb-2 Ritz method was extended to study the buckling and vibration of thick skew plates based on the Mindlin shear deformable plate theory [241-242]. Singh and Chakraverty [243] used the Rayleigh-Ritz method to determine the frequencies of skew plates with all possible combinations of boundary conditions and various skew angles. The boundary characteristic orthogonal polynomials were used to determine the transverse vibration of a rectangular or skew plate under different boundary conditions. Their results appeared not converged when the skew angle of
the plates becomes large. Hadid and Bashir [244] employed the spline-integral method to calculate the natural frequencies of beams, rectangular and skew plates with different skew angles and simply supported edges. Han and Dickinson [245] studied the vibration of thin, symmetrically laminated skew plate by the Ritz method. Zitnan [246] studied the transverse vibration of rectangular and skew plates by the Rayleigh-Ritz method using B-spline trial functions. Recently, Woo et al. [247] carried out a study on the free vibration of skew Mindlin plates by employing the p-version of finite element method.

Although there are extensive studies on the vibration of skew plates in the open literature, the accuracy of the vibration solutions is not well addressed, especially for skew plates with large skew angles. It is due to the presence of strong stress singularity at the supported obtuse corners in the skew plates. The stress singularities of the skew plates lead to the difficulty of convergence when using numerical methods to determine accurate buckling, vibration and bending results for such plates [248]. Leissa and his colleagues conducted a series of studies on free vibration of skew plates using the Ritz method in association with the corner stress singularity functions to address this problem [248-252]. Their studies showed that the inclusion of the stress singularity functions improves the convergence of vibration frequencies significantly and they were able to obtain accurate vibration frequencies for skew plates with large skew angles.

This chapter employs the MLS-Ritz to analyse the challenging problem of vibration of rhombic plates with large skew angles. Through the analysis of rhombic plates with large skew angles, the validity and accuracy of the MLS-Ritz method are further tested and confirmed. Due to the flexibility of the arrangement of the MLS-Ritz grid points, more grid points can be placed around the obtuse corners of a skew plate so as to address the stress singularity problem at the corners. Section 5.2 briefly presents the modeling of skew plates with the MLS-Ritz method. The distribution of the MLS-Ritz grid points and the domain decomposition of the plates are discussed.
Section 5.3 presents the numerical results obtained using the MLS-Ritz method. The influence of the MLS-Ritz grind points on the convergence and accuracy of the method, and a series of cases for rhombic plates of various edge support conditions are presented to demonstrate the efficiency and accuracy of the MLS-Ritz method. Section 5.4 concludes this chapter.

5.2 Problem Definition and Modelling of Skew Plates

5.2.1 Problem definition

Figure 5.1 shows an isotropic, elastic skew plate of length $a$, width $b$, skew angle $\beta$ and uniform thickness $h$ in a Cartesian coordinate system. The plate is of Young’s modulus $E$, Poisson’s ratio $\nu$, the shear modulus $G = E/(2(1+\nu))$ and the mass density $\rho$, respectively.

![Figure 5.1 Dimensions and coordinate system for a skew plate](image-url)
Based on the Kirchhoff plate theory, the total potential energy functional of the plate in harmonic vibration can be obtained as shown in Eq. (4.26) in Chapter 4. The circular frequency $\omega$ needs to be determined.

The boundary conditions of the plate are the same as the ones defined in Section 4.2.3. As the Ritz method will be applied to obtain the vibration solutions for the skew plate, only essential boundary conditions need to be satisfied.

The same procedure employed in Chapter 4 for rectangular and triangular plates is used in this chapter to obtain the governing eigenvalue equation for vibration of skew plates. The point substitution approach is used to process the boundary conditions of the skew plates. Virtual points are introduced outside the clamped edges to assist the implementation of the edge support conditions. The vibration frequency $\omega$ can then be determined by solving the generalized eigenvalue equation.

Due to the special consideration for the stress singularities around the obtuse corners in skew plates, two schemes are proposed to address this problem, namely, (a) one calculation domain with uniform MLS-Ritz grid points; and (b) multiple calculation domains with more flexibility in MLS-Ritz grid point distribution. The details of the two schemes will be discussed in Sections 5.2.2 and 5.2.3, respectively.

5.2.2 One calculation domain
Figure 5.2(a) shows a typical rhombic plate with one calculation domain and uniform distribution of MLS-Ritz grid points (except for the boundary points). The plate is rotated clockwise by an angle of $(\pi/2 - \beta)/2$ as shown in Figure 5.2 for the convenience of implementation of the MLS-Ritz method. The domain is divided into two regions as shown in different colours in Figure 5.2(b) for the Gaussian integration.
Figure 5.2 A rhombic plate (\( \beta = 60 \) degrees) with (a) 293 MLS-Ritz grid points; and (b) 898 Gaussian integration points

5.2.3 Multiple calculation domains

To better address the stress singularities around the obtuse corners, multiple calculation domains can be used in the MLS-Ritz modelling. To illustrate the procedure in the domain decomposition process, a rhombic plate with two subdomains is discussed in this section.

Figure 5.3(a) shows the two subdomains of a rhombic \( SSFF \) plate, where the two lower edges are simply supported and the associated obtuse corner has stress singularity problem. More MLS-Ritz grid points can be arranged in the lower domain
as shown in Figure 5.3(a). The domain is divided into six regions (four in the upper domain and two in the lower domain) as shown in Figure 5.3(b) for the Gaussian integration.

The interface conditions between two adjacent subdomains must be enforced. It requires the following conditions to be imposed at the interface MLS-Ritz grid points:

\[ w^1 - w^2 = 0 \]  
\[ \frac{\partial w^1}{\partial s} - \frac{\partial w^2}{\partial s} = 0 \]

where the superscript 1 and 2 of \( w \) denotes subdomains 1 and 2, respectively, and \( s \) is the normal direction to the interface.

The point substitution approach proposed in Chapter 4 is employed to enforce the boundary and the interface conditions of the plate. We can group the nominal displacements on the grid points of the plate into two categories \( w_B \) and \( w_I \), i.e.

\[
\mathbf{w} = \begin{bmatrix} \mathbf{w}_B \\ \mathbf{w}_I \end{bmatrix}
\]

where \( \mathbf{w}_B \) contains all nominal displacements of the points on the interface between subdomains, on the simply supported and clamped edges and on the outer domain, and \( \mathbf{w}_I \) contains all nominal displacements of the points on the free edge/s and the inner subdomains, respectively.
(a) An SSFF plate with two computational domains. There are 90 MLS-Ritz grid points in the upper domain and 86 MLS-Ritz grid points in the lower domain.

(b) The distribution of Gaussian integration points in the computational domains. There are 208 Gaussian points in the upper domain and 1026 Gaussian points in the lower domain.

Figure 5.3 The distribution of MLS-Ritz grid points and Gaussian integration points in the computational domains of an SSFF rhombic plate.

The geometric boundary condition for a grid point \((x_j, y_j)\) on a simply supported edge is given by
Chapter 5

MLS-Ritz Method for Skew Plates

\[ w^q(x_j, y_j) = \sum_{i=1}^{N^q} R_i(x_j, y_j) w^q_i = 0 \]  \hspace{1cm} (5.4)

If the point is on a clamped edge, the geometric boundary conditions are

\[ w^q(x_j, y_j) = \sum_{i=1}^{N^q} R_i(x_j, y_j) w^q_i = 0 \]  \hspace{1cm} (5.5)

\[ \frac{\partial w^q(x_j, y_j)}{\partial s} = \sum_{i=1}^{N^q} \frac{\partial R_i(x_j, y_j)}{\partial s} w^q_i = 0 \]  \hspace{1cm} (5.6)

And if the two points are at the same location but on the interface between two subdomains, the conditions are

\[ w^1(x_j, y_j) - w^2(x_j, y_j) = \sum_{i=1}^{N^1} R_i(x_j, y_j) w^1_i - \sum_{i=1}^{N^2} R_i(x_j, y_j) w^2_i = 0 \]  \hspace{1cm} (5.7)

\[ \frac{\partial w^1(x_j, y_j)}{\partial s} - \frac{\partial w^2(x_j, y_j)}{\partial s} = \sum_{i=1}^{N^1} \frac{\partial R_i(x_j, y_j)}{\partial s} w^1_i - \sum_{i=1}^{N^2} \frac{\partial R_i(x_j, y_j)}{\partial s} w^2_i = 0 \]  \hspace{1cm} (5.8)

where \( q \) denotes subdomains 1 or 2, and \( N^1 \) and \( N^2 \) are the number of MLS-Ritz grid points on subdomains 1 and 2, respectively. Applying boundary and interface conditions for all grid points on the simply supported and clamped edges and on the interface, we can obtain a system of linear equations given by

\[ [Q \ S] [w_B \ w_I]^T = 0 \]  \hspace{1cm} (5.9)

The nominal displacements of the grid points on the simply supported and clamped edges and on the outer domain can be expressed as

\[ w_B = -Q^{-1} Sw_I \]  \hspace{1cm} (5.10)

The nominal displacements for all grid points can then be expressed in terms of \( w_I \) as follows:

\[ w = \begin{bmatrix} w_B \\ w_I \end{bmatrix} = \begin{bmatrix} -Q^{-1} S \\ I \end{bmatrix} w_I = Tw_I \]  \hspace{1cm} (5.11)
Note that the geometric boundary conditions and interface conditions between subdomains of the plate are effectively enforced by applying the above point substitution approach.

The governing eigenvalue equation for the rhombic plate with subdomains can be obtained using the same procedure as in Section 4.5. Plates with multiple subdomains can be modeled using the procedure described in this section.

5.3 Results and Discussion

5.3.1 Plates with a single computational domain

The characteristics of the convergence and accuracy of the MLS-Ritz method is studied in this section for the vibration analysis of rhombic plates. The plates are rotated clock-wisely by an angle of \((\pi/2 - \beta)/2\) as shown in Figure 5.4 for the convenience of implementation of the MLS-Ritz method. Rhombic plates with nine combinations of boundary conditions will be considered, namely \(SFSF, SFFS, SFFF, SSSS, SSFF, CCCC, CFFF, CSCS, \) and \(CSSS\) as shown in Figure 5.4, where \(S\) denotes a simply supported edge, \(F\) a free edge and \(C\) a clamped edge, respectively. The sequences of the four-letter symbol are marked on the first case in Figure 5.4. The Poisson ratio \(\nu = 0.3\) is used in the calculation. The frequency parameter \(\lambda = (\omega b^2 / \pi^2)\sqrt{\rho h / D}\) is adopted in this chapter.

The 2-D complete polynomial function is used as the basis function in the MLS fitting scheme. In this chapter, the degree of the 2-D polynomial is taken to be \(P = 2\). The number of MLS-Ritz grid points \(N\), the number of Gaussian points \(N_g\) and the radius of support in the MLS fitting scheme \(d\) are the major influence parameters on the convergence and accuracy of the computational results. Detailed studies on the influence of Gaussian points, the degree of the 2-D polynomial basis function and the
value of $k$ in the weight function have been presented in Chapter 4. We concentrate our study herein on the influence of the number of MLS-Ritz points $N_g$ when using the MLS-Ritz method for the vibration analysis of rhombic plates.

![Rhombic plates with different combinations of boundary conditions](image)

Figure 5.4 Rhombic plates with different combinations of boundary conditions

It is well-known that many numerical methods have encountered serious convergence problems when dealing with skew plates with large skew angles. This is caused by the stress singularity at the obtuse angles of the skew plates, especially when the two adjacent edges that form the obtuse angle are supported. A detailed convergence and comparison study on isotropic elastic rhombic plates will be carried out to validate the MLS-Ritz method for the vibration analysis of skew plates. The
calculated results are mainly based on a single computational domain, although one case for rhombic plates based on two computational domains is also discussed.

The first set of plates considered herein are rhombic plates involving free and simply supported edges, i.e. $SFSF$, $SFFS$, $SSF$ and $SSFF$ plates. The skew angle $\beta$ is set to be 30, 60 and 75 degrees, respectively. Uniformly distributed MLS-Ritz grid points (except for the boundary points) are used in the calculation.

Tables 5.1 to 5.3 presents the frequency parameters $\lambda$ for $SFSF$, $SFFS$ and $SSFF$ rhombic plates with skew angle $\beta = 30$, 60 and 75 degrees, respectively. The convergence of the results does not show a monotonically increasing or decreasing pattern when the number of MLS-Ritz grid points increases. It is observed that the frequency parameters for some modes decrease monotonically, for other modes increase monotonically, and for the remaining modes show an oscillating pattern with the increasing of the number of MLS-Ritz grid points. With a small skew angle, i.e. $\beta = 30$ degrees, the convergence of the frequency parameter is easily obtained and good accuracy is achieved when comparing with the ones obtained by McGee et al. [248], Bardell [238] and Singh and Chakraverty [243]. As the skew angle $\beta$ increases, more MLS-Ritz grid points and Gaussian points are required to reach a converged value as shown in Tables 4.1 to 4.3. It is noted that as the number of MLS-Ritz grid points increases, the radius of support $d$ needs to be reduced accordingly to ensure the accuracy and stability of the results. We observe that even with a small number of MLS-Ritz grid points, reasonably accurate frequency parameters can be obtained for the three considered cases. It is also noted that the current results for rhombic plates with large skew angles agree well with the ones from McGee et al. [248] and Bardell [238], but not the ones from Singh and Chakraverty [243]. It is obvious that the results from Singh and Chakraverty [243] have not converged yet when the skew angle becomes large and their results for
plates with skew angle \( \beta = 75 \) degrees are far from converged and are not included in the tables.

Table 5.1 Convergence and comparison study for an isotropic SFSF rhombic plate with skew angle \( \beta = 30, 60 \) and 75 degrees \(( \nu = 0.3, k = 15, P = 2)\)

<table>
<thead>
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<th>(\beta)</th>
<th>(N)</th>
<th>(d/b)</th>
<th>(N_s)</th>
<th>Mode Sequence</th>
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<td>178</td>
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<tr>
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<td>396</td>
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<tr>
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</tr>
<tr>
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</tr>
<tr>
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<td>[248]</td>
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<td>1537</td>
<td>12.246 17.875 36.424 50.352 65.496</td>
</tr>
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<td></td>
<td>[248]</td>
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<td>46.025 108.10 111.78 172.70</td>
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Table 5.2 Convergence and comparison study for an isotropic SFFS rhombic plate with skew angle $\beta = 30$, 60 and 75 degrees ($\nu = 0.3$, $k = 15$, $P = 2$)

<table>
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<tr>
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<th>$d/b$</th>
<th>$N_e$</th>
<th>Mode Sequence</th>
</tr>
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</tr>
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Table 5.3 Convergence and comparison study for an isotropic SFFF rhombic plate with skew angle $\beta = 30$, 60 and 75 degrees ($\nu = 0.3$, $k = 15$, $P = 2$)

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Table 5.4 shows that for an SSSS rhombic plate, the present MLS-Ritz results converge well for the plate with a small skew angle ($\beta = 30$ degrees). The frequency parameters are in good agreement with the ones from Mizusawa et al. [82], Liew et al. [241], Liew and Lam [239], Woo et al. [247], Huang et al. [253] and Durvasula [254], but not the ones from Singh and Chakraverty [243]. Note that the results from Liew et al. [241] and Woo et al. [247] are based on the Mindlin shear deformable
plate theory with the plate thickness ratio $h/b = 0.001$ to simulate the behaviour of thin plates. For the plate with a large skew angle ($\beta = 60$ degrees), the current results agree well with the ones by Huang et al. [253] and Mizusawa et al. [82] and are slight lower than the ones by Liew et al. [241], Bardell [238] and Woo et al. [247]. For the plate with a very large skew angle ($\beta = 75$ degrees), the current results still converge well and are in close agreement with the ones from Huang et al. [253] and are slight smaller than the ones from Bardell [238]. Note that for the $SSSS$ rhombic plate with different skew angles, only a small number of MLS-Ritz grid points is required to obtain converged frequency parameters as shown in Table 4.4. The effect of the stress singularity at the obtuse corners has not substantially affected the convergence of the MLS-Ritz method for the first four cases.

