CHAPTER 1
INTRODUCTION

Controlled thermonuclear fusion may some day provide a clean, safe, and abundant source of energy. The difficulties involved in demonstrating the scientific feasibility of this process, in igniting and controlling the fusion fire is formidable. Most fusion research to date has been directed at confining a dilute fusion fuel using cleverly designed magnetic fields. However, during the past decade an alternate approach known as inertial confinement fusion has begun to receive a considerable amount of attention. In this approach, intense laser or charged particle beams are used to rapidly compress a tiny pellet of fusion fuel typically from one to five millimetres in diameter, for the enormous densities and temperatures required for efficient thermonuclear burn. If the fuel pellet is compressed to sufficient densities, then it will burn so rapidly that appreciable fusion energy will be released before it can blow apart. It will be confined during the fusion burn by its own inertia.

Chapter 2 of this thesis continues to discuss the characteristics of the high powered laser used in inertial confinement fusion research, and reviews the mechanisms of fusion energy and its application to inertial confinement fusion.

Progress in plasma research has revealed that plasma is an active medium which exhibits a wide variety of nonlinear phenomena. Response to an external disturbance in a plasma is typically represented by excitation of a wave or a collective motion of charged particles, which is a result of the long range character of the Coulomb interaction. Since collisional dissipation effects are weak in high temperature plasmas, the excited waves can easily grow to a high level and exhibit a variety of nonlinear behaviour. Chapter 3 of this thesis describes the nature and properties of a plasma and will sketch the kinetic theory of plasmas for some background understanding.

Historically, the first quantitative theory of self-focusing of laser radiation resulting in a threshold for the process was successful for the dielectric materials (non-ionised solids, liquids and gases). The essential mechanism was the nonlinearity of the dielectric constant. For the
self-focusing in plasmas, the nonlinearity of the dielectric constant was not the first successful way for deriving the self-focusing thresholds, but an application of the nonlinear forces. It has to be mentioned that a first concept to discuss self-focusing of lasers in plasma was published by (Askaryan 1088, 1962). Askaryan considered the energy momentum flux density of the laser beam at self-focusing, where the whole plasma has been expelled and where the pressure is balanced by the plasma pressure profile, acting against the centre of the laser beam. Askaryan was able to compare the necessary optical intensities for compensating the gas dynamic pressure. This compensation was proposed by (Schlüter 471, 1968) to study the self-focusing of laser beams in plasma.

In the last decade, with the availability of high power laser beams, a large number of interesting nonlinear optical phenomena have been studied. The self-focusing of intense laser beams occupies an important place because of its relevance to other nonlinear effects. The self-focusing effect described in Chapter 5 of this thesis arises primarily due to the dependence of the refractive index of a material on the intensity of the propagating electromagnetic wave. This nonlinear dependence may arise, among others, from the following mechanisms:

1. Electrostriction in dielectrics.
2. Thermal Effects in an absorbing media.
5. Relativistic self-focusing in plasmas.

The research of the present thesis is a theoretical investigation of the last two mechanisms, ponderomotive and relativistic self-focusing in plasmas. These mechanisms are investigated in Chapter 5.

Relativistic self-focusing is one of the self-focusing processes of laser beams in plasmas resulting from the relativistic mass and energy dependence of the refractive index at high laser intensities. It is obvious that such phenomena would be apparent near the relativistic threshold, where the electron oscillation energy equals to \( m_0 c^2 \). However this mechanism becomes significant at values exceeding \( 3.66 \times 10^{18} \) W/cm\(^2\) for a Neodymium glass laser. Chapter 5 of this thesis presents a numerical program in C++, which incorporates both ponderomotive force in self-focusing mechanism and relativistic effects, to evaluate various
parameters of refractive index, power threshold and self-focusing of the beam as a function of laser intensity.

Self-focusing and filamentation of laser beams have been experimentally observed in short wavelength laser-plasma interactions (Willi and Lee 120, 1985). It was shown that when the laser intensity was between $10^{13}$ to $10^{14}$ W/cm$^2$, for a short wavelength laser and high Z plasma, thermal self-focusing could initiate the self-focusing process, but as self-focusing decreased, the filament size had increased the intensity in the filament, the ponderomotive force effect was important and rapidly became the dominant self-focusing mechanism. Chapter 6 of this thesis shows the derivation of the nonlinear plasma dielectric function due to the relativistic electron motion. From that, one can obtain the nonlinear refractive index of the plasma and estimate the importance of relativistic self-focusing as compared to ponderomotive self-focusing at very high laser intensities.

The field of nonlinear optics has developed in recent years as nonlinear materials have become available and widespread applications have become apparent. This is particularly true for solitons and other types of nonlinear pulse transmission in optical fibres. Since this form of light propagation can be utilised in the future for very high capacity dispersion-free communications.

Chapter 7 of this thesis explores a numerical study of the semiclassical limit of the linear and the nonlinear paraxial equation in both the defocusing and focusing cases, along with an analytical study of the semiclassical limit of the defocusing cases. Comparisons of both dimension and nonlinearity are identified in Chapter 7 of this thesis, for the defocusing and focusing cases.

The soliton concept is a sophisticated mathematical construct based on the integrability of a class of nonlinear differential equations. Integrable nonlinear differential equations have one feature in common, they are all conservative and thus derivable from a Hamiltonian. The integration is performed via the method of inverse scattering. The nonlinear Schrodinger equation is one member of the class of integrable equations (Zakharov and Shabat 62, 1972). Chapter 8 of this thesis describes the use of a very powerful tool to solve the nonlinear paraxial equation that has stable solutions called solitons. The physical origin of the solutions is the Kerr effect, which relies on a nonlinear dielectric constant that can balance the group dispersion in the optical propagation medium. The resulting effect of this balance is the
propagation of solitons, which has the form of a secant. Chapter 8 of this thesis shows that the formation and propagation of solitons can be observed and understood by using a numerical routine to solve the generalised nonlinear paraxial equation. A sequence of code has been developed in mathematica to explore in depth several features of the soliton's formation and propagation.
CHAPTER 2
LASER PLASMAS AND NUCLEAR ENERGY

2.1 Introduction

The problems of the interaction of high intensity laser radiation with plasma represent a very new, fascinating, and fundamental field of research which could be an important key for solving the energy crisis. Whatever other results may come from this field, the technique of inertial confinement by laser radiation may be the only clean, inexpensive and inexhaustible nuclear energy source of the future.

This chapter of the thesis reviews briefly the essential physics of laser operation and continues to discuss the most common high-powered laser used in inertial confinement fusion (ICF) research.

Fusion energy is the most abundant energy source in the universe. It is the source of energy in the sun and stars. This chapter of the thesis reviews the mechanism of fusion energy and applications of inertial confinement fusion.

2.2 Laser Physics

To introduce the basic concepts involved in laser operation, consider the interaction of light where a single atom can exist in one of only two possible energy states, $E_1$ and $E_2$. Incident light photons with a frequency $v_{12}$ such that $h v_{12} = E_2 - E_1$ can be absorbed by atoms in the ground state $E_1$. However, photons incident on atoms in the excited state $E_2$ can stimulate the emission of a second photon of frequency $v_{12}$, which appears in phase (in coherence) with the incident photon. If there are more atoms in the excited state $E_2$, then photons incident upon a medium containing such atoms can stimulate a growing cascade of photons of frequency $v_{12}$ via the stimulated emission process. That is, the medium could be used to achieve light amplification by stimulated emission of radiation, hence the acronym "laser". This process is shown schematically in Figure (1).
Under normal circumstances there will be many more atoms in the lower energy state (usually the ratio of population densities goes as the Boltzmann factor, \( \exp [- (E_2 - E_1) / KT] \)). Hence to achieve laser process, one must achieve a "population inversion" in which there are more atoms in the upper than lower state, so that an incident photon will stimulate the emission of other photons rather than being absorbed. A variety of excitation mechanisms can be used, including irradiating the medium with intense light at another frequency, using electrical discharges, or chemical reactions.

A key feature in laser operation involves the fact that the photons emitted in the stimulated process appear in phase with one another. This is in sharp contrast to the light emitted from conventional sources of intense light in which the photons are not only distributed over a relatively broad spectrum but furthermore are incoherent in the sense that there is no correlation between the phases of radiation emitted from two different points of the source. An important theorem of classical optics states that it is impossible to devise an optical system that could focus an incoherent beam of light, to an intensity higher than that characterising the source from which the light originates. That is, the incoherent radiation from an extended source of light cannot be imaged with an increase in brightness.

This is in sharp contrast to the coherent beam of light emitted by a laser. It is possible to concentrate or focus this beam in such a way as to increase the light intensity or brightness to a level greater than that of the original source. In fact, the coherence in many laser systems
is sufficient to allow the beam to be focused down to spot sizes on the order of the wavelength of the light (so-called diffraction limited). Furthermore, in sharp contrast with incoherent light sources, the coherent light in a laser beam can be propagated large distances without appreciable losses from geometric spreading or diffraction effects.

2.3 Neodymium Laser

Presently the most common used laser for high power research is the neodymium glass laser. Its wavelength is 1.06 µm, which is in the near infrared region. Its pulse duration can be anywhere between 170 fsec (= 0.17 psec) and continuous-wave operation. Neodymium lasers operate essentially as the four level scheme described in Figure (2).

![Figure 2: Four level laser.](image)

In this particular laser, the lower level lies far above the ground state of the system and is generally unpopulated. Thus, \( n_1 = 0 \), so the normalised population inversion \( n \) is just equal to \( n_2 \). Any population in level 2 gives rise to an inversion with \( n > 0 \), because it is not necessary to pump a four level laser from \( n = -1 \) to \( n = 0 \) before achieving again. Such lasers are much more efficient than three level lasers. If level 1 decays to level 0 rapidly enough, \( n_1 = 0 \) under all conditions, and the four level laser never exhibits absorption of the laser light itself.

The advantage of the neodymium glass laser is the highly developed technology, which makes its use preferable to other systems. Despite the well known disadvantages, nonlinear refractive index, thermal birefringence of laser glass and the low efficiency of the laser system, which does not transfer more than 1% of electrical energy into laser energy.
Lasers, which avoid the solid state problems of nonlinear refractive index and thermal birefringence, and provide high repetition rates up to 1kHz and more at high power outputs, are based on gaseous media. Attempts have been made to keep the ideal properties of the Nd\textsuperscript{3+} laser action by using vapour containing molecules of neodymium compounds.

\subsection*{2.4 Fusion Energy}

As the limitations of the Earth’s resources of conventional fuels have become more apparent, scientists have turned their attention towards the stars for a new source of energy. It has been known for several decades that nuclear fusion reactions are a major energy source in stars. In this process the nuclei of light elements are fused together at very high temperatures to produce more tightly bound, heavier nuclei, releasing energy in the process.

An example of such a reaction is that which occurs when the two heavier isotopes of hydrogen, deuterium (D) and tritium (T), combine to produce helium plus a neutron. This fusion reaction releases 17.6 MeV of energy, which is carried off as kinetic energy by the reaction products. The energy content of such fusion fuels is truly enormous. A thimbleful of deuterium would release as much energy from fusion as the combustion of 20 tons of coal. The natural deuterium contained in one litre of water would produce the fusion energy equivalent of 300 litres of gasoline.

The potential of such reactions for generating large amounts of energy is evident. One can look at any star to see a massive example of fusion energy release. In a sense, nuclear fusion can be regarded as the most primitive form of solar power, since it is also the energy source of our sun. Hence it was natural for scientists to wonder whether fusion might be employed as a terrestrial energy source. The awesome potential of this quest was demonstrated by the development of nuclear fusion weapons - the hydrogen bomb - in the early 1950s. Since that time, proponents of fusion power have predicted that some day this nuclear process would provide us with a safe, clean, and abundant source of energy.

The difficulties involved in igniting and controlling a fusion reaction are formidable. The light nuclei that must fuse together are positively charged and strongly repel one another. To overcome this repulsion, the two nuclei must fuse together at very high velocities. One way of doing this is to take a mixture of deuterium and tritium and heat it to such high temperatures that the velocities of thermal motion of the nuclei are sufficient to overcome
charge repulsion and initiate the fusion reaction. Such a scheme is referred to as a thermonuclear fusion reaction. The temperature required is quite high - roughly 100 million degrees (or 10 keV, where 1 keV corresponds to $1.16 \times 10^7$ K). Until quite recently scientists had imitated the sun only in a rather violent fashion by using a nuclear fission explosion to create temperatures high enough to ignite the fusion reaction in the hydrogen bomb.

Heating the fusion fuel to enormous temperatures is not enough to ignite the fusion reaction. For most of the time, when the nuclei run into each other, they simply bounce off or scatter without fusing together. Indeed, such scattering collisions are a million times more probable than fusion events. Although holding the high temperature fusion fuel together long enough to allow the nuclei to collide millions and millions of times, is necessary to include the fusion reactions.

Therefore to achieve thermonuclear fusion energy we must solve two problems:

1. Produce and heat a plasma fuel to thermonuclear temperatures.

2. Confine it long enough to produce more fusion energy than we have expended in heating and containing the fuel.

These twin requirements are usually quantified by a mathematical relation known as the Lawson criterion, which essentially reflects the balance between thermonuclear energy production and heating energy. This criterion can be expressed as a condition on the product of the fuel density $n$ and the time of fusion fuel containment $\tau$. If we express $n$ in units of number of nuclei per cm$^3$ and $\tau$ in seconds, then the Lawson criterion demands that the product $n\tau$ exceed roughly $10^{14}$ s/cm$^3$ for a D-T fusion reaction and $10^{16}$ s/cm$^3$ for the D-D reaction.

In thermonuclear weapons no attempt is made to confine the reacting fuel. Instead one attempts to heat the fuel to thermonuclear temperatures so fast that an appreciable number of fusion reactions occur before it is blown apart. This scheme is known as inertial confinement, since it is the inertia of the reacting fuel that keeps it from blowing apart prematurely. Although to heat an appreciable mass of fuel to such high temperatures requires an extremely large energy source, and the source used in thermonuclear weapons is an explosive fission chain reaction. That is, an atomic bomb is used to heat the thermonuclear
fuel to ignition temperatures. Again this approach is highly unsuited for a controlled application.