The frequency parameters for an $SSFF$ rhombic plate with a small skew angle ($\beta = 30$ degrees) converges well and are in close agreement with the ones from McGee et al. [248] (see Table 5.5). However, we observe that for an $SSFF$ rhombic plate with larger skew angles ($\beta = 60$ and 75 degrees), although the current method produces converged frequency parameters, the frequency parameters of the first mode for both skew angles are 5.6% and 22% smaller than the ones from McGee et al. [248] and the frequency parameter of the fifth mode for the case with $\beta = 75$ degrees is 8.7% smaller than the one from McGee et al. [248]. We will study this case further in the next subsection with two and three calculation domains with which more MLS-Ritz points can be arranged near the lower obtuse corner of the plate.
Table 5.4 Convergence and comparison study for an isotropic SSSS rhombic plate with skew angle $\beta = 30, 60$ and $75$ degrees ($\nu = 0.3$, $k = 15$, $P = 2$)

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Table 5.5 Convergence and comparison study for an isotropic SSFF rhombic plate with skew angle $\beta = 30, 60$ and $75$ degrees ($\nu = 0.3, k = 15, P = 2$)

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Next, we study the convergence patterns of rhombic plates involving free and clamped edges. Table 4.6 presents the convergence and comparison study for a CCCC rhombic plate with different skew angles. The frequency parameters for this case converge rapidly and are in close agreement with the ones from Liew et al. [241], Bardell [238] and Zitnan [246] and in good agreement with the ones from Mizusawa et al. [78]. The results from Singh and Chakraverty [243] and Hadid and Bashir [244] are not converged when the skew angle becomes large.
Table 5.6 Convergence and comparison study for an isotropic $CCCC$ rhombic plate with skew angle $\beta = 30, 60$ and $75$ degrees ($\nu = 0.3$, $k = 15$, $P = 2$)

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<td>406.75 520.66 619.55 723.30 827.17</td>
</tr>
<tr>
<td></td>
<td>1059</td>
<td>0.20</td>
<td>1766</td>
<td>407.21 520.61 619.74 723.28 827.32</td>
</tr>
<tr>
<td></td>
<td>1425</td>
<td>0.15</td>
<td>1766</td>
<td>407.27 520.61 619.76 723.28 827.34</td>
</tr>
<tr>
<td></td>
<td>1849</td>
<td>0.10</td>
<td>2740</td>
<td>407.32 520.62 619.78 723.28 827.34</td>
</tr>
<tr>
<td></td>
<td>[238]</td>
<td></td>
<td></td>
<td>407.66 520.64 619.94 723.39 827.95</td>
</tr>
<tr>
<td></td>
<td>[246]</td>
<td></td>
<td></td>
<td>407.40 520.61 619.81 723.28</td>
</tr>
</tbody>
</table>
Table 5.7 presents frequency parameters for a CFFF rhombic plate having various skew angles and computed with different number of MLS-Ritz grid points. The current results converge rapidly as the number of MLS-Ritz grid points increases. The MLS-Ritz frequency parameters compare well with the ones from Bardell [238], McGee et al. [250] and Han and Dickinson [245]. The results from Liew et al. [241] have not fully converged yet and the ones from Singh and Chakraverty [243] are far from converged when the skew angle is large.

Finally, the convergence patterns of rhombic plates involving simply supported and clamped edges are studied. Table 4.8 presents the frequency parameters for a CSCS rhombic plate. With a small skew angle ($\beta = 30$ degrees), the current results converge well and are in good agreement with Liew et al. [241], Woo et al. [247], Mizusawa and Kajita [235] and McGee et al. [251]. The current converged results still agree well with McGee et al. [251] and Liew et al. [241] when the skew angle $\beta = 60$ degrees, while the results from Woo et al. [247], Mizusawa and Kajita [235] and Singh and Chakraverty (1994) [243] are not fully converged. The MLS-Ritz results for the CSCS plate with the skew angle $\beta = 75$ degrees show good convergence. However, the current frequency parameters of the fourth and fifth modes are lower than the ones from McGee et al. [251].
Table 5.7 Convergence and comparison study for an isotropic CFFF rhombic plate with skew angle $\beta = 30, 60$ and 75 degrees ($\nu = 0.3, k = 15, P = 2$)

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$N$</th>
<th>$d/b$</th>
<th>$N_s$</th>
<th>Mode Sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>$30^\circ$</td>
<td>72</td>
<td>0.75</td>
<td>178</td>
<td>3.9551 9.5378 25.649 26.289 41.651</td>
</tr>
<tr>
<td></td>
<td>228</td>
<td>0.50</td>
<td>396</td>
<td>3.9255 9.4257 25.344 25.942 41.367</td>
</tr>
<tr>
<td></td>
<td>562</td>
<td>0.40</td>
<td>1072</td>
<td>3.9257 9.4129 25.289 25.928 41.339</td>
</tr>
<tr>
<td></td>
<td>1188</td>
<td>0.30</td>
<td>2384</td>
<td>3.9259 9.4100 25.279 25.926 41.333</td>
</tr>
<tr>
<td></td>
<td>2046</td>
<td>0.25</td>
<td>4220</td>
<td>3.9268 9.4096 25.280 25.928 41.332</td>
</tr>
<tr>
<td></td>
<td>[241]</td>
<td></td>
<td></td>
<td>3.9883  10.208 26.089  27.401 43.688</td>
</tr>
<tr>
<td>$60^\circ$</td>
<td>52</td>
<td>0.75</td>
<td>92</td>
<td>8.2544 20.434 30.587 33.116 46.885</td>
</tr>
<tr>
<td></td>
<td>314</td>
<td>0.40</td>
<td>898</td>
<td>5.2959 16.014 30.567 45.513 59.316</td>
</tr>
<tr>
<td></td>
<td>630</td>
<td>0.30</td>
<td>1990</td>
<td>5.2701 16.013 30.471 45.426 59.193</td>
</tr>
<tr>
<td></td>
<td>1588</td>
<td>0.20</td>
<td>3508</td>
<td>5.2545 16.017 30.419 45.361 59.111</td>
</tr>
<tr>
<td></td>
<td>2224</td>
<td>0.15</td>
<td>3508</td>
<td>5.2515 16.017 30.408 45.348 59.094</td>
</tr>
<tr>
<td></td>
<td>[238]</td>
<td></td>
<td></td>
<td>5.251   16.061 30.415   45.340</td>
</tr>
<tr>
<td></td>
<td>[241]</td>
<td></td>
<td></td>
<td>5.4628  16.899 32.431  50.030 61.676</td>
</tr>
<tr>
<td></td>
<td>[243]</td>
<td></td>
<td></td>
<td>6.1126  17.223 35.375  59.757 69.554</td>
</tr>
<tr>
<td></td>
<td>[250]</td>
<td></td>
<td></td>
<td>5.2436  16.027 30.408   45.339 59.587</td>
</tr>
<tr>
<td>$75^\circ$</td>
<td>42</td>
<td>0.75</td>
<td>52</td>
<td>19.162 26.781 34.724 55.225 72.107</td>
</tr>
<tr>
<td></td>
<td>208</td>
<td>0.40</td>
<td>468</td>
<td>6.1598 25.381 48.876 73.091 95.983</td>
</tr>
<tr>
<td></td>
<td>388</td>
<td>0.30</td>
<td>1006</td>
<td>6.1051 25.145 48.773 72.902 95.750</td>
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<tr>
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<td>910</td>
<td>0.20</td>
<td>1766</td>
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<tr>
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</tr>
<tr>
<td></td>
<td>1640</td>
<td>0.10</td>
<td>2740</td>
<td>6.0589 24.941 48.725 72.826 95.536</td>
</tr>
<tr>
<td></td>
<td>[238]</td>
<td></td>
<td></td>
<td>6.064   24.936 49.383   73.955</td>
</tr>
<tr>
<td></td>
<td>[245]</td>
<td></td>
<td></td>
<td>6.0638  24.936 49.382   73.956 95.551</td>
</tr>
</tbody>
</table>
Table 5.8 Convergence and comparison study for an isotropic CSCS rhombic plate with skew angle $\beta = 30, 60$ and 75 degrees ($\nu = 0.3, k = 15, P = 2$)

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$N$</th>
<th>$d/b$</th>
<th>$N_s$</th>
<th>Mode Sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
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<td>1</td>
</tr>
<tr>
<td>$30^\circ$</td>
<td>79</td>
<td>0.75</td>
<td>178</td>
<td>20.206</td>
</tr>
<tr>
<td></td>
<td>241</td>
<td>0.50</td>
<td>396</td>
<td>36.923</td>
</tr>
<tr>
<td></td>
<td>583</td>
<td>0.40</td>
<td>1072</td>
<td>36.945</td>
</tr>
<tr>
<td></td>
<td>1219</td>
<td>0.30</td>
<td>2384</td>
<td>36.949</td>
</tr>
<tr>
<td></td>
<td>2087</td>
<td>0.25</td>
<td>4220</td>
<td>36.951</td>
</tr>
<tr>
<td></td>
<td>[235]</td>
<td></td>
<td></td>
<td>37.16</td>
</tr>
<tr>
<td></td>
<td>[241]</td>
<td></td>
<td></td>
<td>36.963</td>
</tr>
<tr>
<td></td>
<td>[243]</td>
<td></td>
<td></td>
<td>37.193</td>
</tr>
<tr>
<td></td>
<td>[247]</td>
<td></td>
<td></td>
<td>37.016</td>
</tr>
<tr>
<td></td>
<td>[251]</td>
<td></td>
<td></td>
<td>36.954</td>
</tr>
<tr>
<td>$60^\circ$</td>
<td>59</td>
<td>0.75</td>
<td>92</td>
<td>50.248</td>
</tr>
<tr>
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<td>898</td>
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<td></td>
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<td>0.30</td>
<td>1990</td>
<td>95.928</td>
</tr>
<tr>
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<td>3508</td>
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</tr>
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<td>2285</td>
<td>0.15</td>
<td>3508</td>
<td>96.099</td>
</tr>
<tr>
<td></td>
<td>[235]</td>
<td></td>
<td></td>
<td>101.0</td>
</tr>
<tr>
<td></td>
<td>[241]</td>
<td></td>
<td></td>
<td>97.272</td>
</tr>
<tr>
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<td>[243]</td>
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<td>104.53</td>
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<tr>
<td></td>
<td>[247]</td>
<td></td>
<td></td>
<td>99.920</td>
</tr>
<tr>
<td></td>
<td>[251]</td>
<td></td>
<td></td>
<td>96.209</td>
</tr>
<tr>
<td>$75^\circ$</td>
<td>49</td>
<td>0.75</td>
<td>52</td>
<td>109.38</td>
</tr>
<tr>
<td></td>
<td>229</td>
<td>0.40</td>
<td>468</td>
<td>311.49</td>
</tr>
<tr>
<td></td>
<td>419</td>
<td>0.30</td>
<td>1006</td>
<td>313.34</td>
</tr>
<tr>
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<td>961</td>
<td>0.20</td>
<td>1766</td>
<td>314.85</td>
</tr>
<tr>
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<td>1307</td>
<td>0.15</td>
<td>1766</td>
<td>315.03</td>
</tr>
<tr>
<td></td>
<td>1711</td>
<td>0.10</td>
<td>2740</td>
<td>315.32</td>
</tr>
<tr>
<td></td>
<td>[251]</td>
<td></td>
<td></td>
<td>316.40</td>
</tr>
</tbody>
</table>
Table 5.9 shows the convergence pattern of the frequency parameters for a CSSS rhombic plate with different skew angles. We observe that when the number of MLS-Ritz grid points is small, the MLS-Ritz method generates erroneous results for the plate with small and large skew angles. As the number of MLS-Ritz grid points increases, the frequency parameters show rapid convergence. The current results for the CSSS rhombic plate with the skew angle $\beta = 30$ degrees are in close agreement with the ones from McGee et al. [251] and in good agreement with Nair and Durvasula [255], Liew and Lam [239] and Singh and Chakraverty [243]. The MLS-Ritz results are still in close agreement with the results from McGee et al. [251] when the skew angle increases to 60 degrees. Further increasing the skew angle to 75 degrees, the current results appear converged. However, the MLS-Ritz frequency parameters of the third, fourth and fifth modes are smaller than the ones from McGee et al. [250] and it shows the same trend as for the CSCS and SSFF rhombic plates (see Tables 5.5 and 5.8).
5.3.2 SSFF plate with two and more computational domains

As the MLS-Ritz method is developed for analysing plates with complicated shapes and support conditions, we demonstrate herein that the method can be used for the vibration analysis of rhombic plates with multiple computational domains. The SSFF rhombic plate is selected for the illustration purpose as it is found that some of the
first five frequency parameters for the plate do not compare well with the ones from McGee et al. [248] when the skew angle of the plates becomes large (see Table 5.5). It is hoped that by increasing the MLS-Ritz grid points around the obtuse corner where stress singularity exists, better converged results can be obtained.

Figure 5.3 shows the MLS-Ritz grid point distribution for an SSFF plate with two computational domains. The height of the lower domain is 1/4 of the total height of the plate. The density of the MLS-Ritz grid points in the lower domain is three times of the density in the upper domain (see Figure 5.3).

Table 5.10 presents the convergence and comparison study for the first 5 frequency parameters of the SSFF rhombic plate using two computational domains, where $N_u$ and $N_l$ denote the number of MLS-Ritz grid points, $d_u$ and $d_l$ are the radii of support, and $N_{gu}$ and $N_{gl}$ are the number of Gaussian points used in the upper and lower domains, respectively.

Comparing with results in Table 5.5, it is observed that less MLS-Ritz grid points are needed when using two computational domains to achieve the same level of convergence as using one computational domain. However, the present fundamental frequency parameters for $\beta = 60$ and 75 degrees and the fifth frequency parameter for $\beta = 75$ degrees are still substantially lower than the ones from McGee et al. [248].
Table 5.10 Convergence and comparison study for an isotropic SSFF rhombic plate with skew angle $\beta = 30, 60$ and $75$ degrees ($\nu = 0.3$, $k = 15$, $P = 2$) and two calculation domains

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$N_u \ N_l$</th>
<th>$d_u/b$, $d_l/b$</th>
<th>$N_{gu} \ N_{gl}$</th>
<th>Mode Sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>60°</td>
<td>40, 44</td>
<td>0.75, 0.45</td>
<td>80, 108</td>
<td>6.5249</td>
</tr>
<tr>
<td></td>
<td>115, 147</td>
<td>0.50, 0.30</td>
<td>296, 390</td>
<td>5.1339</td>
</tr>
<tr>
<td></td>
<td>115, 210</td>
<td>0.50, 0.20</td>
<td>296, 670</td>
<td>4.6205</td>
</tr>
<tr>
<td></td>
<td>172, 212</td>
<td>0.45, 0.20</td>
<td>454, 670</td>
<td>4.6010</td>
</tr>
<tr>
<td></td>
<td>172, 286</td>
<td>0.45, 0.15</td>
<td>454,1026</td>
<td>4.5055</td>
</tr>
<tr>
<td></td>
<td>172, 375</td>
<td>0.45, 0.125</td>
<td>454,1202</td>
<td>4.4004</td>
</tr>
<tr>
<td></td>
<td>172, 401</td>
<td>0.45, 0.10</td>
<td>454,1388</td>
<td>4.3669</td>
</tr>
<tr>
<td></td>
<td>[248]</td>
<td></td>
<td>4.6157</td>
<td>22.629</td>
</tr>
<tr>
<td>75°</td>
<td>80, 88</td>
<td>0.50, 0.30</td>
<td>160, 210</td>
<td>16.445</td>
</tr>
<tr>
<td></td>
<td>178, 224</td>
<td>0.50, 0.15</td>
<td>410, 536</td>
<td>7.5658</td>
</tr>
<tr>
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<td>178, 255</td>
<td>0.50, 0.10</td>
<td>410, 720</td>
<td>6.5727</td>
</tr>
<tr>
<td></td>
<td>178, 311</td>
<td>0.50, 0.075</td>
<td>410, 816</td>
<td>6.4079</td>
</tr>
<tr>
<td></td>
<td>313, 321</td>
<td>0.40, 0.075</td>
<td>884, 816</td>
<td>6.4081</td>
</tr>
<tr>
<td></td>
<td>[248]</td>
<td></td>
<td>8.2184</td>
<td>39.935</td>
</tr>
</tbody>
</table>

Figure 5.5 shows the MLS-Ritz grid point distribution for an SSFF plate with three computational domains. The height of both the lower domain and the middle domain is $1/8$ of the total height of the plate. The density of the MLS-Ritz grid points in the lower domain is the highest, followed by the middle domain and the upper domain (see Figure 5.5).

Table 5.11 shows the convergence and comparison study for the first 5 frequency parameters of the SSFF rhombic plate using three computational domains, where $N_u$, $N_m$ and $N_l$ denote the number of MLS-Ritz grid points, $d_u$, $d_m$ and $d_l$ are the radii of support, and $N_{gu}$, $N_{gm}$ and $N_{gl}$ are the number of Gaussian points used in the upper, middle and lower domains, respectively.
Comparing with results in Tables 5.5 and 5.10, we can demonstrate that the efficiency of the MLS-Ritz method is further improved with three calculation domains as more MLS-Ritz grid points can be arranged around the supported obtuse corner where the stress singularity occurs. The same level of convergence can be achieved with less MLS-Ritz grid points using three calculation domains when comparing with the same plate with the single domain or two domain computations.

Figure 5.5 An SSFF rhombic plate ($\beta = 60$ degrees) with 172, 82 and 116 MLS-Ritz grid points in the upper, middle and lower domains, respectively
Table 5.11 Convergence and comparison study for an isotropic SSFF rhombic plate with skew angle $\beta = 30$, 60 and 75 degrees ($v = 0.3$, $k = 15$, $P = 2$) and three calculation domains

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$N_m$</th>
<th>$N_m$</th>
<th>$N_l$</th>
<th>$d/b$</th>
<th>$d_u/b$</th>
<th>$d_l/b$</th>
<th>$N_{gu}$</th>
<th>$N_{gm}$</th>
<th>$N_{gl}$</th>
<th>Mode Sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>60°</td>
<td>172</td>
<td>62</td>
<td>34</td>
<td>1.0</td>
<td>0.8</td>
<td>0.6</td>
<td>1752</td>
<td>812</td>
<td>722</td>
<td>6.5148, 22.628, 43.148, 68.959, 90.441</td>
</tr>
<tr>
<td></td>
<td>172</td>
<td>62</td>
<td>86</td>
<td>1.0</td>
<td>0.8</td>
<td>0.2</td>
<td>1752</td>
<td>1348</td>
<td>1800</td>
<td>4.3681, 22.624, 42.908, 68.956, 87.968</td>
</tr>
<tr>
<td></td>
<td>172</td>
<td>82</td>
<td>90</td>
<td>1.0</td>
<td>0.6</td>
<td>0.2</td>
<td>1752</td>
<td>1348</td>
<td>1800</td>
<td>4.3628, 22.626, 42.913, 68.947, 87.970</td>
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<td></td>
<td>172</td>
<td>82</td>
<td>116</td>
<td>1.0</td>
<td>0.6</td>
<td>0.2</td>
<td>1752</td>
<td>1348</td>
<td>1800</td>
<td>4.5425, 22.627, 42.931, 68.948, 88.149</td>
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<tr>
<td></td>
<td>[248]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>4.6157, 22.629, 42.959, 69.091, 88.491</td>
</tr>
<tr>
<td>75°</td>
<td>104</td>
<td>73</td>
<td>86</td>
<td>1.0</td>
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<td>0.25</td>
<td>884</td>
<td>738</td>
<td>926</td>
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</tr>
<tr>
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<td>104</td>
<td>75</td>
<td>105</td>
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<td>0.65</td>
<td>0.19</td>
<td>1960</td>
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<td>926</td>
<td>6.6472, 39.938, 68.983, 104.85, 144.04</td>
</tr>
<tr>
<td></td>
<td>178</td>
<td>81</td>
<td>105</td>
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<td>0.65</td>
<td>0.19</td>
<td>1960</td>
<td>1496</td>
<td>926</td>
<td>6.6453, 39.923, 68.934, 104.76, 143.95</td>
</tr>
<tr>
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<td>243</td>
<td>85</td>
<td>105</td>
<td>0.9</td>
<td>0.65</td>
<td>0.19</td>
<td>1960</td>
<td>1496</td>
<td>926</td>
<td>6.41758, 39.925, 68.939, 104.77, 143.96</td>
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<td>[248]</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>8.2184, 39.935, 69.306, 105.34, 157.64</td>
</tr>
</tbody>
</table>
5.4 Conclusions

The MLS-Ritz method has been applied to study the challenging problem of free vibration of rhombic plates with large skew angles and different combinations of boundary conditions. Extensive convergence and comparison studies have been carried out to check the validity and accuracy of the MLS-Ritz method for the analysis of rhombic plates. It has been found that the MLS-Ritz method is very stable and can generate converged vibration results for rhombic plates with various combinations of edge support conditions.

It should be noted that although the current study has not incorporated the corner stress singularity functions proposed by Lessia and his associates [248-252], the MLS-Ritz method is able to generate accurate frequency parameters for rhombic plates with very large skew angles. Future study will need to be carried out to examine if the efficiency of the MLS-Ritz method can be further improved by the inclusion of the corner stress singularity functions.

This study has also revealed that some of the previous studies on the vibration of rhombic plates did not provide converged results. The MLS-Ritz method will be further applied to analyse 3-D vibration of isotropic elastic plates in Chapter 6.
CHAPTER 6

APPLICATION OF MOVING LEAST SQUARE RITZ METHOD FOR 3-D PLATE ANALYSIS

6.1 Introduction

As stated in the previous chapters, the vibration analysis of plates has been extensively studied over the last 100 years and most of the studies are for plates based on the classical plate theory [72]. When the thickness of a plate becomes large comparing to its length and width, the vibration results based on the classical plate theory are over-predicted and are unreliable in engineering practice. The problem is due to the negligence of the transverse shear deformation of the plate in the model of the classical plate theory. Several researchers proposed modified plate theories to overcome the problem caused by the classical plate theory, e.g. the first order shear deformable plate theories by Reissener [256] and Mindlin [257] and the higher-order plate theory by Reddy [220]. These theories partially addressed the problem associated with the transverse shear deformation in the plates and are reasonably accurate in predicting the behaviour of plates with moderate thickness. However, the simplifications in the modified plate theories also introduce inaccuracies when the thickness of a plate becomes large. Therefore, a three-dimensional analysis based on the theory of elasticity is required to deal with plates with large thickness.
In the past few decades, many studies have been carried out for the 3-D vibration analysis of rectangular plates by applying/developing various analytical and numerical methods. Due to the complexity of the 3-D problem, very limited analytical/exact vibration solutions exist in the open literature [258]. Most 3-D vibration results for thick plates are obtained by various numerical methods. Cheung and Chakrabarti [259] extended the finite element method to develop a finite layer method in the study of vibration of thick rectangular plates with various combinations of edge support conditions. The beam functions were employed in the two in-plane directions and a simple polynomial function was used in the thickness direction in their modelling. Leissa and Zhang [95] employed a set of simple algebraic polynomial based displacement fields for the 3-D free vibration analysis of cantilevered parallelepiped. McGee and Giaimo [97] presented the first-known 3-D vibration solutions for cantilevered right-angled triangular plates with varying thickness. The Ritz method was employed in their study together with a simple polynomial function as the trial functions for the displacement fields of the triangular plates. Liew and his colleagues conducted a series of studies on the 3-D vibration analysis of rectangular, skew trapezoidal and skew plates using the Ritz method in conjunction with the orthogonal polynomial functions [90-92]. Liew and Teo [178] also applied the differential quadrature (DQ) method on the 3-D vibration analysis of rectangular plates. Malik and Bert [260] employed the DQ method to analyse the 3-D vibration behaviour of thick rectangular plates. Recently, Zhou and his associates [261-265] developed a Ritz method with Chebyshev polynomials to study 3-D vibration problems for thick circular, triangular and rectangular plates with various complications.

This chapter extends the MLS-Ritz method for the 3-D vibration analysis of isotropic elastic thick plates. The analysis is based on the linear elasticity theory. The MLS data interpolation technique presented in Chapter 3 is utilized to establish the Ritz trial functions for the displacement fields of a thick plate. Vibration frequencies for thick square and right-angled isosceles triangular plates are obtained by the MLS-
Ritz method. The implementation of the boundary conditions is discussed in details for the 3-D analysis of thick plates. The reliability and accuracy of the presented method are examined. Extensive convergence and comparison studies are performed to evaluate the validity and accuracy of the MLS-Ritz method in the 3-D vibration analysis of thick square and right-angled isosceles triangular plates with various combinations of edge support conditions.

This chapter is organised as follows. Section 6.2 presents the mathematical modelling of the 3-D thick plates. Section 6.3 shows the 3-D vibration solutions for square and right-angled isosceles triangular thick plates. Extensive convergence and comparison studies are carried out in this section. Section 6.4 concludes this chapter.

6.2 Mathematical Formulation

6.2.1 Problem definition

Figure 6.1 shows the dimensions of a thick rectangular plate and a thick right-angled triangular plate in the Cartesian coordinate system \((x, y, z)\). The free vibration frequencies of the plates based on 3-D elasticity theory are to be determined using the MLS-Ritz method.
6.2.2 Three-dimensional energy functional for a plate

In the small strain 3-D elasticity theory, the strains and displacement fields have the following relationship:

\[ \varepsilon_x = \frac{\partial u}{\partial x} \]  \hspace{1cm} (6.1a)

\[ \varepsilon_y = \frac{\partial v}{\partial y} \]  \hspace{1cm} (6.1b)

\[ \varepsilon_z = \frac{\partial w}{\partial z} \]  \hspace{1cm} (6.1c)

\[ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \]  \hspace{1cm} (6.1d)

\[ \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \]  \hspace{1cm} (6.1e)
where \( u(x, y, z) \), \( v(x, y, z) \) and \( w(x, y, z) \) are the displacements along the \( x \), \( y \) and \( z \) directions, respectively, \( \varepsilon_x \), \( \varepsilon_y \) and \( \varepsilon_z \) are the normal stresses in the \( x \), \( y \) and \( z \) directions, and \( \gamma_{xy} \), \( \gamma_{yz} \) and \( \gamma_{zx} \) are the shear strains in the \( x-y \), \( y-z \) and \( z-x \) planes, respectively.

For isotropic linear elastic material, the relationship of stresses and strains is defined by the generalised Hooke’s law as given by Eq. (4.11). The strain energy in a deformed body can be calculated by

\[
U = \frac{1}{2} \int \left( \sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy} \gamma_{xy} + \tau_{yz} \gamma_{yz} + \tau_{zx} \gamma_{zx} \right) dV
\]

where \( V \) is the volume of the elastic body.

Substituting Eqs. (4.11) and (6.1) into Eq. (6.2), we can obtain the strain energy of a 3-D body in terms of the displacement fields:

\[
U_{\text{max}} = \Delta \int \left\{ \nu \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)^2 + (1 - 2\nu) \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial z} \right)^2 \right] \right\} dV
\]

\[
\frac{1}{2} (1 - 2\nu) \left[ \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)^2 \right] dV
\]

and the kinetic energy of the 3-D body in harmonic vibration can be obtained as:

\[
T_{\text{max}} = \frac{\rho \omega^2}{2} \int \left[ u^2 + v^2 + w^2 \right] dV
\]

where \( \Delta = E/[(1 + \nu)(1 - 2\nu)] \), \( E \) is the Young’s modulus, \( \nu \) is the Poisson ratio and \( \rho \) is the mass density. For the considered two plate systems, we can use the non-dimensionalised coordinate system by using the transformation below:

\[
X = \frac{x}{a}, \quad Y = \frac{y}{b}, \quad Z = \frac{z}{h}
\]
Eqs. (6.3) and (6.4) can be expressed as follows:

\[
U_{\text{max}} = \frac{\Delta}{2} \int_{\mathcal{C}} \left( \frac{1}{a} \frac{\partial u}{\partial X} + \frac{1}{b} \frac{\partial u}{\partial Y} + \frac{1}{h} \frac{\partial u}{\partial Z} \right)^2 + (1-2\nu) \left( \frac{1}{a} \frac{\partial v}{\partial X} \right)^2 + \left( \frac{1}{b} \frac{\partial v}{\partial Y} \right)^2 + \left( \frac{1}{h} \frac{\partial v}{\partial Z} \right)^2 \\
+ \frac{1}{2} (1-2\nu) \left[ \left( \frac{1}{b} \frac{\partial u}{\partial Y} + \frac{1}{a} \frac{\partial v}{\partial X} \right)^2 + \left( \frac{1}{h} \frac{\partial v}{\partial Z} + \frac{1}{b} \frac{\partial w}{\partial Y} \right)^2 + \left( \frac{1}{h} \frac{\partial u}{\partial Z} + \frac{1}{a} \frac{\partial w}{\partial X} \right)^2 \right] \, abh \, d\mathcal{V} \tag{6.6}
\]

\[
T_{\text{max}} = \frac{\rho \omega^2}{2} \int_{\mathcal{C}} [u^2 + v^2 + w^2] \, abh \, d\mathcal{V} \tag{6.7}
\]

where \( \mathcal{V} \) is the non-dimensional volume of the plate. The total potential energy functional (or the Lagrangian) of the plate in harmonic vibration can be expressed as

\[
F = U_{\text{max}} - T_{\text{max}} \tag{6.8}
\]

As the Ritz method will be employed to derive the governing eigenvalue equation for the 3-D plate system, only essential boundary conditions of the plate need to be satisfied:

For a simply supported edge:

\[
u \cos \alpha + v \sin \alpha = 0, \quad w = 0 \tag{6.9}
\]

where \( \alpha \) is the tangential angle of the boundary point on the edge.