The approach to fusion power that has been most extensively studied to date works with far smaller quantities of thermonuclear fuel. In particular, it takes advantage of the fact that at the high temperatures necessary for fusion to occur, the fuel becomes an ionised or charged gas known as plasma. Since such charged particles have difficulty moving across magnetic-field lines instead of tending to spiral along them. The primary approach has been to design a "magnetic bottle" composed of strong magnetic fields to contain the fuel. Traditionally these magnetic confinement fusion schemes have worked with very low fuel densities ($\sim 10^{14}$ cm$^{-3}$) and have attempted to achieve confinement times of the order of a second to satisfy the Lawson criterion. After two decades of intensive research, magnetic confinement fusion has reached the threshold of achieving the goal of scientific breakeven, in which the Lawson criterion is satisfied and the fusion energy produced by the fuel exceeds the energy necessary to heat and confine it.

Recently, scientists have become aware about an alternative approach to controlled thermonuclear fusion based on inertial confinement. In this approach intense laser or charged particle beams can be used to rapidly compress a tiny pellet of deuterium-tritium fuel to tremendous densities and temperatures, and ignite a thermonuclear fusion reaction. If the fuel pellet is compressed to sufficient densities, then it will burn so rapidly that appreciable fusion energy will be released before it can blow apart. This is illustrated in Figure (3).
When the outer surface of the pellet blows away, an enormous pressure is generated (as much as by a rocket exhaust) that would compress the core of the fuel pellet to densities as high as 1,000 to 10,000 times solid-state density. This compression would also raise the temperature of the core of the pellet to fusion temperature so that a thermonuclear burn is ignited. This burn would then propagate outward through the rest of the fuel pellet, igniting and burning it, which would result in an explosive release of fusion energy. The process of compression, thermonuclear ignition and burn would occur in a time much shorter than the time required for the pellet to blow apart (~10⁻⁹ s). Hence a premium is placed on developing driver beams capable of delivering large quantities of energy on to tiny targets (1 to 10mm in diameter) in a very short pulse (0.1 to 20ns).

In a sense, the inertial confinement approach to controlled fusion represents a scaling down of the hydrogen bomb over a millionfold to a tiny microexplosion. For a brief instant, the driver beams compress or implode the fuel pellet to produce conditions similar to those found in stars. A tiny sun is produced, which bursts in an instant, releasing its fusion energy. If this is captured, then it can be converted into useful purposes.

Inertial confinement fusion schemes are based on the following analogous sequence:
1. A tiny pellet of deuterium-tritium isotopes is injected into a blast chamber.

2. The pellet is compressed to very high density with intense laser or charged particle beams.

3. The high density and compression heat induce the ignition of a thermonuclear reaction producing a microscopic thermonuclear explosion.

4. The thermonuclear energy carried by the reaction products including neutrons, X rays and charged particles, is deposited as heat in a blanket, which then acts as a heat source in a steam thermal cycle to produce electricity known as the process of conversion of nuclear energy into electric energy.

The inertial confinement fusion and internal combustion engine would use a series of microthermonuclear explosions (from 1 to 100 per second, each generating the energy equivalent of several kilograms of high explosive) to generate power.

The applications of inertial confinement fusion fall into several categories: power production, weapons applications, and fundamental physics studies. Much of the funding for research activities in this area has been stimulated by the recognition that the environment is created by the implosion and thermonuclear burn of a tiny fuel pellet, is similar in some respects to that of a thermonuclear weapon. Hence there has been considerable interest in using inertial confinement fusion targets to simulate weapons physics and effects on a microscopic scale.

The most immediate application of inertial confinement fusion will be in basic physics studies. The imploded fuel pellet produces conditions of temperature and pressure that are quite unusual, at least on a terrestrial scale. Inertial confinement fusion implosions can be used to study properties of matter under extreme conditions, the interaction of intense radiation with matter, and aspects of low energy nuclear physics. Indeed, inertial confinement fusion presents us with a unique opportunity to study certain aspects of astrophysics such as stellar interiors on a laboratory scale.

Significant applications of inertial confinement fusion will be towards the production of energy that can then be used for a variety of purposes such as the generation of electricity, the production of process heat or synthetic fuels, or propulsion. The importance of this
application is apparent when it is recognised that there are only three major inexhaustible energy source options available to our civilisation at the present time; the nuclear fission breeder reactor, solar energy, and nuclear fusion.

2.5 Laser-Plasma Interaction

In inertial confinement fusion, an intense laser light shines on a solid target to produce an expanding plasma. The interaction of the laser light with the plasma is quite a complicated physical problem. The most widely used laser at present is the neodymium glass laser whose frequency $\omega_0$ is about $1.8 \times 10^{15}$ s$^{-1}$ at a wavelength of 1.06 $\mu$m, while the plasma frequency at solid density is typically one order of magnitude higher than $\omega_0$. Therefore the incident laser light cannot propagate inside the solid target, but first ionises the target surfaces. For a power density of $10^{15}$ W/cm$^2$, the associated electric field is about 15 V/$\mu$m. The ionised gas or plasma expands outward with approximately the ion sound speed, which is about $10^6$ m/s.

During the ~$1$ ns laser pulse the plasma expands about 1 mm. This is much longer than the laser wavelength, so the subsequent laser light pulses interact with the expanding low density plasma, of density gradient length $L$, much longer than the wavelength. In the classical theory, laser light absorption takes place by a process which is the inverse of bremsstrahlung, that is absorption of a photon accompanied by an electron-ion collision.

For $\omega_0 \gg \omega_{pe} \gg v_{ei}$ the absorption length is given by:

$$l_{ab} = \frac{\left(\frac{\omega_0}{\omega_{pe}}\right)^2}{\sqrt{1 - \frac{\omega_{pe}^2}{\omega_0^2} \frac{c}{v_{ei}}}}$$ (2.1)

where $\omega_{pe}$ is the electron plasma frequency, $v_{ei}$ is the electron-ion collision frequency. Since $\omega_{pe}^2 v_{ei}$ is proportional to $n^2 T_e^{-3/2}$, $l_{ab}$ depends sensitively on the plasma density and temperature. The parameter $n$ is chosen to be one half the critical density, i.e. $(\omega_0 / \omega_{pe})^2 = 2$. The temperature depends on the incident laser power density.

At high power density $T_e$ becomes a few keV. Then $l_{ab}$ becomes several centimetres which is much larger than the plasma depth $L$. The classical inverse bremsstrahlung is then ineffective and most of the laser energy arrives at the critical density. Therefore the above formula is no longer valid.
The oscillating electromagnetic field tunnels into the region of the critical density, where it excites an electrostatic electron plasma wave known as the "linear mode conversion" which in turn is absorbed by the Landau damping. This is reflected in Figure (4). This process is called the resonance absorption of the electromagnetic wave and is very effective near the critical density. The laser energy is then absorbed by the electrons in a thin layer at the critical density producing energetic electrons which expand outward. This expanding hot electron flow produces two important effects. One is a current which produces a magnetic field. The strength of the magnetic field becomes on the order of megagauss. The other is to produce fast ions by the space charge field. If the laser energy is carried by a small fraction of ions, then the resultant ablation pressure is reduced. The hot electrons are brought back to the target by the space charge field and penetrate into the high density region. Since the mean free path of the hot electrons is long, they penetrate into the central core region and deposit their energy there, causing preheating of the central core of the plasma. Experiments show that the heat conduction due to the hot electrons in the region of over critical density is substantially reduced below the classical value. Various mechanisms have been proposed to explain this reduction of the hot electron heat conductivity, such as a self-generated magnetic field, two-stream instability due to the cold electron return current, nonlocal heat transport due to the steep temperature gradient, etc. In any case, reduction of the heat conductivity further enhances the hot electron energy at the critical region.
CHAPTER 3
NATURE AND PROPERTIES OF PLASMA

3.1 Introduction

A plasma can be derived in various ways. It has been called the fourth state of matter as distinct from the solid, liquid and gaseous states. More than ninety nine percent of the cosmos consists of plasma in the stars and to a large extent in the interstellar matter. Since there are no such sharp distinguishing marks as a melting point or boiling point, but only the fact that all matter is ionised at high temperatures (above 10,000°K all matter is ionised to some degree).

This chapter of the thesis will describe the nature and properties of a plasma and will sketch the kinetic theory of plasmas for some background understanding. The kinetic theory or the theory of kinetic equations is a link between the single-particle description of the microscopic theory and of the macroscopic continuum theory of hydrodynamic quantities, such as particle densities, net velocities and temperatures.

3.2 Definition of Plasma

Any ionised gas cannot be called a plasma, of course; there is always some small degree of ionisation in any gas. A useful definition is as follows: A plasma is a quasineutral gas of charged and neutral particles which exhibits collective behaviour.

The plasma is "quasineutral", that is, neutral enough so that one can take n_e = n, where n is a common density called the plasma density, but not so neutral that all the interesting electromagnetic forces vanish. What is meant by "collective behaviour" is as follows: consider the forces acting on a molecule of, say, ordinary air. Since the molecule is neutral, there is no net electromagnetic force on it, and the force of gravity is negligible. The molecule moves undisturbed until it makes a collision with another molecule, and these collisions control the particle's motion.
A macroscopic force applied to a neutral gas, such as from a loudspeaker generating sound waves, is transmitted to the individual atoms by collisions. The situation is totally different in a plasma, which has charged particles. Once these charges move around, they can generate local concentrations of positive or negative charge, which gives rise to electric fields. Motion of charges also generates currents and hence magnetic fields. These fields affect the motion of other charged particles far away.

Let us consider the effect on each other of two slightly charged regions of plasma separated by a distance \( r \). The Coulomb force between \( A \) and \( B \) diminishes as \( 1/r^2 \). However, for a given solid angle (that is, \( \Delta r/r = \text{constant} \)), the volume of plasma in \( \beta \) that can affect \( A \) increases as \( r^3 \). Therefore, elements of plasma exert a force on one another even at large distances. It is this long-ranged Coulomb force that gives the plasma a large repertoire of possible motions and enriches the field of study known as plasma physics. In fact, the most interesting results concern so-called "collisionless" plasmas, in which the long-range electromagnetic forces are so much larger than the forces due to ordinary local collisions that the latter can be neglected altogether. By "collective behaviour", we mean motions that depend not only on local conditions but on the state of the plasma in remote regions as well.

The word "plasma" seems to be a misnomer. It comes from the Greek word which means something moulded or fabricated. Due to collective behaviour, a plasma does not tend to conform to external influences, instead it often behaves as if it had a mind of its own.

### 3.3 Plasma Parameters

One important parameter is the plasma frequency \( \omega_p \) which determines the oscillations of the electron gas between the ions in a plasma. Electrons in a homogeneous plasma with a density \( n_e \) are removed from their equilibrium position (by any process: incident electromagnetic wave or else). By simple geometry, the relative change of the electron density is found by:

\[
\frac{dn_e}{n_e} = \frac{d\xi}{dx}
\]  

(3.1)

The electric field \( E \) produced by this change is given by the differential change \( dn_e \) of the electron density according to Gauss' law and using Equation (3.1):
\[
\frac{dE}{dx} = -4\pi e \varepsilon n_e = 4\pi e n_e \frac{d\xi}{dx}
\]  

(3.2)

This field produces a force to each electron which can be used in a Newton eqn of motion, describing the temporal dependence of the disturbance length \( \xi(t) \):

\[
m \frac{d^2 \xi}{dt^2} = -e E = -4\pi e n_e e^2 \xi(t)
\]

(3.3)

This second order differential equation can be solved by the ansatz:

\[
\xi(t) = \text{const } \exp \left( i \omega_p t \right)
\]

(3.4)

and describes an oscillation with a frequency:

\[
\omega_p^2 = \frac{4\pi e^2 n_e}{m}
\]

\[
\omega_p = 5.65 \times 10^4 \sqrt{n_e}
\]

(3.5)

which is called the plasma frequency. The units used are the Gaussian magnetic cgs-system where the electron charge is \( e=4.803\times10^{-10} \text{cm}^2\text{s}^{-1}\text{g}^{1/2} \) and the electron mass is \( m=0.90109\times10^{-27} \text{g} \). The electron density \( n_e \) is given in particles per cm\(^3\).

Another plasma parameter is the Debye length \( \lambda_D \), which is given as a kind of wavelength to the electrostatic Langmuir oscillations of the plasma frequency. Although the product of wavelength time frequency is a wave speed, we have now the product of wavelength \( \lambda_D \) times the radian frequency (2\( \pi \)-times the frequency) to take where the wave velocity is identical with the averaged thermal velocity \( V_e \) of the electrons, which is related to the electron temperature \( T \) by the Boltzmann constant \( K=1.38 \times 10^{-16} \text{ erg/degree} \) through \( V_e^2/2m = 3K T/2 \). Then the Debye length is found as:

\[
\lambda_D = \frac{V_e}{\omega_p} = \left[ \frac{KT}{4\pi n_e e^2} \right]^{1/2}
\]

(3.6)

\[
\lambda_D (\text{cm}) = 6.9 \left[ \frac{T(K)}{n_e (\text{cm}^{-3})} \right]^{1/2} = 743 \left[ \frac{T(\text{ev})}{n_e (\text{cm}^{-3})} \right]^{1/2}
\]

(3.7)
In the first part of Equation (3.7) the temperature is given in degree Kelvin and in the second part given in electron volts with the relation $T(\text{Kelvin}) = 1.16 \times 10^4 \ T(\text{ev})$.

An important quantity in plasma is the collision frequency $\nu$ between the electrons and ions. This process involves an electron moving towards an ion of positive charge (the ion may be a nucleus with $Z$ protons) at a lateral separation $r_o$ at a large distance called the impact parameter. It will be attracted by a Coulomb force $f$ where $r$ is the distance-vector between the electron and the ion:

$$f = -\frac{ze^2 r}{r^3}$$ (3.8)

The main interaction occurs during a time:

$$t = \frac{r_o}{v}$$ (3.9)

if the electron has a velocity $V$.