For a clamped edge:

\[
u = 0, \quad v = 0, \quad w = 0 \tag{6.10}
\]

No constraint needs to be imposed for a free edge.

### 6.2.3 MLS-Ritz formulations

A set of MLS-Ritz grid points is selected in a 3-D rectangular plate in the non-dimensionalised coordinate system as shown in Figure 6.2. The MLS interpolation technique for a 3-D function has been discussed in Chapter 3. Using the technique
described in Chapter 3, the displacement fields of the 3-D plate can be approximately expressed by

\begin{align*}
\mathbf{u}(X,Y,Z) &= \sum_{i=1}^{N} R_i(X,Y,Z) u_i = R\mathbf{u} = \mathbf{u}^T \mathbf{R}^T \tag{6.11a} \\
v(X,Y,Z) &= \sum_{i=1}^{N} R_i(X,Y,Z) v_i = R\mathbf{v} = \mathbf{v}^T \mathbf{R}^T \tag{6.11b} \\
w(X,Y,Z) &= \sum_{i=1}^{N} R_i(X,Y,Z) w_i = R\mathbf{w} = \mathbf{w}^T \mathbf{R}^T \tag{6.11c}
\end{align*}

where

Figure 6.2 Boundary conditions for a cantilevered rectangular plate in the non-dimensionalised coordinate system
\[ \mathbf{R} = \begin{bmatrix} R_1 & R_2 & \cdots & R_N \end{bmatrix} \]  
(6.12a)

\[ \mathbf{u} = \begin{bmatrix} u_1 & u_2 & \cdots & u_N \end{bmatrix}^T \]  
(6.12b)

\[ \mathbf{v} = \begin{bmatrix} v_1 & v_2 & \cdots & v_N \end{bmatrix}^T \]  
(6.12c)

\[ \mathbf{w} = \begin{bmatrix} w_1 & w_2 & \cdots & w_N \end{bmatrix}^T \]  
(6.12d)

and the shape function \( R_i \) has been defined in Eq. (3.43) in Chapter 3, \( u_i, v_i \) and \( w_i \) are the nominal values of \( u, v \) and \( w \) at the \( i \)th grid point, and \( N \) is the total number of MLS-Ritz grid points in the plate.

Substituting Eqs. (6.11) and (6.12) into Eq. (6.8), we obtain

\[
F = \frac{1}{2} \begin{bmatrix} \mathbf{u}^T & \mathbf{v}^T & \mathbf{w}^T \end{bmatrix} \begin{bmatrix} k_{uu} & k_{uv} & k_{uw} \\ k_{vu} & k_{vv} & k_{vw} \\ k_{wu} & k_{vw} & k_{ww} \end{bmatrix} \begin{bmatrix} m_{uu} & 0 & 0 \\ 0 & m_{vv} & 0 \\ 0 & 0 & m_{ww} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{bmatrix} - \omega^2 \begin{bmatrix} m_{uu} & 0 & 0 \\ 0 & m_{vv} & 0 \\ 0 & 0 & m_{ww} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{bmatrix} \]  
(6.13)

where the terms in the stiffness matrix and the mass matrix can be calculated by

\[
k_{uu} = \Delta \iint_{\bar{V}} \left[ \frac{\nu + (1 - 2\nu)}{a^2} \mathbf{R}_x \mathbf{R}_x^T + \frac{1 - 2\nu}{2b^2} \mathbf{R}_y \mathbf{R}_y^T + \frac{1 - 2\nu}{2h^2} \mathbf{R}_z \mathbf{R}_z^T \right] abh d\bar{V} \]  
(6.14a)

\[
k_{uv} = k_{vu} = \Delta \iint_{\bar{V}} \left[ \frac{\nu}{ab} \mathbf{R}_x \mathbf{R}_y^T + \frac{1 - 2\nu}{2ab} \mathbf{R}_y \mathbf{R}_x^T \right] abh d\bar{V} \]  
(6.14b)

\[
k_{uw} = k_{wu} = \Delta \iint_{\bar{V}} \left[ \frac{\nu}{ah} \mathbf{R}_x \mathbf{R}_z^T + \frac{1 - 2\nu}{2ah} \mathbf{R}_z \mathbf{R}_x^T \right] abh d\bar{V} \]  
(6.14c)

\[
k_{vv} = \Delta \iint_{\bar{V}} \left[ \frac{\nu + (1 - 2\nu)}{b^2} \mathbf{R}_y \mathbf{R}_y^T + \frac{1 - 2\nu}{2a^2} \mathbf{R}_x \mathbf{R}_x^T + \frac{1 - 2\nu}{2h^2} \mathbf{R}_z \mathbf{R}_z^T \right] abh d\bar{V} \]  
(6.14d)

\[
k_{vw} = k_{wv} = \Delta \iint_{\bar{V}} \left[ \frac{\nu}{bh} \mathbf{R}_y \mathbf{R}_z^T + \frac{1 - 2\nu}{2bh} \mathbf{R}_z \mathbf{R}_y^T \right] abh d\bar{V} \]  
(6.14e)

\[
k_{ww} = \Delta \iint_{\bar{V}} \left[ \frac{\nu + (1 - 2\nu)}{h^2} \mathbf{R}_z \mathbf{R}_z^T + \frac{1 - 2\nu}{2a^2} \mathbf{R}_x \mathbf{R}_x^T + \frac{1 - 2\nu}{2b^2} \mathbf{R}_y \mathbf{R}_y^T \right] abh d\bar{V} \]  
(6.14f)

\[
\mathbf{m}_{uu} = \rho \iint_{\bar{V}} \mathbf{R} \mathbf{R}^T abh d\bar{V} \]  
(6.15a)
The numerical evaluation of $R$ and its derivatives with respect to $X$, $Y$ and $Z$ in Eqs. (6.14) and (6.15) can be determined by the procedure described in Chapter 3. The integration in Eqs. (6.14) and (6.15) will be performed by the Gaussian quadrature over the whole calculation domain.

### 6.2.4 Implementation of boundary conditions

The point substitution approach is again employed to implement the boundary conditions when using the MLS-Ritz method to analyse the vibration of 3-D plates. Take a cantilevered rectangular plate as an example, the boundary conditions of the plate can be processed as follows.

Figure 6.2 shows a cantilevered rectangular plate and its boundary conditions in the non-dimensionalised coordinate system $(X, Y, Z)$. To implement the boundary conditions, the displacements at the $j$th MLS-Ritz point on the face of the clamped edge ($X = 0$) must satisfy

$$u(X_j, Y_j, Z_j) = R^T_j u = 0$$  
$$v(X_j, Y_j, Z_j) = R^T_j v = 0$$  
$$w(X_j, Y_j, Z_j) = R^T_j w = 0$$

We can group the nominal values of $u$, $v$ and $w$ on the MLS-Ritz points as boundary and interior parts, respectively:

$$[u] = \begin{bmatrix} u_B \\ u_f \end{bmatrix}$$

$$[v] = \begin{bmatrix} v_B \\ v_f \end{bmatrix}$$
In this case, the boundary part contains all MLS-Ritz points on the face of the fixed edge and the interior part is the remaining MLS-Ritz points in the calculation domain. Applying boundary conditions for all grid points on the face of the fixed edge, we can obtain

$$\begin{bmatrix} Q_u & S_u \end{bmatrix} \begin{bmatrix} u_B \\ u_I \end{bmatrix} = 0$$  \hspace{1cm} (6.18a)

$$\begin{bmatrix} Q_v & S_v \end{bmatrix} \begin{bmatrix} v_B \\ v_I \end{bmatrix} = 0$$  \hspace{1cm} (6.18b)

$$\begin{bmatrix} Q_w & S_w \end{bmatrix} \begin{bmatrix} w_B \\ w_I \end{bmatrix} = 0$$  \hspace{1cm} (6.18c)

The values of the nominal boundary displacements $u_B$, $v_B$ and $w_B$ can be expressed in terms of the values of the nominal interior displacements as

$$u_B = -Q_u^{-1} S_u u_I = T_u u_I$$  \hspace{1cm} (6.19a)

$$v_B = -Q_v^{-1} S_v v_I = T_v v_I$$  \hspace{1cm} (6.19b)

$$w_B = -Q_w^{-1} S_w w_I = T_w w_I$$  \hspace{1cm} (6.19c)

Therefore, the vector of the nominal displacements can be written as

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & 0 & I & 0 & 0 \\ 0 & I & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} T_u & 0 & 0 \\ 0 & T_v & 0 \\ 0 & 0 & T_w \end{bmatrix} \begin{bmatrix} u_I \\ v_I \\ w_I \end{bmatrix} = \Gamma \begin{bmatrix} u_I \\ v_I \\ w_I \end{bmatrix}$$  \hspace{1cm} (6.20)

Substituting Eq. (6.20) into Eq. (6.13) and minimizing the total potential energy functional $F$ with respect to the nominal interior displacements, we obtain
\[
\begin{bmatrix}
(K - \omega^2 M) \\
\end{bmatrix}
\begin{bmatrix}
u_l \\
w_l
\end{bmatrix}
= 0
\]

(6.21)

where the modified stiffness and mass matrices are determined by

\[
K = \Gamma^T \begin{bmatrix}
k_{uu} & k_{uv} & k_{uw} \\
k_{uv} & k_{vv} & k_{vw} \\
k_{uw} & k_{vw} & k_{ww}
\end{bmatrix} \Gamma
\]

(6.22)

\[
M = \Gamma^T \begin{bmatrix}
m_{uu} & 0 & 0 \\
0 & m_{vv} & 0 \\
0 & 0 & m_{ww}
\end{bmatrix} \Gamma
\]

(6.23)

The frequency parameter \(\omega\) can be evaluated by solving the generalised eigenvalue equation as defined in Eq. (6.21).

The same procedure can be applied for plates with other combinations of edge support conditions.

6.3 Results and Discussion

The 3-D MLS-Ritz method is employed to compute the non-dimensional frequency parameters, \(\hat{\lambda} = (\omega h^2 / \pi^2)(\rho h/D)^{3/2}\) for the isotropic, elastic thick rectangular and right-angled isosceles triangular plates subject to different combinations of boundary constraints.

The 3-D complete polynomial function is used as the basis function in the MLS fitting scheme. The number of degrees of the polynomial is taken to be 2, i.e. the terms of the basis function are \(P^2(X,Y,Z) = [1\ X\ Y\ Z\ X^2\ Y^2\ Z^2\ XY\ XZ\ YZ]\). The major influence parameters on the convergence and accuracy of the computational results are the MLS-Ritz grid points, the Gaussian integration points and the radius
of support $d$. The number of MLS-Ritz grid points in the $X$, $Y$ and $Z$ directions are denoted by $N_X$, $N_Y$ and $N_Z$, respectively. The number of Gaussian points in the $X$ and $Y$ directions is the same and is denoted by $N_g$, while the number of Gaussian points in the $Z$ direction is $N_g/2$. The rate of convergence is examined for the plates with thickness ratios $h/b = 0.1$, 0.2, and 0.5. Detailed studies on the influence of Gaussian points, the degree of the 3-D polynomial basis function and the value of $k$ in the weight function can be found in Chapter 4. We concentrate our study herein on the influence of the number of MLS-Ritz points when using the MLS-Ritz method for the vibration analysis of 3-D plates.

6.3.1 Vibration of thick square plates

The characteristics of the convergence and accuracy of the MLS-Ritz method is first studied in this section for the 3-D vibration analysis of thick square plates ($a/b = 1$). Figure 6.3 shows the different combinations of the boundary conditions for the thick square plates considered in this study.

![Figure 6.3 Boundary conditions for the five considered thick square plates](image)

Figure 6.3 Boundary conditions for the five considered thick square plates
The cases considered are for plates with various boundary conditions, namely SSSS, CCCC, CFCF, SCSC, and CFFF as shown in Figure 6.3, where S denotes a simply supported edge, F a free edge and C a clamped edge, respectively. The Poisson ratio $\nu = 0.3$ is used in the calculation.

The first case considered herein is for square plates with simply supported edges (SSSS plates). Uniformly distributed MLS-Ritz grid points are used in the calculation and Figure 6.4 shows a typical MLS-Ritz grid point distribution with $(N_X, N_Y, N_Z) = (7, 7, 5)$. Tables 6.1 to 6.3 present the frequency parameters $\lambda$ for the plates with thickness ratio $h/b = 0.1, 0.2$, and 0.5 respectively. The results are obtained with different combinations of the MLS-Ritz grid points in the X, Y and Z directions, namely $(N_X, N_Y, N_Z) = (7, 7, 5), (9, 9, 5), (11, 11, 5), (11, 11, 7)$, and $(11, 11, 9)$, respectively. The convergence trend shows that, in general, the frequency parameters decrease as the number of MLS-Ritz grid points increases. However, the frequency parameters for some modes show an oscillating pattern with the increasing of the number of MLS-Ritz grid points. The convergence of the frequency parameters is obtained with the MLS-Ritz grid points $(11, 11, 9)$ and good accuracy is achieved when comparing with the ones obtained by Srinivas et al. [258], Leissa and Zhang [95], Liew et al. [90] and Zhou et al. [262]. It is noted that as the number of MLS-Ritz grid points increases, the radius of support $d$ needs to be reduced accordingly to ensure the accuracy and stability of the results. Tables 6.1 to 6.3 also show that better convergence can be achieved for plates with larger thickness ratios. We observe that even with a small number of MLS-Ritz grid points, reasonably accurate frequency parameters can be obtained for the SSSS square plates with thickness ratio $h/b = 0.2$ and 0.5.
Table 6.1 Frequency parameters for SSSS square plate with $h/b = 0.1115$

<table>
<thead>
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<th>$N_x$</th>
<th>$N_y$</th>
<th>$N_z$</th>
<th>$d$</th>
<th>$N_t$</th>
<th>Mode Sequence</th>
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[258] 1.934 4.622 4.622 - - 7.103 8.662 8.662 - -
[95]  1.634 4.622 4.622 6.523 6.523 7.103 8.662 8.662 - -
Table 6.2 Frequency parameters for SSSS square plate with $h/b = 0.2$

<table>
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<th>$N_z$</th>
<th>$d$</th>
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<th>3</th>
<th>4</th>
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<td>4.613</td>
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Table 6.3 Frequency parameters for SSSS square plate with $h/b = 0.5$

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The convergence pattern for cantilevered square plates (CFFF) is presented in Table 6.4. Similar to the SSSS plates, the general trend of the convergence for the CFFF plates is that the frequency parameters decrease as the number of MLS-Ritz grid points increases. Small oscillation of the frequency parameters is observed with the increasing of the number of MLS-Ritz grid points. Good convergence is achieved when the number of MLS-Ritz grid points becomes (11, 11, 9). Note that we were unable to found results in the open literature to compare with the cases presented in Table 6.4. We have used the finite element package ANSYS with 10304 Solid187 elements to generate the vibration frequencies for a CFFF square plate with $h/b =$
0.2. It is observed that the current MLS-Ritz solutions are in close agreement with the ones from ANSYS.

Table 6.5 shows the convergence and comparison studies of the first 10 frequency parameters for square plates with all edges clamped (CCCC plates). Comparing with the SSSS plates, more MLS-Ritz grid points are required for the CCCC plates to achieve the convergence of the frequency parameters. The general trend of the convergence is that the frequency parameters decrease as the number of MLS-Ritz grid points increases, although small oscillation of the frequency parameters is observed. The convergence of the frequency parameters is achieved with the MLS-Ritz grid points (13, 13, 11) and the present MLS-Ritz results are in good agreement with the ones by Liew et al. [90] and Zhou et al [262].

Tables 6.6 to 6.7 present the frequency parameters for CFCF and SCSC square plates. Convergence results are obtained when the number of MLS-Ritz grid points reaches (13, 13, 11) for both plates. Excellent agreement is observed for vibration frequencies from the current MLS-Ritz method and from the Ritz method by Liew et al. [90] when the thickness ratio of the plates is $h/b = 0.5$. For the other two plate thickness ratios ($h/b = 0.1$ and 0.2), the present results are slightly large than the ones from Liew et al. [90].
## Table 6.4 Frequency parameters for CFFF square plates

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6.3.2 Vibration of right-angled isosceles triangular plates

Figure 6.5 shows the six right-angled isosceles triangular plates studied in this chapter, namely, CFF, CCF, CCC, FFC, SSF and SSC plates respectively. The Poisson ratio $\nu = 0.3$ is used for all cases, unless stated otherwise.

![Boundary conditions for the six considered thick right-angled isosceles triangular plates](image)

Table 6.8 presents the convergence study on the frequency parameters for CFF right-angled isosceles triangular plates with plate thickness ratio $h/b = 0.1, 0.2$ and 0.5, respectively. The results are obtained with different combinations of the uniform MLS-Ritz grid points in the $X$, $Y$ and $Z$ directions, namely, $(7, 7, 5)$, $(9, 9, 5)$, $(11, 11, 5)$, $(11, 11, 7)$, $(11, 11, 9)$, and $(13, 13, 11)$, respectively. The grid points along the $X$ and $Y$ directions in the bracket are the number of points along the two perpendicular edges. Figure 6.6 shows the MLS-Ritz grid point distribution of the $(7, 7, 5)$ in the non-dimensionalised calculation domain. In general, the vibration frequency parameters decrease as the number of MLS-Ritz grid points increases. The frequency parameters converge to a satisfactory level when the number of the MLS-Ritz grid points reaches $(13, 13, 11)$ in the calculation.
To verify the correctness of the MLS-Ritz method for the vibration analysis of right-angled isosceles triangular plates, a comparison study is carried out. We found that there were very limited 3-D vibration results for triangular plates in the open literature. The comparison study is performed against the 3-D Ritz results by McGee and Giaimo [97] and the results generated by ANSYS for a CFF right-angled isosceles triangular plate with $h/b = 0.061$ and the Poisson ratio $\nu = 0.25$. The MLS-Ritz results are obtained with the MLS-Ritz grid points $(13, 13, 9)$, the radius of support $d = 1.2$ and the Gaussian points $N_g = 40$, respectively. The ANSYS results are based on 5053 Solid187 elements. Table 5.9 shows that the current MLS-Ritz results are in close agreement with the ANSYS results and are in good agreement with the first two frequency parameters from McGee and Giaimo [979]. It appears that the frequency parameters for the higher modes by McGee and Giaimo [97] have not yet fully converged.

Figure 6.6 A typical MLS-Ritz grid point distribution, $(N_X, N_Y, N_Z) = (7, 7, 5)$, for a thick right-angled isosceles triangular plate in the non-dimensionalised coordinate system.
Table 6.10 presents the frequency parameters for $CCF$, $CCC$, $FFC$, $SSF$ and $SSC$ right-angled isosceles triangular plates with plate thickness ratio $h/b = 0.1$, $0.2$ and $0.5$ respectively. The MLS-Ritz grid points $(13, 13, 11)$, the radius of support $d = 1$ and the Gaussian points $N_g = 40$ are used for obtaining all the results in Table 6.10. Most of the results in Table 6.10 are first-known values for the 3-D right-angled isosceles triangular plates and may be used as benchmark values for future reference.
### Table 6.8 Frequency parameters for CFF right-angle isosceles triangular plate

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146
Table 6.9 Comparison study for the frequency parameters of the CFF right-angle isosceles triangular plate \((h/b = 0.061, \nu = 0.25)\)

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### Table 6.10 Vibration frequency parameters for \( CCF, CCC, FFC, SSF, SSC \) right-angled isosceles triangular plates

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6.4 Conclusions

This Chapter has presented a 3-D vibration analysis for thick rectangular and triangular plates. The theory of elasticity has been employed to derive the strain energy equation for the plates and the MLS-Ritz method has been used to obtain the solutions for the 3-D vibration of thick square and right-angled isosceles triangular plates. Extensive convergence and comparison studies have been carried out to establish the number of MLS-Ritz grid points required for generating converged solutions and to verify the accuracy of the method. The study has shown that the MLS-Ritz method can be applied to investigate the 3-D vibration behaviour of thick plates with high accuracy.
CHAPTER 7

APPLICATION OF MLS-RITZ METHOD FOR
ANALYSIS OF 2-D ELECTROMAGNETIC
FIELDS

7.1 Introduction

The theory of electromagnetic fields is a discipline that studies the effects of electric charges, at rest and in motion, which produces currents and electromagnetic fields. It is, therefore, fundamental to the study of electrical engineering and physics, and indispensable to the understanding, design, and operation of many practical systems using electromechanical energy conversion, antennas, scattering, microwave circuits and devices, communications, and computers. The study of electromagnetics includes both theoretical and applied concepts. The theoretical concepts are defined by a set of basic laws formulated primarily through experiments conducted during the nineteenth century by many scientists and were widely acclaimed as Maxwell’s equations. The applied concepts of electromagnetics are formulated by applying the theoretical concepts to the design and operation of practical systems.

There are some simple cases in electromagnetic field analysis that can be solved analytically and these analytical/exact solutions serve as important benchmark values for checking the validity and accuracy of various numerical methods in the analysis
of electromagnetic fields. As we have discussed in Chapter 1, for most of the complex electromagnetic (EM) related problems closed form analytical solutions are either intractable or do not exist. These EM problems have to be solved approximately by different numerical methods. There are many numerical methods that are available for the analysis of EM problems. However, no single numerical method has proved to be the best for all EM problems encountered in the scientific and engineering applications.

The development of numerical methods for solving EM problems emerged in the mid-1960s with the availability of modern high-speed digital computers. Researchers such as Winslow, Silvester, Yee and Wexler are among the earliest to develop numerical methods for the solutions of electromagnetic fields and waveguide problems at that time [36, 266-272]. Since then, a large number of methods have been developed in the past four decades.

The most well developed and trusted method is the finite element method (FEM). It can be applied to solve EM problems with complex domains, different mediums and various boundary conditions. The applications of the FEM in electromagnetics deal with magneto-static field calculations, the electrostatic potential problems, waveguide analysis, the analysis of microwave and eddy currents, transformer and motor design etc. [19, 273-283].

The finite difference method is another important numerical method in the analysis of electromagnetic fields [284-287] and several schemes have been developed over the last few decades [288-289]. The finite difference time-domain method (FDTD) originally developed by Yee in 1966 [36] is still popular in dealing with time varying field problems.

In recent years, the meshless method has attracted much attention in the analysis of electromagnetic field problems [150, 290-292] as an alternative to the FEM. This
method has the advantage of no need for generating meshes in the analysis domains and it is efficient to analyze problems with large geometrical deformation or with discontinuities in fields. However, it has its own disadvantages such as the difficulty in enforcing the geometric boundary conditions as the nominal function values in the meshless method are not the same as the nodal values of the function. Many schemes have been proposed to solve this problem, including the use of Langrangian multiplier method and meshless-FEM method, etc. [291].

There are also hybrid numerical methods that combine different numerical methods such as the FEM with boundary element method, or the FEM with meshless method [192, 293-298]. These methods attempt to improve the efficiency and accuracy of the original methods by taking advantages of the other methods in processing boundary conditions, and treating interface conditions etc.

The Ritz method has been widely applied in the analysis of electromagnetic field problems, especially in waveguide eigenvalue problems [19-20, 290-291]. However, the success of the method in this type of calculations is limited due to the difficulties in establishing the Ritz trial functions that must satisfy the Dirichlet and Neumann boundary conditions of the calculation domain if the domain is irregular and contains multiple mediums.

This chapter presents the analysis of electromagnetic field problems by the MLS-Ritz method. The objective of the chapter is to demonstrate that the MLS-Ritz method enhances the conventional Ritz method by adding the flexibility to handle domains of different mediums, arbitrary shapes and any combination of boundary conditions. Three applications are presented in this chapter to demonstrate the flexibility and accuracy of the MLS-Ritz method.

This chapter is organised as follows. Section 7.2 briefly reviews the electromagnetic field theory. Section 7.3 presents the variational formulation of the electrical
potential. Section 7.4 applies the MLS-Ritz method to solve the static electrical field problems of a 2-D electrical potential in a trough with uniform medium and a domain with two different mediums. The waveguide eigenvalues for a double ridged waveguide are calculated in Section 7.5. The results of the three selected cases are compared with the benchmark values available in the open literature and the comparative studies confirm the applicability and accuracy of the MLS-Ritz method. Section 7.6 concludes this chapter.

7.2 Theory of Electromagnetic Fields

7.2.1 Maxwell’s equations
Electromagnetics is a subject that is concerned with interrelated electric and magnetic fields, an effect which occurs when the two fields are time-varying. In general, electrical and magnetic fields are vector quantities that have both magnitude and direction. The relations and variations of the electric and magnetic fields, charges, and currents associated with electromagnetic waves are governed by physical laws – Maxwell equations, which are the fundamental equations governing all macroscopic electromagnetic phenomena. For general time-varying fields, Maxwell’s equations in differential forms can be described as [19]:

**Faraday’s law:**
\[
\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0
\]
\[(7.1a)\]

**Maxwell-Ampere’s law:**
\[
\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}
\]
\[(7.1b)\]

**Gauss’s law:**
\[
\nabla \cdot \mathbf{D} = \rho
\]
\[(7.1c)\]

**Gauss’s law-magnetic:**
\[
\nabla \cdot \mathbf{B} = 0
\]
\[(7.1d)\]
where \( \mathbf{E} \) is the electric field intensity, \( \mathbf{D} \) the electric flux density, \( \mathbf{B} \) the magnetic flux density, \( \mathbf{H} \) the magnetic field intensity, \( \mathbf{J} \) the electric current density and \( \rho \) the electric charge density, respectively. \( \mathbf{E} \) and \( \mathbf{D} \) represent the electric field and \( \mathbf{B} \) and \( \mathbf{H} \) the magnetic field. \( \nabla \) is the differential operator (\( \nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j \) for 2-D fields and \( \nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \) for 3-D fields).

Another fundamental equation, the equation of continuity, is defined by:

\[
\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0
\]

which expresses the conservation (or indestructibility) of electric charges.

Among the above five equations, only three of them are independent. Since the number of the independent equation is less than the number of unknowns (five unknowns), the Maxwell’s equations become definite when constitutive relations between the field quantities are specified.

The constitutive relations which describe the macroscopic properties of the medium can be defined by the following equations:

\[
\mathbf{D} = \varepsilon \mathbf{E}
\]

\[
\mathbf{B} = \mu \mathbf{H}
\]

and

\[
\mathbf{J} = \sigma \mathbf{E}
\]

where \( \varepsilon \) is the permittivity, \( \mu \) the permeability, and \( \sigma \) the conductivity.

When the field quantities do not vary with time, the field is static. In the cases of electrostatic and magnetostatic fields, Maxwell’s equations (6.1a), (6.1b) and (7.2) reduce to
\[ \nabla \times \mathbf{E} = 0 \quad (7.4a) \]
\[ \nabla \times \mathbf{H} = \mathbf{J} \quad (7.4b) \]
and
\[ \nabla \cdot \mathbf{J} = 0 \quad (7.4c) \]

When a system is excited sinusoidally, only the steady state of the problem need to be considered, and it is convenient to represent the variables in the complex phasor form for such cases. When the field quantities in Maxwell’s equations are harmonically oscillating functions with a single frequency, the field is time-harmonic. For the time-harmonic fields, Eqs. (7.1a), (7.1b) and (7.2) can be written as:

\[ \nabla \times \mathbf{E} + j \omega \mathbf{B} = 0 \quad (7.5a) \]
\[ \nabla \times \mathbf{H} - j \omega \mathbf{D} = \mathbf{J} \quad (7.5b) \]
and
\[ \nabla \cdot \mathbf{J} + j \omega \rho = 0 \quad (7.5c) \]

where the time convention \( e^{j \omega t} \) is used and suppressed and \( \omega \) the angular frequency of excitation.

### 7.2.2 Electrical and magnetic potentials and wave equations

Generally, the electrical and magnetic field intensities (\( \mathbf{E} \) and \( \mathbf{H} \)) are the ones we are interested in, as \( \mathbf{E} \) and \( \mathbf{H} \) are physically measurable quantities. It is often convenient to use auxiliary functions in analysing an EM field. Because the first-order form of the Maxwell’s differential equations involves these two field quantities, it is often convenient to use auxiliary functions in analysing an EM field. By converting Eqs. (7.1a) and (7.1b) into the second-order differential equations, only one field quantity is required. The analysis of electromagnetic field problems can be carried out by introducing electromagnetic potential functions instead of dealing with the fields directly.
For any twice differentiable vector \( \mathbf{A} \), the following mathematical identity exists
\[
\nabla \cdot \nabla \times \mathbf{A} = 0
\]
(7.6)

Since the field vector \( \mathbf{B} \) satisfies a zero divergence condition (see Eq. (7.4)), it can be expressed in terms of an auxiliary vector \( \mathbf{A} \) as follows:
\[
\nabla \times \mathbf{A} = \mathbf{B}
\]
(7.7)

Substituting Eq. (7.7) into Eq. (7.1a), we have
\[
\nabla \times \left( \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0
\]
(7.8)

By applying another mathematical identity
\[
\nabla \times \nabla \phi = 0
\]
(7.9)
from the Eq. (7.8) and Eq. (7.9), we have
\[
\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi
\]
(7.10)

The variable \( \mathbf{A} \) and \( \phi \) are usually referred to as the magnetic vector potential and the electric scalar potential. Therefore, if we know \( \mathbf{A} \) and \( \phi \), the fields \( \mathbf{E} \) and \( \mathbf{B} \) can be obtained from Eqs. (7.7) and (7.10).