During the interaction time $t$, a change of the electron momentum, resulting in an angle $\phi_0$ occurs of the value:

$$\Delta(mv) = |ft| = \frac{ze^2}{r_o v}$$ (3.10)

If we restrict only to $90^\circ$ collisions as the only interesting collision process, then the relation results:

$$\Delta(mv) = mv = \frac{ze^2}{r_o v}$$ (3.11)

$$r_o = \frac{ze^2}{mv^2}$$ (3.12)

defining the value $r_o$ of the impact parameter for the Coulomb collision.
The Coulomb collision cross section is given by:

\[ \bar{\rho} = \pi r_0^2 = \frac{z^2\pi\theta^4}{m^2v^4} \]  

(3.13)

Which results in the collision frequency in a plasma of an ion density \( n_i \):

\[ v_{ei} = n_i \bar{\rho} v = \frac{zn_e\pi\theta^4}{m^2v^3} \]  

(3.14)

Expressing the electron velocity \( v \) by the electron temperature \( T \) arrives at:

\[ v_{ei} = \frac{zn_e\pi\theta^43^{-3/2}}{m^{1/2}(KT)^{3/2}} \]  

(3.15)

### 3.4 Euler’s Equation of Motion

The hydrodynamic equation of motion is the field-theoretical generalisation of Newton’s single-particle equation of motion:

\[ ma = F = -\nabla \phi \]  

(3.16)

The product of mass \( m \) times the acceleration \( a \) of a body equals the force \( F \) which can be expressed by the gradient of a potential \( \phi \). In the case of a fluid, there is a velocity field \( v(x,y,z,t) \), of which the temporal derivative corresponds to the mass \( m \), and a force density, which is given by the gradient of the pressure field \( p(x,y,z,t) \). The fluid can be considered as being composed of electrons of mass \( m_e \), density \( n_e \), temperature \( T_e \) and ions of mass \( m_i \), density \( n_i \), and charge \( Z \). Assuming space charge neutrality \( n_e = n_i/Z \), the mass density field is given by:

\[ \rho(x,y,z,t) = m_n \rho_e(x,y,z,t) + mn_e(x,y,z,t) \]  

(3.17)

The pressure field (also as function of \( x,y,z,t \)) is:

\[ p = n_e KT_e + n_i KT_i = (1+z) n_i KT_e, \text{ if } T_e = T_i \]  

(3.18)
The equation of motion corresponding to Newton’s Eqn (3.16), the Euler equation, is:

$$\rho \frac{dv}{dt} = -\nabla p + \eta \nabla^2 V$$  \hspace{1cm} (3.19)

The last term on the right-hand side is added to the original Euler equation and is called the Navier-Stokes term. This includes the hydrodynamic viscosity $\eta$, which determines the internal friction of the fluid. The operation on the left-hand side of Euler’s equation (3.19) is:

$$\rho \frac{dv}{dt} = \rho \frac{\partial v}{\partial t} + \rho \frac{\partial v}{\partial x} \frac{dx}{dt} + \rho \frac{\partial v}{\partial y} \frac{dy}{dt} + \rho \frac{\partial v}{\partial z} \frac{dz}{dt}$$  \hspace{1cm} (3.20)

Combining the last three terms to $\rho v \cdot \nabla v$, which is a nonlinear term. Remember that the spatial “del” operator is:

$$\nabla = i_x \frac{\partial}{\partial x} + i_y \frac{\partial}{\partial y} + i_z \frac{\partial}{\partial z}$$  \hspace{1cm} (3.21)

where the unit vectors $i_x$, $i_y$ and $i_z$ are of the modulus 1 and are in the direction of the Cartesian coordinates $x$, $y$ and $z$ respectively. Using this way of writing, the Euler equation is then:

$$\rho \frac{\partial v}{\partial t} + \rho v \cdot \nabla v = -\nabla p$$  \hspace{1cm} (3.22)

Where the viscosity has been dropped. This equation of motion corresponds to the conservation of momentum.

### 3.5 Bernoulli’s Stationary Solution

From Euler’s equation (3.22) Bernoulli’s equation can be derived as a stationary solution. Because of the necessary time independence:

$$\frac{\partial}{\partial t} = 0$$  \hspace{1cm} (3.23)

the Euler equation (3.22) reduces to:

$$\rho v \cdot \nabla v + \nabla p = 0$$  \hspace{1cm} (3.24)
By using the vector identity:

\[ \nu \nabla \nu = \frac{1}{2} \nabla \nu^2 - \nu \times (\nabla \times \nu) \]  

(3.25)

The last term can be dropped for vortex-free motion, giving:

\[ \frac{\rho}{2} \nabla \nu^2 + \nabla p = 0 \]  

(3.26)

Under the further special assumption that the spatial variation of the density \( \rho \) is zero:

\[ \rho = \text{const} \]  

(3.27)

Eqn (3.26) can be written as:

\[ \nabla \left( \frac{\rho}{2} \nu^2 + p \right) = 0 \]  

(3.28)

When integrated, yields the Bernoulli equation:

\[ \frac{\rho}{2} \nu^2 + p = \text{const} \]  

(3.29)

Note: That the Bernoulli equation is a very particular case of the general Euler hydrodynamic equation, not only because of the stationary condition, Eqn (3.23), but also because of the neglect of the last term in Eqn (3.25) (vortex-free motion) and by condition in Eqn (3.27), which corresponds to a chemical homogeneity and incompressibility of the fluid.

3.6 Equation of Continuity

The Euler equation is one of the three basic equations of hydrodynamics and corresponds to the conservation of momentum. The next basic equation is the conservation of mass, which is sometimes called the equation of continuity. It can be derived from the following geometrical consideration. Take a constant volume \( V \) with a density \( \rho \), which is moved during a time interval \( dt \) along the large arrow of Figure (5), (Hora 21, 1996).
Figure 5: Geometrical variation with conservation of mass for the derivation of the equation of continuity.

After the time \( dt \) the volume \( V \) has changed its density to the value \( \rho + \delta \rho \). The increase of mass in the volume \( V \) can only be due to material streaming into the volume \( V \); therefore it has a negative divergence of the velocity field (expressed by \( -\nabla \cdot \mathbf{v} \)):

\[
\frac{d\rho}{dt} = -\rho \nabla \cdot \mathbf{v} \tag{3.30}
\]

Gauss' law, expressing the converging or diverging velocity field at a closed surface by the divergence \( (\nabla \cdot \mathbf{v}) \) of a volume integral within a closed area, gives:

\[
\int \mathbf{v} d^2a = \iiint \nabla \cdot \mathbf{v} d^3x \tag{3.31}
\]

The left-hand side of equation (3.31) can be expressed by the partial differentiation:

\[
\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{v} \tag{3.32}
\]

Using the differentiation relation:

\[
\nabla \cdot \rho \mathbf{v} = \rho \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \rho \tag{3.33}
\]
Eqn (3.32) can be rewritten as an equation of continuity:

$$\frac{\partial p}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$  \hspace{1cm} (3.34)

For the special case of incompressible fluids, Eqn (3.27), the special formulation of the equation of continuity is obtained:

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{v} = 0$$  \hspace{1cm} (3.35)

This is related to a fast change of the density within a very large volume.

### 3.7 Concept of Microscopic Plasma Theory

In the hydrodynamic plasma theory, further forces have to be included into the equation of motion in Eqn (3.22). For example: the Lorentz force, can be based or derived also from a microscopic theory similar to Lorentz' attempt. To describe the metals from an ensemble of a large number of electrons and ions. In contrast to the purely phenomenological hydrodynamics based on Euler's equation of motion, Gauss' clarification of continuity and the energy conservation elaborated from the classical gas laws.

The model considers a large number of electrons and ions giving them classically numbers from 1 to N. Each of these particles follows an equation of motion, where the accelerations of the n-th particle with mass \(m_n\) is given by the forces from all the other particles 1 to N, given by their position coordinates \((x,y,z)\) and their velocities. A set of 3N differential equations hence can be achieved:

$$m_n \frac{d^2 x_n}{dt^2} = f_{x_n} \left( x_1, \ldots, x_N, \frac{\partial x_1}{\partial t}, \ldots, \frac{\partial x_N}{\partial t} \right)$$

$$n = 1, \ldots, N$$

$$m_n \frac{d^2 y_n}{dt^2} = f_{y_n} \left( y_1, \ldots, y_N, \frac{\partial y_1}{\partial t}, \ldots, \frac{\partial y_N}{\partial t} \right)$$

$$n = 1, \ldots, N$$

$$m_n \frac{d^2 z_n}{dt^2} = f_{z_n} \left( z_1, \ldots, z_N, \frac{\partial z_1}{\partial t}, \ldots, \frac{\partial z_N}{\partial t} \right)$$

$$n = 1, \ldots, N$$  \hspace{1cm} (3.36)
This type of equation is indeed being treated by computers for describing plasmas with $N$ up to one million and more. Indeed the forces $f$ are simplified and only the interaction of the particles with their immediate neighbours are covered. While a considerable number of plasma phenomena like oscillations, instabilities etc can be perfectly described, others where the Coulomb collision frequency is important, cannot.

The next step for a microscopic description of plasmas is to summarise the large number of particle by introducing a distribution function. To understand a distribution function one may start from a number $N$ of elements $a_i$ ($i = 1$ to $N$) of which an average value $M$ has to be found.

$$M = \frac{\sum_{i=1}^{N} a_i}{N} \quad (3.37)$$

Instead of this procedure, one may bundle elements of equal value where $f_i$ describes the number of the elements of each bundle. With this weighting numbers one can then perform the same task as in Eqn (3.37):

$$M = \frac{\sum_{i=1}^{n} f_i a_i}{\sum_{i=1}^{n} f_i} \quad (3.38)$$

The numbers $f$ describe the distribution of all the elements, and this can also be expressed for a continuous (instead of discrete) manifold by a distribution function $f(x)$ in order to get the expectation value $a$, and the weighted average value of the manifold $a(x)$.

$$\langle a \rangle = \frac{\int f(x) a(x) \, dx}{\int f(x) \, dx} \quad (3.39)$$

Our ensemble of $N$ particles can be described by a distribution function $f$ depending on the coordinates $x_i$ and the velocity components $v_i$ ($i = 1, 2, 3$) giving the number of particles of these (6-dimensional) coordinates, which are located in the differential element $dx_1 \, dx_2 \, dx_3 \, dv_1 \, dv_2 \, dv_3$:

$$f \left( x_{11}, x_{22}, x_{33}, v_{11}, v_{22}, v_{33}, t \right) \, d^6x d^3v \quad (3.40)$$
One then has to realise that instead of talking about quantities $Q$ directly, one can look into the distribution function and derive from it the expectation (average) values of $Q$ from the procedures given by Eqn (3.39):

$$Q(r,t) = \int \int \frac{Q(r,\omega_1,\omega_2,\omega_3) f(r,\omega_1,\omega_2,\omega_3) \, d^3\omega}{\int \int f(r,\omega_1,\omega_2,\omega_3) \, d^3\omega}$$  \hspace{1cm} (3.41)

If $Q = 1$, one receives the particle density $n_0$ for electrons or for ions of the plasma. If $Q$ is the (microscopic) velocity $\omega$ one arrives at the macroscopic plasma velocity $v$ (or drift velocity). The difference $\mu$ is the random (thermal) velocity $W = v + \mu$, $\mu$ can be a Maxwellian distribution, but can describe any other non-equilibrium distribution.

The advantage of the description with the distribution function $f$, called kinetic theory, which consists of the fact that non-equilibrium states and their relaxation can be described, whilst this is not possible with a simple one fluid macroscopic hydrodynamics, where temperatures $T$ are defined always from an equilibrium state.

The intriguing property of the distribution function $f$ consists of the fact that there is a way for deriving the macroscopic hydrodynamics. This is given from any temporal change of $f$ that has to be described by the collisions between the particles:

$$\frac{d}{dt} f = \left( \frac{\partial f}{\partial t} \right)_{\nu}$$  \hspace{1cm} (3.42)

The right hand side is called Boltzmann's collision term. Eqn (3.42) is Boltzmann's equation which will now be written in more detail. The total differentiation in Eqn (3.42) results in:

$$\frac{d}{dt} f = \frac{\partial}{\partial t} f + \frac{\partial}{\partial x_1} f \frac{\partial x_1}{\partial t} + \frac{\partial}{\partial x_2} f \frac{\partial x_2}{\partial t} + \frac{\partial}{\partial x_3} f \frac{\partial x_3}{\partial t}$$

$$+ \frac{\partial}{\partial \omega_1} f \frac{\partial \omega_1}{\partial t} + \frac{\partial}{\partial \omega_2} f \frac{\partial \omega_2}{\partial t} + \frac{\partial}{\partial \omega_3} f \frac{\partial \omega_3}{\partial t}$$  \hspace{1cm} (3.43)

Using the nabla operator for the spatial differentiation and the nabla operator with an index $\omega$ for the differentiation towards the velocity components and remembering that the acting force
F can be compressed according to Newton's law by:

\[ f = m \frac{\partial}{\partial t} \omega \]  \hspace{1cm} (3.44)

Hence the usual formulation of Boltzmann's equation is finally arrived at:

\[ \frac{\partial f}{\partial t} + \omega \cdot \nabla f + \frac{F}{m} \nabla_\omega f = \left( \frac{\partial f}{\partial t} \right)_c \]  \hspace{1cm} (3.45)

The intriguing property of \( f \) can be seen when integrating Eqn (3.45) by \( d^3\omega \) over the whole velocity space. The integration over Boltzmann's collisional term is zero and the other terms immediately result in the macroscopic hydrodynamic equation of continuity Eqn (3.35).