For the time harmonic case, the vector wave equation, which only involves either an electric or magnetic field, can be derived from Eqs. (7.5a), (7.5b) and the aid of the constitutive relations in Eqs. (7.3a) to (7.3c)
\[
\nabla \times \left( \frac{1}{\mu} \nabla \times \mathbf{E} \right) - \omega^2 \varepsilon_0 \mathbf{E} = -j \omega \mathbf{J}
\]
(7.11a)
\[
\nabla \times \left( \frac{1}{\varepsilon_0} \nabla \times \mathbf{H} \right) - \omega^2 \mu \mathbf{H} = \nabla \times \left( \frac{1}{\varepsilon_0} \mathbf{J} \right)
\]
(7.11b)
where $\mathbf{J}_i$ is an impressed or source current, and $\varepsilon_c = \varepsilon - j\sigma/\omega$ results from the combination of the induced current ($\sigma\mathbf{E}$) and displacement current ($j\omega\mathbf{D}$). For simplicity, we will henceforth use $\varepsilon$ to denote $\varepsilon_c$. These two equations are called the inhomogeneous vector wave equations.

The $z$-components of Eqs. (7.11a) and (7.11b) become

$$
\left[ \frac{\partial}{\partial x} \left( \frac{1}{\mu_r} \frac{\partial}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{1}{\mu_r} \frac{\partial}{\partial y} \right) + k_0^2 \varepsilon_r \right] E_z = jk_0 z_0 J_z 
$$

(7.12a)

and

$$
\left[ \frac{\partial}{\partial x} \left( \frac{1}{\varepsilon_r} \frac{\partial}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{1}{\varepsilon_r} \frac{\partial}{\partial y} \right) + k_0^2 \mu_r \right] H_z = \frac{\partial}{\partial x} \left( \frac{1}{\varepsilon_r} J_y \right) + \frac{\partial}{\partial y} \left( \frac{1}{\varepsilon_r} J_x \right) 
$$

(7.12b)

where $\varepsilon_r (= \varepsilon / \varepsilon_0)$ and $\mu_r (= \mu / \mu_0)$ denote the relative permittivity and relative permeability, respectively, which are assumed here to be complex scalar functions of position; $k_0 (= \omega \sqrt{\varepsilon_0 \mu_0})$ is the wave number in free space, and $Z_0 (= \sqrt{\mu_0 / \varepsilon_0})$ the intrinsic impedance of free space. These equations are called the inhomogeneous scalar wave equations.

### 7.2.3 Boundary conditions

The electromagnetic problems are boundary value problems. Therefore, a complete description of an electromagnetic problem should include both the differential equation and boundary conditions. The boundary conditions at the interface between two different media 1 and 2 with parameters of ($\varepsilon_1$, $\sigma_1$, $\mu_1$) and ($\varepsilon_2$, $\sigma_2$, $\mu_2$) can be expressed mathematically as

$$
n \cdot (\mathbf{E}_1 - \mathbf{E}_2) = 0 \quad (7.13a)
$$

$$
n \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s \quad (7.13b)
$$

$$
n \cdot (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s \quad (7.13c)
$$

and
\[ \mathbf{n} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0 \quad (7.13d) \]

where \( \mathbf{n} \) is the unit vector normal to the interface, pointing from medium 1 into medium 2, \( \mathbf{J}_s \) the surface electric current density, and \( \rho_s \) the surface charge density as shown in Figure 7.1.

![Figure 7.1 Interface between two media](image)

### 7.3 Variational Formulation

The Ritz method is the direct application of the variational principle. In order to apply the Ritz method to solve the partial differential equations derived from the Maxwell’s equations, the variational principle has to be applied first to transform the differential equations into weak forms.
The variational principle is a technique for solving boundary value problems that replace the problem of integrating a differential equation by the equivalent problem of seeking a function that gives a minimum value of the integral.

A typical boundary value problem can be defined as

\[ L\varphi = f \]  

(7.14)

where \( \varphi \) is the unknown function, which satisfies the essential boundary conditions on the boundary \( \Gamma \) in a domain \( \Omega \); \( L \) the differential operator and \( f \) the forcing function or excitation function.

If the operator is self-adjoint and positive definite, the solution of Eq. (7.14) can be obtained by minimizing the potential functional

\[ F(\psi) = \frac{1}{2} \int (L\psi)\psi d\Omega - \frac{1}{2} \int (\psi f) d\Omega - \frac{1}{2} \int (f \psi) d\Omega \]  

(7.15)

where \( \psi \) is the trial function for the unknown function \( \varphi \) in Eq. (7.14). The trial function must satisfy the essential boundary conditions on the boundary \( \Gamma \).

For a 2-D boundary value problem, the second-order differential equation (wave equation) can be defined as [19]

\[ -\frac{\partial}{\partial x}\left( \alpha_x \frac{\partial \phi}{\partial x} \right) - \frac{\partial}{\partial y}\left( \alpha_y \frac{\partial \phi}{\partial y} \right) + \beta \phi = f \quad (x, y) \in \Omega \]  

(7.16)

where \( \Omega \) is the domain of concern, \( \phi \) the unknown function, \( f \) the source or excitation function, and \( \alpha_x, \alpha_y \) and \( \beta \) are unknown parameters associated with the physical properties of the domain.

The boundary conditions to be considered for the electromagnetic problems are given by

\[ \phi = p \quad \text{on} \quad \Gamma_1 \]  

(7.17a)

and
\[
\left( \alpha_x \frac{\partial \phi}{\partial x} + \alpha_y \frac{\partial \phi}{\partial y} \right) \cdot \hat{n} + \gamma \phi = q \quad \text{on} \quad \Gamma_2
\]

where \( \Gamma (= \Gamma_1 + \Gamma_2) \) denotes the boundary enclosing the area \( \Omega \), \( \hat{n} \) is its outward normal unit vector, and \( \gamma, p, \) and \( q \) are known parameters associated with the physical properties of the boundary.

Equation Eq. (7.17a) is the Dirichlet condition, and Eq. (7.17b) is referred to as a boundary condition of the third kind. When \( \gamma = 0 \), Eq. (7.17b) is known as the Neumann boundary condition.

Applying the variational principle, we obtain the variational problem equivalent to the above boundary value problem given by [19]:

\[
\begin{cases}
\partial F(\phi) = 0 \\
\phi = p \quad \text{on} \quad \Gamma_1
\end{cases}
\]  

(7.18)

where \( F(\phi) \) is formed by the variational principle, which can be expressed in the weak form as:

\[
F(\phi) = \frac{1}{2} \int_{\Omega} \left[ \alpha_x \left( \frac{\partial \phi}{\partial x} \right)^2 + \alpha_y \left( \frac{\partial \phi}{\partial y} \right)^2 + \beta \phi^2 \right] d\Omega \\
- \int_{\Gamma_1} \left( \frac{\gamma}{2} \phi^2 - q \phi \right) d\Gamma - \int_{\Omega} f \phi \, d\Omega
\]

(7.19)

7.4 MLS-Ritz Method for Analysis of Static Electrical Field Problem

In this section, the MLS-Ritz method is applied to two static electrical field (EF) problems as examples to demonstrate the applicability of the method.
7.4.1 Problem definition

Figure 7.2 defines the dimensions and boundary conditions of an infinitely long rectangular conducting trough of uniform media, where $a = 10$, $b = 10$ and $p = 100$ [20]. The electrical potential of the trough at various locations will be determined by the MLS-Ritz method.

Figure 7.2 Dimension and boundary conditions of a conducting rectangular trough

Figure 7.3 shows the geometry and material properties of another electrical potential problem, where $a = b = 0.5$, $h = w = 1.0$, $\varepsilon_1 = \varepsilon_0$, $\varepsilon_2 = 3\varepsilon_0$ and $p = 100$, respectively. The charge density $\rho_c = 0$. One quarter of the square is filled with dielectric electrical material. The boundary conditions and the interface conditions between the two mediums are also specified in Figure 7.3, respectively.
For 2-D static electrical field boundary value problems, the governing differential equation can be expressed by the Poisson equation as shown below [19]

\[
-\frac{\partial}{\partial x}(\varepsilon_r \frac{\partial \phi}{\partial x}) - \frac{\partial}{\partial y}(\varepsilon_r \frac{\partial \phi}{\partial y}) = \frac{\rho_e}{\varepsilon_0} \quad (x, y) \in \Omega
\]  

(7.20)

where \( \Omega \) is the domain of concern, \( \phi \) the electrical potential function, \( \rho_e \) the charge density, \( \varepsilon_0 \) and \( \varepsilon_r \) are the permittivity of the vacuum and the relative permittivity of the medium, respectively.
Comparing Eq. (7.20) with Eq. (7.16), the variational formulation of $F(\phi)$ for the 2-D static electrical field boundary value problem with Neumann boundary condition ($\gamma = 0$ and $q = 0$), can be derived from Eq. (7.19) as:

$$F(\phi) = \frac{1}{2} \int_\Omega \left[ \varepsilon_r \left( \frac{\partial \phi}{\partial x} \right)^2 + \varepsilon_r \left( \frac{\partial \phi}{\partial y} \right)^2 \right] d\Omega - \int_\Omega \frac{\rho}{\varepsilon_0} \phi d\Omega$$

(7.21)

The electrical potential function $\phi$ can be determined by minimizing Eq. (7.21) and enforcing the boundary conditions of the domain of concern.

### 7.4.2 MLS-Ritz formulation for 2-D static electric field problem

The MLS-Ritz method is employed to calculate the distribution of the electrical potential. The calculation domain $\Omega$ is firstly divided into two sub-domains to accommodate the two mediums. The domains are then discretized by a set of pre-selected points. The distribution of the points can be either regular or irregular, depending on the requirement of the problem. For convenience and simplicity, uniformly distributed grid points are used in this study.

The Ritz trial function is firstly established by the MLS technique as described in Chapter 3. Using the MLS technique, the value of potential function $\phi(x, y)$ can be approximately calculated by

$$\phi^h(x, y) = \sum_{i=1}^{N} R_i(x, y) \phi_i = \mathbf{R} \phi = \mathbf{\phi}^T \mathbf{R}^T$$

(7.22)

where $\phi^h(x, y)$ is the approximate value of $\phi(x, y)$, $N$ the total number of the MLS-Ritz grid points in the calculation domain, $R_i(x, y)$ the MLS shape function, which is derived in Section 3.3. $\mathbf{R} = [R_1(x, y) \quad R_2(x, y) \quad \cdots \quad R_N(x, y)]$ is a row vector of the shape functions, and $\mathbf{\phi} = [\phi_1 \quad \phi_2 \quad \cdots \quad \phi_N]^T$ a column vector with the nominal potential values at the grid points, respectively. In this study, the 2-D complete polynomial was employed as the basis functions (see Section 3.3).
Substituting Eq. (7.22) into Eq. (7.21), the variational functional for calculating the 2-D electromagnetic field can be expressed as follows

\[ F = \frac{1}{2} \phi^T K \phi - \phi^T L \]  
(7.23)

where the matrices \( K \) and \( L \) of dimension \( N \times N \) are given by

\[ K = \int_\Omega \epsilon_r \left[ \nabla_x \nabla_x^T + \nabla_y \nabla_y^T \right] d\Omega \]  
(7.24)

and

\[ L = \frac{\rho}{\epsilon_o} \int_{\Omega_1} d\Omega \]  
(7.25)

### 7.4.3 Boundary conditions

In the MLS approximation, the nominal function value \( \phi_i \) at the pre-selected point \((x_i, y_i)\) in the calculation domain is not the same as the required approximate function value \( \phi^h(x_i, y_i) \). This causes a difficulty in enforcing boundary conditions in the calculation. A point substitution technique is employed in this study to process the boundary/interface conditions of the selected problems.

The pre-selected points in a calculation domain can be grouped into boundary and inner points, and the corresponding nominal function values can then be expressed as

\[ \phi = \begin{bmatrix} \phi_B \\ \phi_I \end{bmatrix} \]  
(7.26)

where \( \phi_B \) is a column vector of all nominal function values on the boundary and interface points between the two calculation domains, and \( \phi_I \) a column vector with all nominal function values of the inner points. To satisfy the boundary and interface conditions as defined in Figure 7.2, a system of linear equations can be obtained

\[ \begin{bmatrix} Q & S \end{bmatrix} \begin{bmatrix} \phi_B \\ \phi_I \end{bmatrix} = T \]  
(7.27)
The nominal function values on the boundary points can be expressed in terms of nominal function values of the inner points as

\[ \varphi_B = Q^{-1}(T - S\varphi_I) \]  \hspace{1cm} (7.28)

The nominal function values for all grid points can then be expressed in terms of \( \varphi_I \) as follows

\[
\varphi = \begin{bmatrix} \varphi_B \\ \varphi_I \end{bmatrix} = \begin{bmatrix} Q^{-1}T \\ 0 \end{bmatrix} + \begin{bmatrix} Q^{-1}S \\ -I \end{bmatrix} \varphi_I = U + V\varphi_I 
\]  \hspace{1cm} (7.29)

The interface conditions between two different mediums can be processed using the same approach. It is noted that for the waveguide problem in Section 7.5, all the elements in the column vector \( U \) are zero. Substituting Eq. (7.29) into Eq. (7.23) and minimizing the functional with respect to the nominal function values, \( \varphi_I \), one can obtain a system of linear equations for the electromagnetic field problem as follows:

\[ \mathbf{K}\varphi_I - \mathbf{F} = 0 \]  \hspace{1cm} (7.30)

where

\[ \mathbf{K} = V^T\mathbf{K}V \]  \hspace{1cm} (7.31)

and

\[ \mathbf{F} = -V^T\mathbf{K}U + V^T\mathbf{L} \]  \hspace{1cm} (7.32)

The nominal values of the electrical potential at the inner points, \( \varphi_I \), can be obtained by solving the system of linear equations defined by Eq. (7.30). The nominal values of the potential at the boundary points, \( \varphi_B \), are then computed by Eq. (7.28). The solutions for electrical potential defined by Eq. (7.20) at any point can finally be obtained by Eq. (7.22).
7.4.4 Numerical results

The basis function in the MLS fitting is taken to be \( p(x, y) = [1 \ x \ y \ x^2 \ xy \ y^2]^T \). The value \( k \) in the weight function in Eq. (3.8) is set to be 15.

For the problem defined in Figure 7.2, a single computational domain is adopted. The Gaussian integration is also performed over the single calculation domain. We will study the influence of the number of Gaussian points, \( N_g \), the number of the MLS-Ritz grid points, \( N \), and the radius of support, \( d \), on convergence and accuracy of the electrical potential of the given problem.

Note that this problem is governed by the Laplace’s equation:

\[
\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0
\]  

(7.33)

The analytical solution of this problem is [20]

\[
\phi(x, y) = \frac{4p}{\pi} \sum_{n=1,3,5} \sin \frac{n\pi x}{b} \sinh \frac{n\pi y}{b} \frac{n\pi a}{n \sinh \frac{n\pi a}{b}}
\]  

(7.34)

which will be used as the benchmark values to check the validity and accuracy of the MLS-Ritz method.