Multiplying Eqn (3.45) by \( mw \) and integrating over the whole velocity space, one arrives at the macroscopic equation of motion Eqn (3.22). When multiplying Eqn (3.45) with \( m\omega^2 \) and integrating over the whole velocity space, one arrives at the hydrodynamic equation of energy conservation. The advantage of the kinetic theory with the Boltzmann equation consists of the fact that the general properties for non-equilibrium and of numerous nonlinear properties are included in a much more sophisticated way than in the macroscopic theory. This is the reason why plasma instabilities and other complex phenomena are treated preferably by the kinetic theory.
CHAPTER 4
ELECTRODYNAMIC PHENOMENA

4.1 Introduction

There is existence of electric charges as an established fact, whether it is produced by rubbing a piece of amber, the godparent of electricity, or recognising them from the spark when connecting the poles of a battery. This is interpreted by the observed attraction, repulsion, and heat generation as the result of the charges which have been produced. This is taken with care, not to determine the charge verbally or to describe a derived dimension to it by some arbitrary procedure. Instead this is regarded as having its own dimension as an entity beyond the range of mechanics. This quantity is called Q. This unit of charge can be chosen to be either negative or positive, the familiar universal charge of the electron.

However Q is preferred to stand for the Coulomb, the accepted unit in the practical system, in terms of which the electron charge is expressed by $e = 1.60 \times 10^{-19}$ Coulomb. The atomistic nature of charge is disregarded in the Maxwell theory property. The charge of the atoms, and elementary particles is, to a much higher degree an absolute constant than the mass.

In addition to the electric unit Q this would normally employ as mechanical units of length, mass, and time the Giorgi units M (metre), K (kilogram mass), and S (second).

There is advantage that in this system the units of energy and power correspond exactly to the joule and watt. This is designated as:

\[
\begin{align*}
1 \text{ joule} &= 1 \text{M}^2 \text{KS}^2 = 10^7 \text{ cm}^2.\text{g}.\text{sec}^2 \\
&= 10^7 \text{ erg} \\
1 \text{ joule}/\text{s} &= 1 \text{M}^2 \text{KS}^{-3} = 10^7 \text{ cm}^2.\text{g}.\text{sec}^{-3} \\
&= 10^7 \text{ erg/sec} \\
&= 1 \text{ watt}
\end{align*}
\]
This defines correspondingly:

\[
1 \text{ newton} = 1 \text{ MKS}^{-2} = 10^5 \text{ cm.g.sec}^{-2} = 10^5 \text{ dynes}
\]

This unit of force "newton" is seen to be conveniently comparable in size to the practical unit of force known as the "kilogram" = 9.81.10^5 dynes.

This chapter of this thesis proceeds to examine in sequence the basic electromagnetic concepts. In most cases a dimensional description is concerned rather than with a complete definition. The latter will be derived from their interrelation through the basic equations of the theory.

4.2 Concepts of the Electromagnetic Field

Before describing Maxwell's equations, the following elementary definitions of electromagnetic field have to be given. Electrodynamics indeed begins with Newton. Newton revealed that there is a force from the amber (Greek: elektra) acting on the paper against the gravitational force of the earth. Coulomb could realise and elaborate the attraction of repulsive electrostatic forces between two charges \( q_1 \) and \( q_2 \) in a distance \( r \) which is given by the equation:

\[
|F| = \frac{q_1 q_2}{r^2} c_o
\]  

(4.1)

Describing the force in the units of dyn = gcms\(^{-2}\), \( r \) in cm and using a constant \( c_o = 1 \). the units of the charge is \( [q] = g^{1/2} \text{ cm}^{3/2} \text{ s}^{-1} \).

In this thesis the use of mostly (Gaussian) cgs system will be used, because it is usual in plasma physics. The charge of an electron is then \( 4.803 \times 10^{-10} \) cgs units. Coulomb's constant \( c_o \) in Eqn (4.1) is:

\[
c_o = \frac{1}{4\pi \epsilon_o}
\]  

(4.2)

Where \( \epsilon_o \) is a constant.

From the Coulomb force Eqn (4.1) one derives the definition of the electric field strength \( E \) as
the force per unit charge $Q$:

$$E = \frac{\text{Force}}{\text{Charge}} = \frac{\text{newton}}{Q}$$

(4.3)

In such a temporally constant electrostatic field $E$, the line integral between two points A & B:

$$\int_{A}^{B} E_x ds = \int_{A}^{B} E ds$$

(4.4)

$E_x$ is the perpendicular projection of $E$ on the direction of the line element $d\mathbf{s}$, which is regarded as a vector; $E.d_s$ denotes, as usual the scalar product. This line integral is called the "voltage" $V$:

$$V = \int_{A}^{B} E ds = \frac{\text{newton} M}{Q} = \frac{\text{dyne} \cdot \text{cm}}{Q} \approx 10^7 \frac{\text{erg}}{Q}$$

(4.4a)

The conversion of the units from MKSQ - to the cgs - system shows that our unit of voltage is identical with:

$$1 \text{ volt} = 10^8 \text{ cgs units}$$

(4.4b)

$$1 \text{ coulomb} = \frac{1}{10} \text{ cgs unit}$$

(4.4c)

From the definition of the voltage it is necessary that in addition to the terminal points A, B the path between them be prescribed. Only in lamellar fields is the independence of the line integral with respect to the path guaranteed by Stokes' law.

A companion is introduced to the field strength $E$ as a second electric vector $D$. To make the introduction of $D$ comprehensible by the following consideration: The charge, in its historical origin, is a concept based on the notion of action at a distance. To adopt it to the viewpoint of action by a field it is necessary to imagine an excitation of the surrounding medium proceeding from the charge centres, in which the excitation will be described by the vector $D$. For a single point charge $e$, imagine "D lines" leaving $e$ uniformly in all directions, with such density that the "D-flux" becomes:
\[ \oint D_n \, d\sigma = \varepsilon \quad (4.5) \]

\( d\sigma \) is an element of an arbitrary surface surrounding \( \varepsilon \). If in particular a spherical surface of radius \( r \) is chosen, therefore it is found that:

\[ 4\pi \, r^2 \, D = \varepsilon \quad (4.5a) \]

For arbitrarily, including continuously distributed charges, Eqn (4.5) is replaced by:

\[ \oint D_n \, d\sigma = \bar{\varepsilon} \, , \quad \bar{\varepsilon} = \int d\sigma \quad (4.5b) \]

Where \( \bar{\varepsilon} \) indicates the total charge within \( \sigma \), the algebraic sum of positive and negative charges.

From the preceding equations the dimension of \( D \) is seen to be:

\[ D = \frac{\text{charge}}{\text{area}} = \frac{Q}{M^2} \quad (4.6) \]

This dimension is entirely different from the dimension of the field strength \( E \) given by Eqn (4.3).

Equation (4.6) is compared with the dimension of the electric current density. One knows that this is to be defined as the quantity of electricity traversing unit area in unit time in a conductor. Its dimension is therefore:

\[ J = \frac{\text{charge}}{\text{area.time}} = \frac{Q}{M^2 S} \quad (4.6a) \]

Depending on whether the unit is placed perpendicular to the direction of the current or at an angle thereto, the absolute magnitude of \( J \) or a component of it is obtained. \( J \) is thus a vector similar in character to \( D \). Dimensionally however, \( D \) is not, but the time rate of change of \( D \) which is the so called displacement current \( D \), corresponds to \( J \).

A sharp distinction has been here assumed between conductor and non-conductor dielectric medium. Actually, no perfect insulator exists since even the best nonconductor conducts to
some extent, e.g. under the influence of cosmic radiation.

Maxwell therefore supplements the displacement current to form the total current:

\[ C = \dot{D} + J \quad (4.7) \]

the designation \( C \) (current) was introduced by Maxwell. This notion of the equivalence of \( D \) and \( J \) is a basically new idea of Maxwell's, which is a prerequisite for the unified representation of electromagnetic phenomena.

### 4.3 Maxwell's Equations in Integral Form

The hypothesis of electrodynamics, just as the Newtonian hypothesis of mechanics, rest on experience, more exactly on the ordering of the totality of experience into a simplified and idealised form.

To begin with, two principal hypotheses are set up which are supplemented by secondary hypotheses. One of these shall be called Faraday's law of induction. The other shall be named after Ampere, since he was the first to formulate the relationship between current and magnetic fields. Both hypotheses shall however be stated in the universal form whose possibility was first realised by Maxwell.

For this purpose an arbitrary surface \( \sigma \) is considered with the boundary curve \( S \). Therefore the surface integrals are computed as:

\[ \int B_n \, d\sigma \text{ and } \int C_n \, d\sigma \quad (4.8) \]

Which Equation (4.8) is extended over \( \sigma \), and shall be called the magnetic flux and electric current flux.

Next the following line integrals are computed which are extended over the boundary curve \( S \):

\[ \oint E \cdot ds \text{ and } \oint H \cdot ds \quad (4.9) \]

31.
These are called electric and magnetic loop tension. Now the two principal hypotheses which relate the quantities defined in Eqns (4.8) and (4.9) will be written down in this completely general sense. They are:

\[
\frac{d}{dt} \int B_n \, d\sigma = -\oint E \cdot ds \tag{4.10}
\]

\[
\int C_n \, d\sigma = \oint H \cdot ds \tag{4.11}
\]

The above equations (4.10) and (4.11) can be stated that for every change in the number of magnetic lines of force, which traverse a given surface \( \sigma \) produces in its boundary \( s \) an electric loop tension, which is numerically equal to the rate of change, but opposite in sign (Faraday’s law of induction). The number of electric current lines which traverse an arbitrary surface \( \sigma \) is accompanied by a magnetic loop tension in the bounding curve \( s \) of \( \sigma \), which is equal to it in both magnitude and direction (Ampere’s law relating magnetic field and electric current).

Consider a closed surface \( \sigma \) with a boundary curve which has contracted to a point. This does not contribute to the line integrals in Eqns (4.10) and (4.11). If the integration is indicated over the now unbounded surface with \( \oint \), thus obtain:

\[
\frac{d}{dt} \oint B_n \, d\sigma = 0 \quad \text{and} \quad \oint C_n \, d\sigma = 0 \tag{4.12}
\]

From Equation (4.7) the second equation may also be written:

\[
\oint j_n \, d\sigma + \frac{d}{dt} \oint D_n \, d\sigma = 0 \tag{4.12a}
\]

More particularly, if the surface \( \sigma \) lies entirely in nonconducting material and is hence traversed by no conduction currents:

\[
\frac{d}{dt} \oint D_n \, d\sigma = 0 \tag{4.12b}
\]

The first equation (4.12) and equation (4.12b) state, in integrated form:

\[
\oint B_n \, d\sigma = \text{const}, \quad \oint D_n \, d\sigma = \text{const} \tag{4.13}
\]
The second Eqn (4.12) and Eqn (4.12a) show that the total electric current is always closed. In Maxwell's theory: the quantities entering and leaving just compensate each other, the current lines traversing our surface \( \sigma \) form closed loops somewhere outside of it. Furthermore, the magnetic lines of force also are always closed. It follows that the constant in the first Eqn (4.13) must be zero, while in the second equation this constant is the algebraic sum \( \bar{e} \) of the charges \( e \) enveloped by the surface \( \sigma \). According to the above this must be a constant in time for a nonconductor:

\[
\oint B_n \, d\sigma = 0, \quad \oint D_n \, d\sigma = \bar{e},
\]

\[
\bar{e} = \int d\bar{e} = \text{const}
\]  

(4.13a)

The first Eqn (4.13a) is a supplementary hypothesis. The second Eqn (4.13a) agrees with our earlier Eqn (4.5b) and states the constancy in time of the charge in nonconductors. The \( D \)-lines and the \( E \)-lines coinciding with them geometrically originate at points of positive charge and end at points of negative charge. Eqn (4.12a) generalising the second Eqn (4.13a), may be designated in hydrodynamic terminology as the continuity equation of electricity. If the definition of \( \bar{e} \) in Eqn (4.13a) is employed, it takes on the form:

\[
\frac{d \bar{e}}{dt} + \oint j_n \, d\sigma = 0
\]  

(4.13b)

### 4.4 Maxwell's Equation in Differential Form

The integral is now passed to the differential form by allowing the loops \( S \) in the integral form, and hence also the surfaces \( \sigma \) passed through them, to become arbitrarily small. If the latter is called \( \Delta \sigma \), therefore can write in the limit:

\[
\oint B_n \, d\sigma = \Delta \sigma B_n, \quad \oint C_n \, d\sigma = \Delta \sigma C_n
\]

(4.14)

Furthermore the definition of the vector operation "\( \text{curl} \)" is recalled by the transition to the limit of a loop integral. For our infinitesimal loops this leads to:

\[
\oint E_n \, ds = \Delta \sigma \text{curl}_n E, \quad \oint H_n \, ds = \Delta \sigma \text{curl}_n H
\]

(4.15)

The time derivative of the first Eqn (4.14) must be form. Imagine here the surface \( \Delta \sigma \) to
remain fixed, which obviously applies to media at rest, to which we shall confine ourselves initially. Then:

\[
\frac{d}{dt} \int B_n \, d\sigma = \Delta \sigma \dot{B}_n, \quad \dot{B} = \frac{\partial B}{\partial t}
\]  

(4.16a)

At the same time, using Eqn (4.7), therefore write Eqn (4.14) in the form:

\[
\int C_n \, d\sigma = \Delta \sigma \left( J_n + \dot{D}_n \right), \quad \dot{D} = \frac{\partial D}{\partial t}
\]  

(4.16b)

With Eqn (4.15) and Eqn (4.16a,b), cancelling the factor $\Delta \sigma$ which is common to all terms, as well as omitting the common index $n$, the principal axioms Eqn (4.10) and Eqn (4.11) lead to the two vectorial differential equations:

\[
\dot{B} = -\text{curl} \ E
\]  

(4.17)

\[
\dot{D} + J = \text{curl} \ H
\]

Now the supplementary hypothesis Eqn (4.13a) will be complete for $B$, and the relation between $D$ and the charge, contained in the same equation.