Table 7.1 presents the variation of the electrical potential at the centre point of the trough \( (x/a = 0.5 \text{ and } y/a = 0.5) \) against the number of Gaussian points, \( N_g \). The number of the MLS-Ritz grid points is set to be 17×17 and the ratio of the radius of support, \( d/a = 0.2 \), is employed in the calculation. We observed that once the number of Gaussian points is greater than 30×30, the electrical potential at the centre of the trough is quite stable with respect to the number of Gaussian points. We choose \( N_g = 50 \times 50 \) in the rest of the calculation for this problem.
Table 7.1 Electrical potentials in the trough with different number of Gaussian points (MLS-Ritz points $N = 17 \times 17$, radius of support $d/a = 0.2$)

<table>
<thead>
<tr>
<th>$N_g$</th>
<th>Point $(x/a = 0.5, y/a = 0.5)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10×10</td>
<td>-</td>
</tr>
<tr>
<td>20×20</td>
<td>25.2277</td>
</tr>
<tr>
<td>30×30</td>
<td>25.0519</td>
</tr>
<tr>
<td>40×40</td>
<td>25.0571</td>
</tr>
<tr>
<td>45×45</td>
<td>25.0571</td>
</tr>
<tr>
<td>50×50</td>
<td>25.0583</td>
</tr>
<tr>
<td>55×55</td>
<td>25.0584</td>
</tr>
<tr>
<td>60×60</td>
<td>25.0582</td>
</tr>
<tr>
<td>70×70</td>
<td>25.0585</td>
</tr>
<tr>
<td>80×80</td>
<td>25.0585</td>
</tr>
<tr>
<td>Analytical Solution [20]</td>
<td>25.000</td>
</tr>
</tbody>
</table>

We will study the range of the radius of support that can be used to generate the electrical potential for the trough. The electrical potentials for three selected points are presented in Table 7.2. The number of MLS-Ritz grid points is set to be $17 \times 17$. It is evident that when the value of $d/a$ is too small ($\leq 0.10$) or too large ($\geq 0.4$), the MLS-Ritz method does not produce correct results. However, over the range of $d/a = 0.18$ to 0.34, the electrical potentials on the three selected points vary over a very small range about the analytical solutions.
Table 7.2 Electrical potentials in the trough with different size of radius of support
(Gaussian points $N_g = 50 \times 50$, MLS-Ritz grid points $N = 17 \times 17$)

<table>
<thead>
<tr>
<th>$d/a$</th>
<th>Point 1 ($x/a = 0.125$, $y/a = 0.125$)</th>
<th>Point 2 ($x/a = 0.25$, $y/a = 0.25$)</th>
<th>Point 3 ($x/a = 0.5$, $y/a = 0.5$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>-4.0994</td>
<td>-291.4776</td>
<td>-1219.3168</td>
</tr>
<tr>
<td>0.12</td>
<td>1.0165</td>
<td>5.9198</td>
<td>24.8903</td>
</tr>
<tr>
<td>0.14</td>
<td>1.5536</td>
<td>6.7290</td>
<td>25.0506</td>
</tr>
<tr>
<td>0.16</td>
<td>1.6403</td>
<td>6.7130</td>
<td>25.0550</td>
</tr>
<tr>
<td>0.18</td>
<td>1.6935</td>
<td>6.7452</td>
<td>25.0575</td>
</tr>
<tr>
<td>0.20</td>
<td>1.7109</td>
<td>6.7903</td>
<td>25.0583</td>
</tr>
<tr>
<td>0.22</td>
<td>1.7137</td>
<td>6.8120</td>
<td>25.0600</td>
</tr>
<tr>
<td>0.24</td>
<td>1.7141</td>
<td>6.8168</td>
<td>25.0632</td>
</tr>
<tr>
<td>0.26</td>
<td>1.7145</td>
<td>6.8181</td>
<td>25.0667</td>
</tr>
<tr>
<td>0.28</td>
<td>1.7152</td>
<td>6.8195</td>
<td>25.0700</td>
</tr>
<tr>
<td>0.30</td>
<td>1.7157</td>
<td>6.8211</td>
<td>25.0725</td>
</tr>
<tr>
<td>0.32</td>
<td>1.7135</td>
<td>6.8192</td>
<td>25.0666</td>
</tr>
<tr>
<td>0.34</td>
<td>1.7092</td>
<td>6.8035</td>
<td>25.0205</td>
</tr>
<tr>
<td>0.36</td>
<td>1.7131</td>
<td>6.8359</td>
<td>25.0335</td>
</tr>
<tr>
<td>0.38</td>
<td>1.7196</td>
<td>6.7633</td>
<td>25.1556</td>
</tr>
<tr>
<td>0.4</td>
<td>1.6414</td>
<td>6.8349</td>
<td>25.0110</td>
</tr>
<tr>
<td>0.45</td>
<td>1.3169</td>
<td>7.4776</td>
<td>25.1926</td>
</tr>
<tr>
<td>0.5</td>
<td>2.0600</td>
<td>6.2689</td>
<td>24.8686</td>
</tr>
<tr>
<td>Analytical Solution [20]</td>
<td>1.709</td>
<td>6.797</td>
<td>25.000</td>
</tr>
</tbody>
</table>

The influence of the number of MLS-Ritz grid points on the convergence of the electrical potentials is also investigated. Table 7.3 presents the variation of the electrical potentials on the three selected points with respect to the number of MLS-Ritz grid points. It is seen that the electrical potentials of the three selected points approach the analytical solutions when the number of MLS-Ritz grid points is greater than $15 \times 15$. It is noted that the radius of support needs to be gradually reduced as
the number of MLS-Ritz grid points increases. The MLS-Ritz solutions are slightly greater than the benchmark analytical solutions [20].

Table 7.3 Electrical potentials in the trough with different MLS-Ritz grid points \( N \) (Gaussian points \( N_g = 50 \times 50 \))

<table>
<thead>
<tr>
<th>( N )</th>
<th>( d/a )</th>
<th>Point 1 ( (x/a = 0.125, \ y/a = 0.125) )</th>
<th>Point 2 ( (x/a = 0.25, \ y/a = 0.25) )</th>
<th>Point 3 ( (x/a = 0.5, \ y/a = 0.5) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>12\times12</td>
<td>0.5</td>
<td>1.7487</td>
<td>6.8330</td>
<td>25.1805</td>
</tr>
<tr>
<td>13\times13</td>
<td>0.5</td>
<td>1.7089</td>
<td>6.8450</td>
<td>25.0368</td>
</tr>
<tr>
<td>14\times14</td>
<td>0.4</td>
<td>1.7184</td>
<td>6.8369</td>
<td>25.1275</td>
</tr>
<tr>
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<td>0.4</td>
<td>1.7012</td>
<td>6.8235</td>
<td>25.0018</td>
</tr>
<tr>
<td>16\times16</td>
<td>0.3</td>
<td>1.7132</td>
<td>6.8198</td>
<td>25.0803</td>
</tr>
<tr>
<td>17\times17</td>
<td>0.2</td>
<td>1.7109</td>
<td>6.7903</td>
<td>25.0583</td>
</tr>
<tr>
<td>18\times18</td>
<td>0.2</td>
<td>1.7159</td>
<td>6.8190</td>
<td>25.0522</td>
</tr>
<tr>
<td>19\times19</td>
<td>0.2</td>
<td>1.7126</td>
<td>6.8150</td>
<td>25.0480</td>
</tr>
<tr>
<td>20\times20</td>
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<td>6.8010</td>
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</tr>
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<td>1.7116</td>
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<td>25.0361</td>
</tr>
<tr>
<td>23\times23</td>
<td>0.17</td>
<td>1.7102</td>
<td>6.8103</td>
<td>25.0327</td>
</tr>
<tr>
<td>Analytical Solutions [20]</td>
<td></td>
<td>1.709</td>
<td>6.797</td>
<td>25.000</td>
</tr>
</tbody>
</table>

Finally, Table 7.4 presents the electrical potentials on various points in the trough obtained by the MLS-Ritz method. The number of MLS-grid points \( N = 17 \times 17 \), Gaussian points \( N_g = 50 \times 50 \) and the radius of support \( d/a = 0.2 \) are employed in the computation. We observed that the MLS-Ritz method can be used to obtain the electrical potentials in the trough with a maximum error against the analytical solutions [20] being 0.74%.
Table 7.4 Analytical and MLS-Ritz solutions of electrical potentials in the trough 
\((N = 17 \times 17, \ N_g = 50 \times 50, \ d/a = 0.2)\)

<table>
<thead>
<tr>
<th>(y/a)</th>
<th>(0.125a)</th>
<th>(0.25a)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Analytical</td>
<td>MLS-Ritz</td>
</tr>
<tr>
<td>0.125</td>
<td>1.709</td>
<td>1.711</td>
</tr>
<tr>
<td>0.250</td>
<td>3.698</td>
<td>3.698</td>
</tr>
<tr>
<td>0.375</td>
<td>6.312</td>
<td>6.311</td>
</tr>
<tr>
<td>0.500</td>
<td>10.070</td>
<td>10.078</td>
</tr>
<tr>
<td>0.625</td>
<td>15.940</td>
<td>15.988</td>
</tr>
<tr>
<td>0.750</td>
<td>26.260</td>
<td>26.453</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(y/a)</th>
<th>(0.375a)</th>
<th>(0.5a)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Analytical</td>
<td>MLS-Ritz</td>
</tr>
<tr>
<td>0.125</td>
<td>4.100</td>
<td>4.111</td>
</tr>
<tr>
<td>0.250</td>
<td>8.834</td>
<td>8.845</td>
</tr>
<tr>
<td>0.375</td>
<td>14.930</td>
<td>14.943</td>
</tr>
<tr>
<td>0.500</td>
<td>23.290</td>
<td>23.327</td>
</tr>
<tr>
<td>0.625</td>
<td>35.070</td>
<td>35.177</td>
</tr>
<tr>
<td>0.750</td>
<td>51.580</td>
<td>51.774</td>
</tr>
</tbody>
</table>

Now the electrical field problem defined in Figure 7.3 is studied with two sub-domains to treat the differences in the mediums as shown in Figure 7.4, where the points in red denote the MLS-Ritz boundary or interface points and the points in black are the inner MLS-Ritz grid points. The calculation domain is divided into four equal squares for the Gaussian integration. We will study the influence of the number of Gaussian points \(N_g\), the number of MLS-Ritz grid points \(N\), and the radius of support \(d\) on convergence and accuracy of the electrical potential of the given problem.
The convergence of the MLS-Ritz method for the electrical potential at the centre point \((w/2, h/2)\) against the MLS-Ritz grid points is investigated. Figure 7.5 shows when the MLS-Ritz grid point size is greater than \(25 \times 25\), a well-converged electrical potential for the centre point \((w/2, h/2)\) is obtained.

Figure 7.4  Typical distribution of the MLS-Ritz grid points for two calculation domains
A set of 29 MLS-Ritz grid point size with radius of support \( d = 0.125w \) is used in the calculation for the results shown in Table 7.5, along with the results from [20] using various other methods. It can be seen that the MLS-Ritz results are in good agreement with those reported in [20].

Table 7.5 Results of 2D electrical potential

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( x/w ) = 0.25</td>
<td>( y/h ) = 0.50</td>
<td>10.2688</td>
<td>10.269</td>
<td>10.166</td>
</tr>
<tr>
<td>( x/w ) = 0.50</td>
<td>( y/h ) = 0.50</td>
<td>16.6667</td>
<td>16.667</td>
<td>16.576</td>
</tr>
<tr>
<td>( x/w ) = 0.75</td>
<td>( y/h ) = 0.50</td>
<td>15.9311</td>
<td>15.931</td>
<td>15.887</td>
</tr>
<tr>
<td>( x/w ) = 0.50</td>
<td>( y/h ) = 0.75</td>
<td>51.0987</td>
<td>51.931</td>
<td>50.928</td>
</tr>
<tr>
<td>( x/w ) = 0.50</td>
<td>( y/h ) = 0.25</td>
<td>6.2163</td>
<td>6.2163</td>
<td>6.1772</td>
</tr>
</tbody>
</table>
7.5 MLS-Ritz Method for Analysis of Waveguide Problems

7.5.1 Problem definition
This section will present an application of the MLS-Ritz method for waveguide problems. In a homogenously filled waveguide, there exist two sets of distinct modes. Assuming that the axis of a waveguide is along the \( z \)-axis and the wave is propagating in the \( z \)-direction, the modes that have no magnetic field component in the propagation direction are referred to as the transverse magnetic (TM) modes. On the other hand, if the modes have no electric field component in the propagation direction, the modes are referred to as the transverse electric (TE) modes. The fields in the waveguide are given by [19]:

\[
E(x, y, z) = E(x, y) e^{j(\omega t - \beta z)}
\]  \hspace{1cm} (7.35)

and

\[
H(x, y, z) = H(x, y) e^{j(\omega t - \beta z)}
\]  \hspace{1cm} (7.36)

where \( \omega \) is the angular frequency, \( \beta \) a propagation constant in the \( z \) direction, and

\[
\beta = \omega \sqrt{\mu \epsilon}.
\]

Substituting Eq. (7.35) and Eq. (7.36) into source-free Maxwell’s equations, the determination of the electromagnetic fields reduces to the determination of two components \( E_z \) and \( H_z \) and they satisfy the homogeneous Helmholtz equation [19]

\[
\nabla^2 \phi + k_T^2 \phi = 0 \quad (x, y) \in \Omega
\]  \hspace{1cm} (7.37)

where \( \Omega \) is the cross section of the waveguide, \( \nabla^2 \) the Laplace operator

\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},
\]

\( k_T \) the waveguide eigenvalue to be determined, and

\[
k_T^2 = \omega^2 \mu \epsilon - \beta^2
\]  \hspace{1cm} (7.38)

For the TM case, \( \phi = E_z \) and \( \phi \) satisfies the homogeneous Dirichlet boundary condition at the waveguide wall,
\[ \phi = 0 \quad \text{at} \quad \Gamma \quad (7.39) \]

For the TE case, \( \phi = H_z \), and \( \phi \) satisfies the Neumann boundary condition at the waveguide wall,

\[ \frac{\partial \phi}{\partial n} = 0 \quad \text{at} \quad \Gamma \quad (7.40) \]

where \( \Gamma \) denotes the conducting boundary of the waveguide wall with \( n \) being its unit normal.

Therefore, the homogeneous waveguide problem is to solve the Helmholtz or wave equation.

A double ridged waveguide is used as an example to demonstrate the applicability of the MLS-Ritz method. The cross-sectional shape and dimensions of the double ridged waveguide are illustrated in Figure 7.6.

![Figure 7.6 Geometry of ridged waveguide (in mm)](image)
Chapter 7  MLS-Ritz Method for Electromagnetic Fields

Considering the TE modes, one can determine the waveguide eigenvalues by solving Eq. (7.37). The boundary conditions for the waveguide problem in the TE model are \( \frac{\partial \phi}{\partial n} = 0 \) on the conductor surface and \( \phi = 0 \) on the magnetic wall. Applying the variational principle to the wave equation, one can express the variational form of \( F(\phi) \) as

\[
F(\phi) = \frac{1}{2} \int_{\Omega} \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial y} \right)^2 - k_f^2 \phi^2 \right] d\Omega \tag{7.41}
\]

The waveguide eigenvalue \( k_f \) can be determined by minimizing (7.41) and enforcing the boundary conditions at the waveguide cross section.