The latter shall now be regarded as continuously distributed in accord with our differential point of view. The infinitesimal charge:

\[
\Delta \theta = \rho \Delta \tau
\]  

(4.18)

is contained in an element of volume $\Delta \tau$ which approaches zero in magnitude. At the same time the vector operation "divergence" is recalled and its representation by the limit of a volume integral. For our present purposes this representation is written as:

\[
\lim_{\Delta \tau \to 0} \frac{1}{\Delta \tau} \oint B_n \, d\sigma = \text{div} B, \quad \lim_{\Delta \tau \to 0} \frac{1}{\Delta \tau} \oint D_n \, d\sigma = \text{div} D
\]  

(4.19)

and obtain from Eqns (4.13a,b), omitting the factor $\Delta \tau$, their differential form:

\[
\text{div} \ B = 0
\]  

(4.17a)
\[ \text{div } D = \rho \quad (4.17b) \]

\[ \frac{\partial \rho}{\partial t} + \text{div } j = 0 \quad (4.17c) \]

Our Eqns (4.17) and (4.17a,b,c) set up the framework into which phenomena of electrodynamics must be fitted. But this phenomena is still too wide. Five vectors \( \mathbf{E}, \mathbf{D}, \mathbf{J}, \mathbf{B} \) and \( \mathbf{H} \) occur in our equations, or altogether 15 unknown functions of time and space. (The scalar \( \rho \) is referred back to the vectors \( \mathbf{D} \) and \( \mathbf{J} \) by the Eqns (4.17b) and (4.17c) respectively.) For their determination, two vector equations are used (4.17) i.e. altogether only six differential equations. The framework must be narrowed down to be able to fill it out with a unified electrodynamic model.

### 4.5 Conductivity and Ohm's Law

The electric current density \( \mathbf{J} \) depends on the electric field strength \( \mathbf{E} \) within the conductor. Assume a linear dependence:

\[ \mathbf{J} = \sigma \mathbf{E} \quad (4.20) \]

The real positive constant \( \sigma \) is the electric conductivity. Eqn (4.20) expresses Ohm's law for unit length of a wire carrying a stationary current. To recognise this, \( \mathbf{J} \) is replaced by the total current \( \mathbf{I} = q \mathbf{J} \) (\( q \) = cross section of the wire) and multiplied by Eqn (4.20) with length of the wire.

\[ RI = V, \quad R = \frac{1}{q} \quad (4.20a) \]
\[ V = IE = \int_{0}^{l} E \, ds = \text{voltage} \]

Ohm's law signifies the introduction of the material constant \( \sigma \). According to Eqn (4.20) its dimension is:

\[ \sigma = \frac{Q^2}{M^2 \, S \, \text{newton}} = \frac{Q^2}{MS \, \text{joule}} \quad (4.20b) \]
According to Eqn (4.20a) \( \sigma \) may also be designated as the reciprocal of the specific electric resistance, i.e. the resistance of a prism of the length \( l = 1 \text{M} \) and of the cross section \( q = 1 \text{M}^2 \). The dimension of the resistance is given by Eqns (4.20a,b):

\[
R = \frac{\text{joule} \, s}{Q^2}
\]  

(4.20c)

The unit of resistance in the practical system of units is the \( \Omega \) (pronounced "ohm") = \( 10^6 \) cgs units. It is identical with the unit in our MKSQ system provided that \( Q \) is equal to 1 Coulomb = \( 1/10 \) cgs unit. Therefore obtain:

\[
1 \frac{\text{joule} \, s}{Q^2} = 10^7 \frac{\text{erg} \, s}{Q^2} = 10^6 \text{ cgs units} = 1 \Omega
\]

(4.20d)

4.6 Dielectric Constant

The displacement \( \mathbf{D} \) depends on the electric field strength \( \mathbf{E} \) at the point in question. The assumption of the dependence to be linear is given by:

\[
\mathbf{D} = \varepsilon \, \mathbf{E}
\]

(4.21)

The real positive constant is called \( \varepsilon \), which is the dielectric constant. Its dimension is given by Eqns (4.6) and (4.3):

\[
\varepsilon = \frac{Q^2}{M \, \text{joule}}
\]

(4.21a)

The dielectric constant of a vacuum is denoted by \( \varepsilon_0 \). It also is a definite quantity of the dimension Eqn (4.21a). The relation is:

\[
\mathbf{D} = \varepsilon_0 \, \mathbf{E}
\]

(4.21b)

valid for vacuum, invariably \( \varepsilon > \varepsilon_0 \).
4.7 Permeability

A relation also exists between the two magnetic vectors \( H \) and \( B \), which, as a first approximation, shall also be assumed to be linear:

\[
H = \mu' \, B \quad (4.22)
\]

Since \( H \) is regarded as analogue of \( D \) and \( B \) as analogue of \( E \).

\[
B = \mu \, H \quad (4.22a)
\]

The material constant \( \mu \) is called permeability and has the dimension:

\[
\mu = \frac{\text{joule}}{\text{Q}^2} \cdot \frac{S^2}{M} \quad (4.22b)
\]

The introduction of \( \mu \), which is illogical in view of Eqn (4.21), leads to the obvious consequence that in later formulas, such as Coulomb's law, not \( \mu \), but its reciprocal \( \mu' \), will take the place of \( \varepsilon \). For vacuum:

\[
B = \mu_0 \, H \quad (4.22c)
\]

\( \mu_0 \) also obviously has the dimension given in Eqn (4.22b). For paramagnetic bodies \( \mu > \mu_0 \), for diamagnetic bodies, \( \mu < \mu_0 \).

4.8 Law of Conservation of Energy and Poynting Vector

Starting from Eqns (4.17), a scalar multiplication is carried out of the first with \( H \), a scalar multiplication of the second with \( E \). Therefore the summation of the two is given by:

\[
H \dot{B} + E \dot{D} + E.J = E \cdot \text{curl} \, H - H \cdot \text{curl} \, E \quad (4.23)
\]

Using the following abbreviation with the dot for partial temporal differentiation:

\[
\dot{B} = \frac{\partial}{\partial t} \, B ; \quad \dot{D} = \frac{\partial}{\partial t} \, D ; \quad \dot{H} = \frac{\partial}{\partial t} \, H ; \quad \dot{E} = \frac{\partial}{\partial t} \, E \quad (4.24)
\]

37.
If $\epsilon$ and $\mu$ are not dependent on time:

$$H\dot{B} = H\mu_0 \dot{H} = \frac{1}{2} \mu_0 \frac{\partial}{\partial t} H^2 \quad (4.25)$$

$$E\dot{D} = E\varepsilon_0 \dot{E} = \frac{1}{2} \varepsilon_0 \frac{\partial}{\partial t} E^2 \quad (4.26)$$

The right hand side of Eqn (4.23) apply the transformation, valid for arbitrary vectors $u,v$:

$$v \cdot \text{curl } u - u \cdot \text{curl } v = \text{div} \ (u \times v) \quad (4.27)$$

This relation is proved by utilising the symbolic "nabla operator" where:

$$\nabla = \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z} \quad (4.28)$$

Interpreting the divergence as scalar multiplication and the curl as a vector multiplication with this vector:

$$\text{div} \ (u \times v) = \nabla \cdot (u \times v) = \nabla_u \cdot (u \times v) + \nabla_v \cdot (u \times v) \quad (4.27a)$$

$$\text{curl } u = \nabla \times u, \quad \text{curl } v = \nabla \times v \quad (4.27b)$$

In Eqn (4.27a) the subscripts $u,v$ indicate that the $\nabla$ differentiations are to be carried out only on the vectors $u$ and $v$, respectively. Since the sequence of the vectors may be cyclically interchanged in the double products, Eqn (4.27a) may also be written as:

$$\text{div} \ (u \times v) = v.\nabla (u \times v) + u.\nabla (v \times v) = v.(\nabla \times u) - u.(\nabla \times v) \quad (4.27c)$$

Here the right side, in view of Eqn (4.27b) is the same as the left side of Eqn (4.27), so that Eqns (4.27c) and (4.27) become identical. This proof of Eqn (4.27) is only an abbreviated form for the direct, but much more involved calculation with rectangular coordinates $x, y, z$. 

38.
Let us now set \( \mathbf{v} = \mathbf{E} \) and \( \mathbf{u} = \mathbf{H} \) in Eqn (4.27) and introduce the abbreviation:

\[
S = \mathbf{E} \times \mathbf{H}
\]  

(4.28)

Then Equation (4.23) becomes:

\[
H \dot{\mathbf{B}} + E \dot{\mathbf{D}} + E \mathbf{J} + \text{div} S = 0
\]  

(4.29)

Eqn (4.29) is Poynting's theorem, \( S \), the Poynting vector.

4.9 Velocity of Light in Electrodynamics

By substituting Eqns (4.18), (4.19) and (4.20a) in Eqn (4.17), one can express the vector equation in differential form:

\[
\mu \frac{\partial \mathbf{H}}{\partial t} = - \text{curl} \ \mathbf{E} , \quad \left( \varepsilon \frac{\partial}{\partial t} + \sigma \right) \mathbf{E} = \text{curl} \ \mathbf{H}
\]  

(4.30)

At the same time the conditions in Eqns 4.30 (a,b,c) take on the form:

\[
\text{div} \left( \mu \mathbf{H} \right) = 0
\]  

(4.30a)

\[
\text{div} \left( \varepsilon \mathbf{E} \right) = \rho
\]  

(4.30b)

\[
\text{div} \left\{ \left( \varepsilon \frac{\partial}{\partial t} + \sigma \right) \mathbf{E} \right\} = 0
\]  

(4.30c)

It appears reasonable to eliminate \( \mathbf{H} \) from Eqn (4.30) and to obtain in this manner a single vector equation for \( \mathbf{E} \). For this purpose the operation curl is applied to the first Eqn (4.30) and the operation \( \mu \partial / \partial t \), to the second. Adding the two equations yields:

\[
\varepsilon \mu \frac{\partial^2 \mathbf{E}}{\partial t^2} + \sigma \mu \frac{\partial \mathbf{E}}{\partial t} = - \text{curl} \ \text{curl} \ \mathbf{E}
\]  

(4.31)

i.e., a linear differential equation of the second order in four coordinates of space and time.
Utilising the general transformation of:

\[
\text{curl curl } E = \text{grad div } E - \Delta E
\]

(4.32)

The Laplace operator \( \Delta \) can, by its definition as div grad, only be applied to the scalar quantities. Incidentally Eqn (4.32) may also be derived from the well known vector formula:

\[
A \times (B \times C) = B (A \cdot C) - C (A \cdot B)
\]

(4.32a)

By symbolic calculation with the nabla operator, where it takes the form:

\[
\nabla \times (\nabla \times E) = \nabla (\nabla \cdot E) - (\nabla \cdot \nabla) E
\]

(4.32b)

This is identical with Eqn (4.32), term for term.

Equation (4.31) is valid in any coordinates, curvilinear as well as cartesian. On the other hand, Eqn (4.32), according to the above, is restricted to the cartesian coordinates \( x,y,z \) and the components \( E_x, E_y, E_z \), since only these may be treated as scalar quantities. With this restriction, Equations (4.31) and (4.32) are given to be:

\[
e_{\mu} \frac{\partial^2 E}{\partial t^2} + \sigma_{\mu} \frac{\partial E}{\partial t} = \Delta E - \text{grad div } E
\]

(4.33)

This can be further simplified. \( E \) represents a solution for a medium of uniform dielectric constant and free of charge. Then Equation (4.17b), with \( E = \text{const} \) and \( \rho = 0 \), becomes:

\[
\text{div } D = \varepsilon \text{ div } E = 0
\]

(4.34)

Thus the last term on the right hand side of Eqn (4.33) vanishes and Eqn (4.33) assumes the form of the wave equation:

\[
e_{\mu} \frac{\partial^2 E}{\partial t^2} + \sigma_{\mu} \frac{\partial E}{\partial t} = \Delta E
\]

(4.35)

The same equation evidently applies, under similar restricting conditions, also for \( H \) (as well as for \( D \) and \( B \)). The first coefficient in Eqn (4.35) is the reciprocal square of a velocity. Therefore one can find from Eqns (4.21a) and (4.22b) that:
\[
\varepsilon \mu = \frac{Q^2}{M \text{joules}} \cdot \frac{joule \ S^2}{Q^2 \ M} = \left(\frac{M}{S}\right)^2
\]  
(4.36)

The velocity of propagation of electromagnetic waves, which in vacuum is identical with that of light:

\[
(\varepsilon_0 \mu_0)^{-1/2} = c = 3.10^8 \frac{m}{s}
\]  
(4.37)

The following corrections to the constants have to be given:

\[
\varepsilon_0 = 8.854187817 \times 10^{-12} \ \frac{S^2 \ C^2}{Kgm^3}
\]  
(4.37a)

\[
\mu_0 = 4\pi \times 10^{-7} = 12.566370614 \times 10^{-7} \ \frac{Kgm}{C^2}
\]  
(4.37b)

Passing from vacuum to an arbitrary electromagnetic medium, the velocity \((\varepsilon \mu)^{-1/2}\) appearing in Eqn (4.36) signifies the velocity of light (more precisely, the "phase velocity of the light") in a ponderable body characterised by \(\varepsilon\) and \(\mu\):

\[
(\varepsilon \mu)^{-1/2} = v
\]  
(4.38)

\[
\frac{c}{v} = n = \text{refractive index}
\]  
(4.39)

Equation (4.37) is evidently a supplementation of Maxwell's theory derived from experiment, which establishes a relationship between the two material constants \(\varepsilon_0, \mu_0\) of vacuum. In this section of the thesis, the constants \(\varepsilon_0, \mu_0\) of vacuum will be determined individually.

The integration of the wave equation in (4.35) is specialised for vacuum:

\[
\frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = \Delta E
\]  
(4.40)

With the auxiliary condition made use of:

\[
\text{div } E = 0
\]  
(4.40a)
The particular solutions of Eqn (4.40a) which are independent of y and z.

For a purely periodic time dependence these represent monochromatic plane waves which advance along the x-axis. In view of the assumed independence of y and z of the function, Equation (4.40a) reduces to:

\[
\frac{\partial E_x}{\partial x} = 0, \quad \frac{\partial^2 E_x}{\partial t^2} = 0
\]

\[(4.40b)\]

\(E_x\) would thus be a linear function of \(t\), which is inconsistent with the periodic dependence on \(t\). Hence \(E_x = 0\).

If the wave has a single electrical component or, in the usual terminology, is plane polarised, its direction of vibration is taken as the y-axis, so that, in addition to \(E_x = 0\), also \(E_z = 0\).

Equation (4.40) then becomes:

\[
\frac{1}{c^2} \frac{\partial^2 E_y}{\partial t^2} = \frac{\partial^2 E_y}{\partial x^2}
\]

\[(4.41)\]

The solution which is purely periodic in time is:

\[E_y = a \cos \left(kx - \omega t + \alpha\right)\]

\[(4.42)\]

According to Eqn (4.41) the wave number \(k\) introduced here and the angular frequency \(\omega\) are related by:

\[\frac{\omega}{k} = c\]

\[(4.42a)\]

Equation (4.42a) is in terms of the wave length \(\lambda\) and the period \(\tau\):

\[k = \frac{2\pi}{\lambda}, \quad \omega = \frac{2\pi}{\tau}\]

\[(4.42b)\]

Omitting the sign Re, denoting "real part of", Equation (4.42) is written as a plane wave solution:
\[ E_y = Ae^{jkx-\omega t}, \quad A = ae^{ik} \]  \hspace{1cm} (4.43)

The magnetic component of the plane wave is next investigated. It may be derived from the first vector equation (4.30), specialised for vacuum:

\[ \mu_0 \frac{\partial H}{\partial t} = -\text{curl} \ E \]  \hspace{1cm} (4.44)

Since \( E_x = E_z = 0 \) and

\[ \frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0 \]

This leads to:

\[ H_x = H_y = 0 \]  \hspace{1cm} (4.45)

This equips the following equation for \( H_z \):

\[ \mu_0 \frac{\partial H_z}{\partial t} = -\frac{\partial E_y}{\partial x} = -ikAe^{jkx-\omega t} \]  \hspace{1cm} (4.46)

For purely periodic time dependence, its integration with respect to \( t \) is carried out by dividing the right side by \(-i\omega\). Accordingly:

\[ \mu_0 H_z = \frac{k}{w} Ae^{jkx-\omega t} = \frac{1}{c} Ae^{jkx-\omega t} \]  \hspace{1cm} (4.47)

In view of Equation (4.37):

\[ H_z = \sqrt{\frac{\epsilon_0}{\mu_0}} Ae^{jkx-\omega t} \]  \hspace{1cm} (4.48)

The dimension of the coefficient \((\epsilon_0 / \mu_0)^{1/2}\) is reciprocal of a resistance, i.e. \( \Omega^{-1} \). From Equations (4.21a), (4.22b) and (4.20c), the following is derived:

\[ \frac{\epsilon}{\mu} = \frac{Q^2}{M \text{ joule}} \left(\frac{Q^2\text{ joule}}{s^2}\right) = \left(\frac{Q^2}{\text{ joule s}}\right)^2 = \frac{1}{\Omega^2} \]  \hspace{1cm} (4.49)

\((\mu_0 / \epsilon_0)^{1/2}\) is designated as "wave resistance of vacuum".

43.
CHAPTER 5
SIMULATION RESULTS OF SELF-FOCUSING

5.1 Introduction

With the availability of high power laser beams, a large number of interesting nonlinear optical phenomena have been studied, both theoretically and experimentally during the last decade. The phenomena of self-focusing of intense laser beams occupies a unique place in nonlinear optics because it considerably influences all other nonlinear phenomena. The self-focusing effect arises primarily due to the dependence of the refractive index of a material on the intensity of the propagating electromagnetic wave.

With the advent of the laser, the various types of forces generated in materials irradiated by electromagnetic radiation hence will be analysed. One must distinguish from purely thermokinetic forces, due to gasdynamic pressures of the generated plasma. The force of electrodynamic nature are expressed by the Poynting vector and gradient of quadratic expressions, and the electric or magnetic laser field strength given by $E$ and $H$. This would include dielectric properties which later are dependent on the gasdynamic properties of the generated plasma or nonlinearity on the actual laser fields. These electrodynamic and essential nonlinear forces will be investigated in this chapter of the thesis to produce nonlinear force (ponderomotive) self-focusing.

Another type of self-focusing happens if the relativistic effects are considered. Relativistic self-focusing is one of the self-focusing processes of laser beams in plasmas, resulting from the relativistic mass and energy dependence of the refractive index at high laser intensities. It is obvious that such phenomena would be apparent near the relativistic threshold, where the electron oscillation energy is equal to $m_e c^2$. However this mechanism becomes significant at values exceeding $3.66 \times 10^{18}$ W/cm$^2$ for a Neodymium glass laser. This chapter of the thesis implements a numerical program in C++, which incorporates both ponderomotive force in self-focusing mechanism and relativistic effects, in order to evaluate various variations of refractive index, power threshold, and self-focusing of the beam as a function of the laser intensity.
5.2 Self-focusing Effect

In its most general definition, self-focusing is a decrease in divergence (or equivalently, an increase in convergence) of powerful radiation, owing to various nonlinear effects that are caused by the beam itself. The change in divergence is precisely what leads to change in cross-section of the beam, formation of filaments, caustics, condensations, foci, etc. That is the different special manifestations of self-focusing are a consequence of change in divergence of radiation.

The spatial distribution of self-focusing action makes it analogous to the appearance of a dielectric waveguide, created by a nonlinear change in the dielectric constant. Here we should take the term waveguide in the usually adopted broad sense. That is, it can have an arbitrary variable cross-section and dielectric-constant distribution, and have any length. The justification of the waveguide description of self-focusing is the extended spatial distribution of the focusing action, while the beam cross-section is comparable with the cross-section of the profile of altered refractive index that it creates.

Two factors hinder concentration and localisation of the beam energy. The first is a change in the nonlinear increment of the refractive index arising from the change in time of the field of the beam or the time development of nonlinear processes, (inertia or onset, relaxation, or secondary processes) that alter the focusing action or cause movement of the focal points. The second factor is the so-called caustic of the focusing action, or focusing of different parts of the beam at different regions of the axis. This caustic can be very strong because the profile of the intensity distribution that determines the nonlinear refraction can be so unstable as to cause aberration or onset of subdivision of the beam.

There is an essential characteristic of self-focusing which is the threshold power, i.e. the power at which the nonlinear refraction is comparable with the diffractional divergence, and can compensate it. Another essential characteristic of self-focusing is the so-called "kelly length", or the distance at which the intensity of a beam of small initial divergence varies substantially. These two characteristics of self-focusing are manifested in many of its varieties, and in particular, they govern the processes of subdivision of a beam. A beam whose power is considerably above the threshold will probably subdivide into several beams of threshold power.
5.3 **Nonlinear Force (Ponderomotive) Self-Focusing**

In order to derive the threshold for self-focusing of a laser beam in a plasma, three physical mechanisms have to be combined. Assuming that the laser beam has a Gaussian intensity profile along the y-axis while propagating in x-direction, the generated nonlinear force \( f_{\text{nl}} \) in the y-direction has to be compensated by the thermokinetic force \( f_{\text{th}} \).

The second physical mechanism is the total reflection of the laser beam components starting under an angle \( \alpha_o \), from the centre of the beam and being bent into a parallel direction to the axis due to the density gradient of the plasma.

The third condition is the diffraction requirement that the main part (e.g. as defined by the first diffraction minimum) of the beam has to have an angle of propagation \( \alpha_o \) which is less than the angle of total reflection. These three conditions are sufficient to calculate the threshold. This is achieved by the derivation from the two fluid equations of motion.

The two fluid equations of motion can be written in terms of the euler equations for the electron ion fluids in a plasma. The indices \( e \) and \( i \) denote the electron and ion parameters respectively.

For ions:

\[
\begin{align*}
\frac{\partial}{\partial t} v_i + v_i \nabla v_i &= -\nabla n_i \frac{m_e v_i}{c^2} - \nabla T_e + \frac{Z_i n_e}{c} E \times H - \frac{m_i n_i m_{n_i}}{m_i n_i + m_{n_i}} (v_i - v_e) + K_i \\
\end{align*}
\]

\[5.1a\]

For electrons:

\[
\begin{align*}
\frac{\partial}{\partial t} v_e + v_e \nabla v_e &= -\nabla n_e \frac{m_e v_e}{c^2} + \frac{Z_i n_i}{c} E \times H - \frac{m_i n_i m_{n_i}}{m_i n_i + m_{n_i}} (v_i - v_e) + K_e \\
\end{align*}
\]

\[5.1b\]

The force densities on the right hand side of Equations (5.1a) and (5.1b) arise from the electric field \( E \), the lorentz force \( V \times H \) and the pressure \( p = n_i K_T \). The net pressure of the plasma:

\[
p = n_i K_T + n_e K_T_e \approx n_i (1 + Z) K_T
\]

\[5.2\]
Where:

\[ T_i = T_e = T \]  \hspace{1cm} (5.2a)

\[ n_e = Zn_i \]  \hspace{1cm} (5.2b)

The net plasma velocity:

\[ \mathbf{v} = \frac{m_i n_i \mathbf{v}_i + mn_e \mathbf{v}_e}{m_i n_i + mn_e} \]  \hspace{1cm} (5.3)

Equation (5.3) can be approximated by:

\[ \mathbf{v} = \frac{m_i n_i \mathbf{v}_i + Zm \mathbf{v}_e}{m_i + Zm} \]  \hspace{1cm} (5.4)

The velocity difference permits a similar sufficient approximation:

\[ \mathbf{v}_e - \mathbf{v}_i = \frac{n_e \mathbf{v}_e - Zn_i \mathbf{v}_i}{n_e} = \frac{j}{\mathbf{e} \mathbf{n}_e} \]  \hspace{1cm} (5.5)

Where the definition of the electric current density \( j \) is:

\[ j = e (Zn_i \mathbf{v}_i - n_e \mathbf{v}_e) \]  \hspace{1cm} (5.6)

Addition of the Equations (5.1a) and (5.1b) and substitution of Equations (5.4) and (5.6) with rearranging terms leads to an equation of motion given by a force density \( f \):

\[ f = m_i n_i \frac{d\mathbf{v}}{dt} = -\nabla p + \frac{1}{c} j \times H + \frac{1}{4\pi} \left( \frac{\omega_p^2}{\omega} \right)^2 E \cdot \nabla E \]  \hspace{1cm} (5.7)

Where \( p \) represents the total gas dynamic pressure in the plasma. The most general equation of motion satisfying all cases of plane optical wave interaction with plasma, (Hora 182, 1969) is:

\[ f = -\nabla p + \frac{1}{c} j \times H + \frac{1}{4\pi} E \nabla E - \frac{1}{4\pi} \frac{\omega_p^2}{\omega^2 + V^2} \left( 1 + \frac{V}{\omega} \right) E \nabla E - \frac{1}{4\pi} \frac{\omega_p^2}{\omega^2 + V^2} \left( 1 + \frac{V}{\omega} \right) EE \nabla E - \frac{1}{4\pi} \frac{\omega_p^2}{\omega^2 + V^2} \left( 1 + \frac{V}{\omega} \right) 
\]  \hspace{1cm} (5.8)
In Equation (5.1a) nonlinear terms follow the thermokinetic force:

\[ f_m = - \nabla p \quad (5.9) \]

These remaining terms in Eqns (5.7) and (5.8) can be seen for the case of electromagnetic waves, to introduce \( j, E \) and \( H \) which are fast oscillating at least with frequency \( \omega \) or as a result of a product, with higher harmonics.

Forces are called ponderomotive forces if they were generated by electric or magnetic fields. Apart from the purely field generated terms, mixed terms exist. These non ponderomotive terms are characterised by dissipative effects and are proportional to the collision frequency. Due to these mixed terms the terminology "nonlinear force" \( f_{NL} \) will be preferred.

\[ f_{NL} = f - f_{th} \quad (5.10) \]

Rewriting the Equation (5.10) from Equation (5.8):

\[ f_{NL} = A + B \quad (5.11) \]

Where:

\[ A = \frac{1}{c} j \times H \quad (5.12) \]

\[ B = \frac{1}{4\pi} E \nabla \cdot E + \frac{1}{4\pi} \nabla \cdot (n^2 - 1) EE \quad (5.13) \]

Using the identity from Maxwell's equation:

\[ \nabla \times H = \frac{4\pi}{c} j + \frac{1}{c} \frac{\partial}{\partial t} E \quad (5.14) \]

From Equation (5.14) \( j \) is eliminated:

\[ j = \frac{c}{4\pi} (\nabla \times H) - \frac{1}{4\pi} \frac{\partial}{\partial t} E \quad (5.15) \]
Equation (5.15) is inserted in Eqn (5.12):

\[
A = \frac{1}{4\pi} (\nabla \times H) \times H - \frac{1}{4\pi c} \left( \frac{\partial}{\partial t} E \right) \times H = -\frac{1}{4\pi} H x (\nabla \times H) - \frac{1}{4\pi c} \left( \frac{\partial}{\partial t} E \right) \times H \tag{5.16}
\]

Using the vector identity:

\[
H \times (\nabla \times H) = \frac{1}{2} \nabla H^2 - H \nabla H \tag{5.17}
\]

The term A from Eqn (5.11) can be written as:

\[
A = -\frac{1}{4\pi} \left[ \frac{1}{2} \nabla H^2 - H \nabla H \right] - \frac{1}{4\pi c} \left( \frac{\partial}{\partial t} E \right) \times H \tag{5.18}
\]