**7.5.2 MLS-Ritz formulation for a double ridged waveguide**

The MLS-Ritz method is used to calculate the waveguide eigenvalue \( k_f \). The procedure of solving the waveguide problem is the same as that for solving the electric static problem described in Section 7.4.2. Substituting Eq. (7.22) into Eq. (7.41), one can express the variational functional for calculating the 2-D waveguide problem as follows;

\[
F = \frac{1}{2} \phi^T (H - k_f^2 M) \phi \tag{7.42}
\]

where the matrices \( H \) and \( M \) have the dimension of \( N \times N \) and are given by

\[
H = \int_{\Omega} \left[ R_x R_x^T + R_y R_y^T \right] d\Omega \tag{7.43}
\]

and

\[
M = \int_{\Omega} R R^T d\Omega \tag{7.44}
\]

Applying the boundary condition for the waveguide problem gives

\[
(H - k_f^2 M) \phi_j = 0 \tag{7.45}
\]

where
\[ \overline{H} = V^T HV \quad (7.46) \]

and

\[ \overline{M} = V^T MV \quad (6.47) \]

The waveguide eigenvalue \( k_T \) can be obtained by solving the generalized eigenvalue problem defined by Eq. (7.45).

### 7.5.3 Numerical Results

The geometry and the MLS-Ritz grid points of a double ridged waveguide [299] are shown in Figure 7.6. Half of the waveguide (symmetric about the \( y \) axis) is used in the computation. A single calculation domain with uniformly distributed MLS-Ritz grid points along the horizontal and vertical directions is employed and Figure 7.7 (a) shows a \( 31 \times 31 \) MLS-Ritz grid point distribution. The basis function in the MLS fitting is taken to be \( p(x, y) = [1 \quad x \quad y \quad x^2 \quad xy \quad y^2]^T \). The value \( k \) in the weight function in (3.2-8) is set to be 15. The Gaussian integration is performed over two areas as shown in Figure 7.7 (b).
Table 7.6 examines the influence of the number of Gaussian points required for producing stable waveguide eigenvalues $T_k$ (rad/mm) by the MLS-Ritz method. The MLS-Ritz grid points are set to be $3^{13} 1$ and the radius of support $d/w = 0.15$ is used in the calculation. $N_{g1}$ and $N_{g2}$ in Table 7.6 denote the number of Gaussian points in the integration areas 1 and 2, respectively. It is observed that when $N_{g1}$ and $N_{g2}$ are greater than 192 and 3240, the waveguide eigenvalues $k_T$ are stabilized.

Figure 7.7  Typical MLS-Ritz grid point distribution and Gaussian integration areas
### Table 7.6 Waveguide eigenvalues $k_F$ of the double ridged waveguide with different number of Gaussian points $N_{g1}$ and $N_{g2} \ (d/w = 0.3 \ and \ N = 31 \times 31)$

<table>
<thead>
<tr>
<th>$N_{g1}$</th>
<th>$N_{g2}$</th>
<th>TE$_{10}$ Hybrid</th>
<th>TE$_{11}$ Trough</th>
<th>TE$_{20}$ Trough</th>
<th>TE$_{30}$ Hybrid</th>
<th>TE$_{11}$ Hybrid</th>
</tr>
</thead>
<tbody>
<tr>
<td>130</td>
<td>2220</td>
<td>0.1464</td>
<td>0.1562</td>
<td>0.2289</td>
<td>0.2813</td>
<td>0.2832</td>
</tr>
<tr>
<td>192</td>
<td>3240</td>
<td>0.1463</td>
<td>0.3184</td>
<td>0.6197</td>
<td>0.6717</td>
<td>0.6987</td>
</tr>
<tr>
<td>266</td>
<td>4368</td>
<td>0.1463</td>
<td>0.3184</td>
<td>0.6197</td>
<td>0.6718</td>
<td>0.6989</td>
</tr>
<tr>
<td>300</td>
<td>5040</td>
<td>0.1463</td>
<td>0.3185</td>
<td>0.6198</td>
<td>0.6717</td>
<td>0.6990</td>
</tr>
<tr>
<td>352</td>
<td>5760</td>
<td>0.1463</td>
<td>0.3184</td>
<td>0.6197</td>
<td>0.6717</td>
<td>0.6988</td>
</tr>
</tbody>
</table>

The influence of the radius of support on the MLS-Ritz waveguide eigenvalues $k_F$ is presented in Table 7.7. The Gaussian points are set to be $N_{g1} = 352$ and $N_{g2} = 5760$ and the MLS-Ritz grid points $31 \times 31$ are employed in the computation. When the radius of support is small, the MLS-Ritz method generates erroneous results. It is seen that the waveguide eigenvalues $k_F$ are stabilized over a range of $d/w$ values from 0.14 to 0.20.
### Table 7.7 Waveguide eigenvalues $k_f$ of the double ridged waveguide with different size of radius of support ($N_{g1} = 352$, $N_{g2} = 5760$, $N = 31 \times 31$)

<table>
<thead>
<tr>
<th>$d/w$</th>
<th>TE$_{10}$ Hybrid</th>
<th>TE$_{11}$ Trough</th>
<th>TE$_{20}$ Trough</th>
<th>TE$_{30}$ Hybrid</th>
<th>TE$_{11}$ Hybrid</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.06</td>
<td>4.0251</td>
<td>4.1205</td>
<td>4.1767</td>
<td>4.4893</td>
<td>4.7237</td>
</tr>
<tr>
<td>0.08</td>
<td>0.2157</td>
<td>0.3952</td>
<td>0.6378</td>
<td>0.6900</td>
<td>0.7685</td>
</tr>
<tr>
<td>0.10</td>
<td>0.1491</td>
<td>0.3174</td>
<td>0.6113</td>
<td>0.6597</td>
<td>0.6882</td>
</tr>
<tr>
<td>0.11</td>
<td>0.1455</td>
<td>0.3150</td>
<td>0.6115</td>
<td>0.6633</td>
<td>0.6881</td>
</tr>
<tr>
<td>0.12</td>
<td>0.1459</td>
<td>0.3167</td>
<td>0.6159</td>
<td>0.6680</td>
<td>0.6939</td>
</tr>
<tr>
<td>0.13</td>
<td>0.1462</td>
<td>0.3180</td>
<td>0.6190</td>
<td>0.6711</td>
<td>0.6979</td>
</tr>
<tr>
<td>0.14</td>
<td>0.1463</td>
<td>0.3183</td>
<td>0.6197</td>
<td>0.6717</td>
<td>0.6987</td>
</tr>
<tr>
<td>0.15</td>
<td>0.1463</td>
<td>0.3184</td>
<td>0.6197</td>
<td>0.6717</td>
<td>0.6988</td>
</tr>
<tr>
<td>0.16</td>
<td>0.1463</td>
<td>0.3185</td>
<td>0.6198</td>
<td>0.6717</td>
<td>0.6990</td>
</tr>
<tr>
<td>0.17</td>
<td>0.1463</td>
<td>0.3185</td>
<td>0.6198</td>
<td>0.6717</td>
<td>0.6990</td>
</tr>
<tr>
<td>0.18</td>
<td>0.1462</td>
<td>0.3186</td>
<td>0.6198</td>
<td>0.6717</td>
<td>0.6990</td>
</tr>
<tr>
<td>0.19</td>
<td>0.1462</td>
<td>0.3186</td>
<td>0.6198</td>
<td>0.6716</td>
<td>0.6990</td>
</tr>
<tr>
<td>0.20</td>
<td>0.1463</td>
<td>0.3187</td>
<td>0.6198</td>
<td>0.6717</td>
<td>0.6992</td>
</tr>
<tr>
<td>0.25</td>
<td>0.1466</td>
<td>0.3194</td>
<td>0.6203</td>
<td>0.6715</td>
<td>0.6997</td>
</tr>
</tbody>
</table>

Finally, the convergence of the waveguide eigenvalues $k_f$ against the number of MLS-Ritz grid points is presented in Table 7.8. The Gaussian points used in the calculation are $N_{g1} = 352$ and $N_{g2} = 5760$. We observe that as the number of MLS-Ritz grid points increases, the waveguide eigenvalues $k_f$ decrease in general except that a small oscillation is observed for the modes TE10 and TE11. The waveguide eigenvalues $k_f$ from the MLS-Ritz method are in good agreement with the ones from [299] and [300].


Table 7.8 Waveguide eigenvalues $k_r$ of the double ridged waveguide with different number of MLS-Ritz grid points

\[ N_{g1} = 352, N_{g2} = 5760 \]

<table>
<thead>
<tr>
<th>$N$</th>
<th>$d/w$</th>
<th>$\text{TE}_{10}$ Hybrid</th>
<th>$\text{TE}_{11}$ Trough</th>
<th>$\text{TE}_{20}$ Trough</th>
<th>$\text{TE}_{30}$ Hybrid</th>
<th>$\text{TE}_{11}$ Hybrid</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 × 20</td>
<td>0.25</td>
<td>0.1503</td>
<td>0.3212</td>
<td>0.6207</td>
<td>0.6728</td>
<td>0.7012</td>
</tr>
<tr>
<td>25 × 25</td>
<td>0.17</td>
<td>0.1472</td>
<td>0.3195</td>
<td>0.6202</td>
<td>0.6718</td>
<td>0.6998</td>
</tr>
<tr>
<td>27 × 27</td>
<td>0.16</td>
<td>0.1479</td>
<td>0.3195</td>
<td>0.6201</td>
<td>0.6722</td>
<td>0.6998</td>
</tr>
<tr>
<td>29 × 29</td>
<td>0.15</td>
<td>0.1474</td>
<td>0.3191</td>
<td>0.6199</td>
<td>0.6721</td>
<td>0.6994</td>
</tr>
<tr>
<td>31 × 31</td>
<td>0.11</td>
<td>0.1455</td>
<td>0.3150</td>
<td>0.6115</td>
<td>0.6633</td>
<td>0.6881</td>
</tr>
<tr>
<td>33 × 33</td>
<td>0.09</td>
<td>0.1465</td>
<td>0.3150</td>
<td>0.6100</td>
<td>0.6613</td>
<td>0.6883</td>
</tr>
<tr>
<td>[6.43]</td>
<td></td>
<td>0.1438</td>
<td>0.3155</td>
<td>0.6215</td>
<td>0.6707</td>
<td>0.6971</td>
</tr>
<tr>
<td>[6.44]</td>
<td></td>
<td>0.1437</td>
<td>0.3166</td>
<td>0.6190</td>
<td>0.6712</td>
<td>0.6973</td>
</tr>
</tbody>
</table>

7.6 Conclusions

This chapter has presented a novel method, the MLS-Ritz method, for the analysis of electromagnetic field problems. The MLS-Ritz method can overcome the difficulties of the conventional Ritz method in establishing trial functions and in dealing with different boundary conditions and geometric shapes of calculation domains. To validate the proposed MLS-Ritz method, three examples including electrical potential problems in a uniform trough and with dielectric medium and a waveguide eigenvalue problem have been analysed and compared with solutions obtained by other methods. Excellent agreement has been achieved.
CHAPTER 8

CONCLUSIONS AND FUTURE WORK

This chapter summarizes the major achievements of the research work presented in the thesis. The merits and limitations of the proposed moving least square Ritz (MLS-Ritz) method in determining the numerical solutions of solid mechanics and electromagnetic field problems are highlighted. Several recommendations are made for the future development of the MLS-Ritz method for the analyses of scientific and engineering problems.

8.1 Conclusions

This thesis has been focused on the development of a novel numerical method, the MLS-Ritz method, to solve boundary value problems encountered in many science and engineering fields. The moving least square data interpolation technique has been utilized to establish the Ritz trial functions in the analysis. A point substitution method has been proposed to enforce the essential boundary conditions of the analysed problems. The MLS-Ritz method has been successfully applied to analyse several selected solid mechanics and electromagnetic field problems. The characteristics of the MLS-Ritz method have been investigated through the detailed convergence and comparison studies presented in the thesis. The major achievements of the research work are summarized as follows.
1. The proposed MLS-Ritz method has expanded the applicability of the conventional Ritz method in solving problems with arbitrary shapes and multiple mediums. As demonstrated in the thesis, the combination of the MLS-Ritz trial functions and the point substitution approach for enforcing boundary conditions has enabled the flexibility of the MLS-Ritz method for analysing problems with complex geometries.

2. The MLS-Ritz method is essentially a meshless method. It is well known that the nominal function value at a grid point in the meshless method is not the approximate function value one wants to obtain. It is difficult to enforce the geometric boundary conditions due to this issue. Many conventional meshless methods employ the penalty approach, such as the Lagrangian multipliers, to process the boundary conditions. However, this increases the number of unknowns to be solved and it could lead to numerical instability due to the matrix ill-conditions. The proposed point substitution method for enforcing the boundary conditions has overcome the difficulty of the conventional meshless methods in processing boundary conditions.

3. The distribution of the MLS-Ritz grid points can be either regular or irregular within the domain of calculation. Therefore, more grid points can be placed in the regions where the function values are expected to have more significant variations than the ones in other regions. Due to this feature, the MLS-Ritz method can achieve high accuracy and at the same time the efficiency of the method is maintained.

4. The domain decomposition technique has been developed and applied in analysing the vibration of skew plates with large skew angles and the electrical potentials of a trough with two mediums. This technique can be further developed to study more complex problems encountered in many science and engineering applications.
5. The MLS-Ritz method has been applied to analyse the free vibration of square and triangular plates. Detailed convergence and comparison studies show that the MLS-Ritz method is highly stable, accurate and efficient in solving plate vibration problems.

6. The MLS-Ritz method has been applied to study the more challenging problem of free vibration of rhombic plates with large skew angles. It is confirmed by the results that the MLS-Ritz method is very stable and can generate converged vibration results for rhombic plates with various combinations of edge support conditions and large skew angles. The study has also revealed that some of the previous studies on the vibration of rhombic plates did not provide converged results.

7. The MLS-Ritz method has also been applied to analyse the 3-D vibration of isotropic elastic plates, including thick square and right-angled isosceles triangular plates. This study shows that the MLS-Ritz method is capable of generating 3-D vibration frequencies for thick plates with high accuracy.

8. The MLS-Ritz method has been further employed to analyse the electromagnetic fields problems. Three cases, namely, electrical potential problems in a uniform trough and a trough with dielectric medium and a waveguide eigenvalue problem, have been studied and compared with solutions obtained by other methods. Excellent agreement has been achieved for all the selected cases.
8.2 Future Work

Due to the limitations of resources and time, the work completed in this thesis is still in the early stage of the development of the MLS-Ritz method. The MLS-Ritz method has a great potential to become a powerful numerical method in science and engineering computations. Much research work needs to be carried out for the further development of this method, including:

1. The domain decomposition technique can be further refined to truly handle subdomains of various shapes and connectivity conditions. This will enable the MLS-Ritz method to be a general numerical method for complex science and engineering applications.

2. The hybrid approach of the MLS-Ritz and the FE methods can be explored to combine the advantages of both methods. The FEM can be used to treat the boundary regions, while the MLS-Ritz method is applied to regions where large deformation or stress concentration occurs.

3. The hybrid approach of the MLS-Ritz and the BE methods can also be studied. It may be applied to study EM wave scattering problems. The MLS-Ritz method can be employed to model the waves in the dielectric scattering body whereas the BE method is used to model the unbounded space.

4. The development and applications of the MLS-Ritz method can be extended to more broad science and engineering computations, such as 3-D electromagnetic field problems, plane stress analysis in solid mechanics, interaction of soils and structures and computational fluid dynamics, to just mention a few.
APPENDIX

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