Turning to the term B, Eqn (5.13):

\[
B = \frac{1}{4\pi} E \nabla \cdot E + \frac{1}{4\pi} \nabla . n^2 E E - \frac{1}{4\pi} E \nabla \cdot E - \frac{1}{4\pi} E \nabla E \tag{5.19}
\]

Equation (5.19) is changed to:

\[
B = \frac{1}{4\pi} n^2 E \nabla E + \frac{1}{4\pi} E \nabla . (n^2 E) - \frac{1}{4\pi} E \nabla E \tag{5.20}
\]

Adding zero by the following additional two last terms and combining the first terms in Eqn (5.20):

\[
B = \frac{1}{4\pi} \left[ \nabla . n^2 E E - E \nabla E + \frac{1}{2} \nabla E^2 - \frac{1}{2} \nabla E^2 \right] \tag{5.21}
\]

The second and the third term are combined to:

\[
\frac{1}{4\pi} E \times (\nabla \times E) = \frac{1}{8\pi} \nabla E^2 - \frac{1}{4\pi} E \nabla E \tag{5.22}
\]

Using the Maxwell Eqn (5.24) in Eqn (5.22) leads to:

\[
B = \frac{1}{4\pi} \left[ \nabla . n^2 E E - \frac{1}{2} \nabla E^2 \right] - \frac{1}{4\pi c} E \times \frac{\partial}{\partial t} H \tag{5.23}
\]
Writing the term $B$ from Eqn (5.23) and the term $A$ from Eqn (5.18), this leads Eqn (5.16) to be given as:

$$f_{NL} = \frac{1}{4\pi} \nabla \left[ EE + HH - \frac{1}{2} (E^2 + H^2) + (n^2 - 1) EE \right] - \frac{1}{4\pi c} \frac{\partial}{\partial t} E \times H$$  
(5.24)

Where use was made of:

$$H \cdot \nabla H = \nabla HH - H \nabla H$$  
(5.25)

With a negligible last term because:

$$\nabla H = 0$$  
(5.26)

Equation (5.23) can be written in the form of:

$$f_{NL} = \nabla \left[ T + \frac{n^2 - 1}{4\pi} EE \right] - \frac{1}{4\pi c} \frac{\partial}{\partial t} E \times H$$  
(5.27)

Where $T$ is the Maxwellian stress tensor:

$$T = \frac{1}{4\pi} \left[ EE + HH - \frac{1}{2} (E^2 + H^2) \right]$$  
(5.28)

The components $T$ are given by the scalar components of $E$ and $H$. The most general equation of motion of a plasma consists of a thermokinetic term, Eqn (5.10) and the following nonlinear terms:

$$f_{NL} = A + B$$  
(5.29)

$$f_{NL} = \frac{1}{c} j \times H + \frac{1}{4\pi} E \nabla E + \frac{1}{4\pi} \nabla (n^2 - 1) EE$$  
(5.30)
Which is identical with the formulation using the Maxwellian stress tensor $T$, Eqn (5.28):

$$ f_{NL} = \nabla \left( \frac{EE + HH - \frac{1}{2} (E^2 + H^2) 1 + (n^2-1)EE}{4\pi} \right) - \frac{1}{4\pi c} \frac{\partial}{\partial t} E \times H $$  \hspace{1cm} (5.31)

For perpendicular incidence (x-direction) of plane waves, the forces in the plasma are especially from Eqn (5.30):

$$ f_{NL} = \frac{1}{c} j \times H $$  \hspace{1cm} (5.32)

From Equation (5.31):

$$ f_{NL} = - \frac{\partial}{\partial x} \frac{E^2 + H^2}{8\pi} $$  \hspace{1cm} (5.33)

The balance is given by the form:

$$ \frac{E^2 + H^2}{8\pi} = n_e KT_e \left(1 + \frac{1}{\epsilon} \right) $$  \hspace{1cm} (5.34)

$$ F_{th} = f_{NL} : \quad \nabla \left( T \cdot \frac{n^2 - 1}{4} EE \right) = \nabla n_e KT(1+\epsilon) $$  \hspace{1cm} (5.35)

Figure (6) demonstrates the force on the plasma is $f_{NL}$. This force causes a radially directed ponderomotive force in a plasma, in which self-focusing is observed.

![Diagram](image)

A laser light of finite diameter that causes a radially directed ponderomotive force in a plasma.

**Figure 6:** Self-focusing of a laser beam is caused by the ponderomotive force.
This force moves plasma out of the beam, so that $\omega_p$ is lower and the dielectric constant $\epsilon$ is higher inside the beam than outside. The plasma then acts as a convex lens, focusing the beam to a smaller diameter.

5.4 Solitons

The properties of the nonlinear force lie in the gradient of the electromagnetic energy density $\nabla (E^2 + H^2)$. Figure (7) demonstrates the case of a symmetric electron density profile $n_e(x)$ of a collisionless plasma. The corresponding penetrating laser beam satisfies the WKB condition, where the density maximum is less than the cut off density $n_{oc}$.

The time averaged nonlinear force for linearly polarised perpendicular light incident is given by:

\[
\frac{1}{8\pi} \nabla (E^2 + H^2)
\]  \hspace{1cm} (5.36)

Using a collisionless WKB approximation where:

\[
E^2 = \frac{E^2_v}{2 |n|} \exp(\pm kx)
\]  \hspace{1cm} (5.37a)

\[
H^2 = \frac{E^2_v |n|}{2} \exp(\pm kx)
\]  \hspace{1cm} (5.37b)

The electromagnetic energy density can be described for the WKB conditions, Eqns (5.37a) and (5.37b) then:

\[
E^2 + H^2 = E^2_v \left( \frac{1}{n} + n \right)
\]  \hspace{1cm} (5.38)

Where $E$ and $H$ describe the electric and magnetic fields, $E_v$ is the vacuum value of the electric field and $n$ the refractive index.

If $n$ is small the electromagnetic field swells as $1/n$. This is also seen via the conservation of energy flux, where if the group velocity is $V_g$ and $\phi$ the illuminated area is related to $E$:
\[ E^2 \nabla \phi = \text{constant} \quad (5.39) \]

With smaller values of the refractive index, the group velocity implying a rise in the amplitude of the electric field is denoted by the swelling.

The essential properties of the nonlinear force can be seen in the following schematic examples. Figure (7) demonstrates a collisionless plasma with a purely propagating laser wave. The nonlinear force \( f_{\text{nl}} \) is caused by the variations, due to the gradient of \( E^2 + H^2 \). The nonlinear force therefore forces the plasma from regions of high densities to low densities. The term soliton describes the solitary electromagnetic wave in Figure (7) (Hora 204, 1970).

\[ \frac{E^2}{\nu} + \frac{H^2}{Z} = E^2 (\frac{1}{\eta} + \eta)/2 \]

\[ |f_{\text{nl}}| = \frac{1}{8\pi} \nabla (E^2 + H^2) \rightarrow x \]

The scheme of the electron density \( n_e \) of a collisionless plasma with a penetrating laser beam (hv) satisfying the WKB condition (\( n_e \) has to be at least a little less than the cut-off density \( n_{\text{ec}} \)). The variation of \( E^2 + H^2 \) causes gradients of this expression to result in nonlinear forces \( f_{\text{nl}} \).

**Figure 7:** The electron density profile of a collisionless plasma with maximum density slightly less than \( n_{\text{ec}} \). The penetrating laser beam satisfying the WKB condition and exists as a solitary electromagnetic wave.

Similarly for a monotonically increasing electron density profile, Figure (8) (Hora 205, 1970), the corresponding approximation for the electromagnetic energy density is shown. Figure (8) demonstrates a plasma with collisions where the WKB approximation is possible even for densities exceeding the cut off density \( n_{\text{ec}} \). The value of the electromagnetic energy density, after approaching \( n_{\text{ec}} \), decreases exponentially due to collisional absorption in the plasma.
Density $n_e(x)$ of a plasma with collisions exceeding the cut-off density $n_{co}$ with an incident purely penetrating laser beam ($h\nu$), which can also be constructed by the WKB approximation. The variation of the quantity $E^2y + H^2z$ results in the nonlinear forces $F_{nl}$.

**Figure 8:** A density profile increasing above the cut off density shows the formation of a soliton which is critically damped at the cut off intensity.

Assuming that the totally penetrating laser beam is collisionally damped at the cut off density. The electromagnetic energy density, where the swelling of the intensity, electric field and wavelength are effectively given by:

$$l = \frac{l_{vac}}{|n|}; \quad E = \frac{E_{vac}}{|n|^{1/2}}; \quad \lambda = \frac{\lambda_{vac}}{|n|} \quad (5.40)$$

The above WKB approximation describes a single wave maximum known as a soliton.

**5.5 Power Threshold**

The Gaussian density profile including the refractive index $n$ is described by the formula:

$$E_y^2 = \frac{E_y^2}{2|n|} \exp \left( -\frac{y^2}{y_0^2} \right); \quad H_z^2 = |n|^2 E_y^2 \quad (5.41)$$

Hence $y_0$ can be interpreted as the radius of the laser beam. This is only an approximation of the exact Maxwellian formulation. In analogy to the suggestions of Askaryan and Schluter, the nonlinear force in the direction of $y$ is given by:
\[ f_{NL} = \frac{1}{8\pi} \nabla (E_y^2 + H_z^2) \]  \hspace{1cm} (5.42)

It is important to note that this formulation is approximately valid only for conditions of strong swelling, that is, for refractive index differing from unity, which is the case for the plasma exceeding the cut off density in the region around the lower beam.

Using Equation (5.41) in Eqn (5.42), the maximum nonlinear force in the y-direction is:

\[ f_{NL} = i \frac{\sqrt{2} Y}{16\pi n} \frac{E_y^2 Y}{y_0} \exp \left( -\frac{1}{2} \right) \]  \hspace{1cm} (5.43)

If this has to be compensated by a thermokinetic force under the assumption of a spatially constant plasma temperature, the formulation is found by:

\[ f_\theta = -i \frac{\sqrt{2}}{16\pi n} \frac{\partial n_e}{\partial y} \]  \hspace{1cm} (5.44)

Equating this force and the nonlinear force of Eqn (5.43) provides an expression for the electron density gradient of the plasma at the laser beam:

\[ \frac{\partial n_e}{\partial y} = \frac{\sqrt{2}}{16\pi K T th} \frac{\exp(1)}{1 + |n|^2} \frac{E_y^2}{y_0 |n| \left( 1 + \frac{1}{z_l} \right)} \]  \hspace{1cm} (5.45)

The second physical condition of total reflection is given by the refractive index in the centre of the beam, \( n \) and its value at \( y_o \), for which Eqn (5.45) is formulated by:

\[ \sin \left( \frac{\pi}{2} - \alpha_o \right) = \frac{|n|}{|n|_{\text{yo}}} \]  \hspace{1cm} (5.46)

Using the following Taylor expansion, for the case of a negligibly small collision frequency:

\[ n_{yo} = n + \frac{\partial n}{\partial y} y_0 ; \quad n^2 = 1 - \frac{\omega_{pe}^2}{\omega^2} \]  \hspace{1cm} (5.47)

55.
Given from Equation (5.47):

$$\sin \alpha_0 = \left( \frac{2}{n} \frac{\partial n}{\partial n_e} \frac{\partial n_e}{\partial y} y_0 \right)^{1/2}$$  \hspace{1cm} (5.48)

If as a third physical condition a particular wave with an angle \( \alpha_0 \), for the first minimum of diffraction has to be reflected totally, the condition is found by:

$$\sin \alpha = \frac{\pi c}{2 \omega y_0} \leq \sin \alpha_0$$  \hspace{1cm} (5.49)

Expressing the right hand side by Eqn (5.48) and using Eqn (5.45) and the relation for the electrical laser field amplitude:

$$E_{\text{vo}} = \frac{c_1 P^{1/2}}{y_0}$$  \hspace{1cm} (5.49a)

(where \( P \) is the averaged laser power and \( c_1 \) is a constant of \( 1.63 \times 10^5 \text{cgs} \)). One arrives at the following:

$$P \geq \frac{(\pi c)^2 n^3 m_e}{\sigma^2 2 \left[ \exp(1) \right]^{1/2} c_1^2 (1+n^2)}$$  \hspace{1cm} (5.50)

This is evaluated in watts. It is remarkable that this threshold for self-focusing is a laser power and not an intensity.

From inspection of these results at different orders of laser intensities, there exists different power thresholds. This is notable due to the presence of the density ratio parameter. The use of the condition of equilibrium between the ponderomotive and hydrostatic forces is necessary to obtain the power threshold to produce self-focusing. It must be presumed that a plasma has been created, and secondarily self-focusing occurs.
5.6 Optical Properties of the Plasma

The dispersion relation of electromagnetic waves in a plasma results in the square of the complex refractive index (Hora 57, 1996):

\[ n^2 = 1 - \left( \frac{\omega_p^2}{\omega^2 + \nu^2} \right) \left( 1 + \frac{\nu}{\omega} \right) \tag{5.51} \]

This has an absolute value of:

\[ |n|^2 = \left( 1 - \frac{\omega_p^2}{\omega^2 + \nu^2} \right)^2 + \left( \frac{\nu}{\omega} \right)^2 \left( \frac{\omega_p^2}{\omega^2 + \nu^2} \right)^2 \tag{5.52} \]

Where \( \omega_p \) is the plasma frequency, \( \nu \) the electron-ion collision frequency, and \( \omega \) is the monochromatic laser frequency. For a collisionless plasma Eqn (5.51) reduces to:

\[ n^2 = 1 - \left( \frac{\omega_p}{\omega} \right)^2 \tag{5.53} \]

Where the plasma frequency is given by the classical Langmuir value \((4\pi e^2 n_e/m_o)^{1/2}\), but with the electron mass possessing the relativistic velocity dependency below:

\[ m_e = m_o \left[ 1 - \left( \frac{v_o}{c} \right)^2 \right]^{1/2} \tag{5.54} \]

With \( m_o \) the electron rest mass, \( v_o \) its oscillation velocity, and \( c \) the speed of light. Furthermore, the \( v_o/c \) velocity ratio is written in terms of the time average laser induced electric field \( E \) within the plasma by:

\[ \left[ 1 - \left( \frac{v_o}{c} \right)^2 \right]^{1/2} = \left[ 1 + \frac{e^2 E^2}{m_o^2 \omega^2 c^2} \right]^{-1/2} \tag{5.55} \]

It is feasible that for a proper low intensity specification of long time ponderomotive effects, a kinetic theory obtained temperature dependency has to be included in the right hand side of Eqn (5.50). Insertion of Equations (5.54) and (5.55) into (5.53) yields:

\[ n^2 = 1 - \left( \frac{\omega_p}{\omega} \right)^2 \left[ 1 + \frac{e^2 E^2}{m_o^2 \omega^2 c^2} \right]^{-1/2} \tag{5.56} \]
With \( \omega_p = (4\pi e^2 n_e/m_0)^{1/2} \). It is interesting to note that for very high irradiative electric fields, such that \( e^2E^2/m_0^2\omega^2c^2 \gg 1 \), Eqn (5.61) becomes:

\[
n^2 = 1 - \left( \frac{\omega_p}{\omega} \right)^2 \left( \frac{m_0\omega c}{eE} \right) \tag{5.57}
\]

The above equation retains only the unity in the binomial expansion. Setting \( E = E_o/\sqrt{2} \) where \( E_o \) is the peak amplitude of the electric field, in which results:

\[
n^2 = 1 - \sqrt{2} \left( \frac{\omega_p}{\omega} \right)^2 \left( \frac{m_0\omega c}{eE_o} \right) \tag{5.58}
\]

It must be added that the high electron number densities created near the focal region may require that a more elaborate quantum mechanical dielectric and collisional model be employed. Nevertheless, the approximations made in the ray focusing solution are probably of a larger magnitude than the neglect of quantum effects. The electron-ion collision frequency \( \nu \) is generalised from:

\[
\nu = \frac{\pi^{3/2} Z_i n_e e^4 \ln \Lambda}{2 m_e^{1/2} (2KT)^{3/2}} \tag{5.59}
\]

to

\[
\nu = \frac{\pi^{3/2} Z_i n_e e^4 \ln \Delta r}{2 m_o^{1/2} \left[ 1 + \frac{e^2E^2}{m_o^2\omega^2c^2} \right]^{1/4} \left[ 2 (KT + \epsilon_{kin})^{3/2} \right]} \tag{5.60}
\]

Where Equations (5.54) and (5.55) have been employed. \( Z_i \) is the average ion charge number over the plasma mixture, \( KT \) is the thermal energy, \( \Delta r \) is the relativistic Coulomb factor, and \( \epsilon_{kin} \), the electron kinetic energy, is given by:

\[
\epsilon_{kin} = m_o c^2 \left\{ \left( 1 + \frac{e^2E^2}{m_o^2\omega^2c^2} \right)^{1/2} - 1 \right\} \tag{5.61}
\]
Where Equation (5.55) has been employed in the relativistic particle energy expression. The electron-electron collision correction factor $\gamma E$, which may be included in the denominator of Eqn (5.59), is really only applicable for electron kinetic energies less than the thermal energy and $\omega \ll \nu$, conditions opposed to the high frequency processes. Furthermore, even if it had applicability in this problem, the production of highly ionised plasmas yield $\gamma E$ values very close to one.

The non-relativistic expression of the Coulomb factor, is given by:

$$\Lambda = \frac{3(KT)^{3/2}}{2Z_i e^2(\pi n_e)^{1/2}} \quad (5.62)$$

Which is equal to the ratio of the Debye length $\lambda_d$, divided by the binary electron-ion collision impact parameter $b_o$, where:

$$\lambda_d = \frac{(KT/4\pi n_e e^2)^{1/2}}{\sqrt{m_e/e}} \quad (5.63)$$

and

$$b_o = Z_i e^2/m_e Ve^2 \quad (5.64)$$

The $\lambda_d$ expression assumes that charged particle screening is entirely caused by the electrons. Their relatively high thermal velocities induce more effectiveness than the comparatively sluggish ions. Strictly speaking, the $m_e$ should be motion-dependent, however, since $\Lambda$ is an average over all electron velocities, the relation $m_e Ve^2 = 3KT$ is employed. $\Lambda$, is then assumed to be generalised to:

$$\Lambda_f = \frac{3(KT + e \text{kin})^{3/2}}{2Z_i e^2(\pi n_e)^{1/2}} \quad (5.65)$$

With $e \text{kin}$ given by Equation (5.61).

It is useful to write the $e^2E^2/m_e^2\omega^2c^2$ term as $l/l_{rel}$ where $l$ is the beam intensity within the plasma, CE$^2$/8II, and $l_{rel}$ is a characteristic relativistic intensity defined by:

$$l_{rel} \equiv \left(\frac{c}{8\Pi I}\right)(m_e c^2 \omega^2/e^2) \quad (5.66)$$

$$= 3.66 \times 10^{18} \text{ W/cm}^2 \text{ for Nd glass lasers}$$

59.
With this substitution, Equations (5.60) and (5.65) become:

\[
\left( \frac{\nu}{\omega} \right)^2 = \frac{C^* n_{\text{ec}} N^2 (\ln \Delta)^2}{\left\{ F \left[ \frac{KT}{\theta^2} \frac{\Gamma - 1}{r_o} \right] \right\}^3} \Gamma
\]

(5.67)

and

\[
D^* = \frac{\left( \frac{KT}{\theta^2} \frac{\Gamma - 1}{r_o} \right)^{3/2}}{(N n_{\text{ec}})^{1/2}}
\]

(5.68)

With \( C^* = \Pi^2 \chi^2 / 128 \), \( D^* = 3/(2\sqrt{2}) \), \( r_o \) (classical electron radius) = 2.81785x10^{-10} cm, \( n_{\text{ec}} \) is the non-relativistic cut-off density \( \omega^2 m_e / 4\Pi \epsilon^2 \) (for Nd glass 10^{21} cm^{-3}), \( N=n_e/n_{\text{ec}} \) is the characteristic density ratio parameter, \( \Gamma \equiv (1 + l/l_{\text{el}})^{1/3} \), and \( F \) is a parametric instability quantity. \( F=1 \) denotes the stable microscopic situation, while values less than unity represent an effective collectively induced absorption that results in a certain level of instability.

As indicated previously, the square of the ratio of the relativistic plasma frequency \( \omega_p \) to laser frequency \( \omega \) is given by:

\[
\left( \frac{\omega_p}{\omega} \right)^2 = \left( \frac{\omega_p}{\omega} \right)^2 \left( 1 + \frac{\epsilon^2 E^2}{m_e^2 \omega^2 c^2} \right)^{-1/2} = N / \Gamma
\]

(5.69)

In terms of our newly defined variables, insertion of this relation into Equation (5.52) gives:

\[
|n| = \left\{ \left[ 1 - \frac{N/\Gamma}{1 + (\nu/\omega)^2} \right] + \left( \frac{\nu}{\omega} \right)^2 \left[ \frac{N/\Gamma}{1 + (\nu/\omega)^2} \right]^2 \right\}^{1/4}
\]

(5.70)

With \((\nu/\omega)^2\) given by Equation (5.67). The absolute value of the refractive index is therefore dependent on the local light intensity, electron number density and temperature, ionisation number and of course, laser frequency.
5.7 Relativistic Self-Focusing

The evaluation of the ratio of self-focusing length $l_{sf}$ to the distance between the half intensity maximum points on the initial Gaussian profile vacuum beam is denoted by $d_o$. This is performed by geometric distortion of an initial plane wave, with the assumption that a ray extends from the target surface to focus a circular arc (Hora 882, 1975).

![Diagram of relativistic self-focusing](image)

Evaluation of the relativistic self-focusing length $l_{sf}$ from the initial beam diameter $d_o$ and from the effective wavelengths. The relativistic effects cause a shorter wavelength at the maximum intensity $l_{max}$ than at the half maximum intensity.

**Figure 9:** Evaluation of the self-focusing length $l_{sf}$ for a laser beam with Gaussian irradiance profile and diameter $d_o$ (half irradiance maximum).

The figure above shows that as the ray approaches the axis to within a distance of the order of the local wavelength. The plasma as a result flares out to form a filamentary structure. The self-focusing is described for a laser beam of a Gaussian irradiance profile with a maximum irradiance $I_{max}$ and a half maximum diameter $d_o$ penetrating a homogeneous plasma that has an electron density $n_e$. If one assumes a plane wave front moves in steps proportional to the effective wavelength:

61.
\[
\lambda = \frac{\lambda_v}{\left| n\left(\frac{l}{n}\right)\right|}
\]  \hspace{1cm} (5.71)

where \(\lambda_v\) is the vacuum wavelength. In the construction \(n\) is an arbitrary radial distance which disappears in the final result. By proportional triangles from Figure (9):

\[
\frac{\lambda(I_{max})}{\varphi_o} = \frac{\lambda}{\left(\frac{l_{max}}{2}\right)} \left(\frac{\varphi_o + \frac{d_o}{2}}{\varphi_o + \frac{d_o}{2}}\right)
\]  \hspace{1cm} (5.72)

Due to the fact that \(|n|\) increases with intensity, the wave front propagates faster at a less intensity radial point than at its maximum intensity \(l_m\) and therefore self-focuses (ideally) without any time delay. Employment of Eqn (5.72) assumes that a relation holding immediately within the plasma surface can be applied throughout the self-focusing length.

From Equations (5.71) and (5.72) the following is given:

\[
\frac{\lambda_v}{\varphi_o |n(lm)|} = \frac{\lambda_v}{\left(\frac{l_{max}}{2}\right) n\left(\frac{l_{max}}{2}\right)}
\]  \hspace{1cm} (5.73)

From the Pythagorean theorem (Figure 9),

\[
\left(\varphi_o + \frac{d_o}{2}\right)^2 = l_{sf}^2 + \varphi_o^2
\]  \hspace{1cm} (5.74)

or

\[
\frac{l_{sf}}{d_o} = \frac{1}{2} \left[ \frac{4}{d_o} \left(\varphi_o + \frac{d_o}{2}\right)^{-1} \right]^{1/2}
\]  \hspace{1cm} (5.75)

Two relations may be derived from Equation (5.73)

\[
4\varphi_o \frac{d_o}{l_{sf}} = \frac{2|n\left(\frac{l}{n}\right)|}{|n(lm)| - n\left(\frac{l}{n}\right)}
\]  \hspace{1cm} (5.76a)
and

$$\phi_o + \frac{d_o}{2} = \phi_o \left| \frac{n(lm)}{n(\frac{lm}{2})} \right|$$

(5.76b)

Insertion of Equations (5.76a) and (5.76b) into Eqn (5.75) gives:

$$\frac{l_{sf}}{d_o} = \frac{1}{2} \left[ \left| \frac{n(lm)}{n(\frac{lm}{2})} \right| + \left| \frac{n(\frac{lm}{2})}{n(lm)} \right| \right]^{1/2}$$

(5.77)

This is independent of $$\phi_o$$. In practice, the beam profile will deviate from the circular configuration owing to stronger narrowing of the cross-section of the beam, by a feedback of the irradiance increase at the centre. This faster focusing will be damped by diffraction so that $$I_{sf}$$ can be assumed as a reasonable approximation of the self-focusing length. It is to be noted from Eqn (5.77) that the solution becomes singular and then imaginary for $$|n(lm/2)| > |n(lm)|$$. There is a threshold value of beam intensity ($$I_s$$) for each combination of parameters below, which physically correct values of $$I_{sf}$$ may not be obtained. Equating the refractive indices at the intensity maximum and the half intensity maximum gives (dropping the $$m = \text{max subscript}$$) the following:

$$\frac{2}{N} \left[ \left( 1 + \frac{I_s}{2I_{rel}} \right)^{-1/2} - \left( 1 + \frac{I_s}{I_{rel}} \right)^{-1/2} \right] + \frac{\left( \frac{v}{\omega} \right)^2 + 1}{1 + \frac{I_s}{I_{rel}}} - \frac{\left( \frac{v}{\omega} \right)^2 + 1}{1 + \frac{I_s}{2I_{rel}}} = G(I_s)$$

(5.78)

This is zero at a threshold intensity root. The term $$(v/\omega)^2$$ has been neglected compared to unity in Eqn (5.70) in the specification of this relation. The proper numerical procedure is to first determine the threshold physical beam intensity $$I_s$$ and then to gradually work upwards in intensity to determine solutions to Eqn (5.77).

A numerical program in C++ is implemented to allow us to consider and evaluate the variations of the refractive index, power threshold and self-focusing of the beam as a function of laser intensity. This C++ program is displayed in Appendix A.
Self-focusing lengths predicted by the computer program, simulating self-focusing of laser beams in plasma, incorporates both ponderomotive force in self-focusing mechanism and relativistic effects.

The advantage of this program is to implement any parameter at any one time to calculate the result over a range of various values. The following program parameters in Table 1 were used for simulating the program results which lead to self-focusing.

<table>
<thead>
<tr>
<th>TABLE 1 - PROGRAM PARAMETERS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vacuum Velocity of Light</td>
</tr>
<tr>
<td>Mass of Electron</td>
</tr>
<tr>
<td>Electron Charge</td>
</tr>
<tr>
<td>Laser Power Threshold</td>
</tr>
<tr>
<td>Relativistic Threshold</td>
</tr>
<tr>
<td>Pi</td>
</tr>
</tbody>
</table>

The numerical program results in c++ are presented in Tables (2.1), (2.2), (2.3) and (2.4) for a range of various values of laser intensity and density. These results investigate various illustrations given for the refractive index, power threshold and self-focusing lengths as a function of laser intensity and density. These illustrations are given in Figures (10), (11) and (12) as an output of the c++ program.
<table>
<thead>
<tr>
<th>Laser Intensity (W/cm²)</th>
<th>1 x 10¹⁴</th>
<th>2 x 10¹⁴</th>
<th>3 x 10¹⁴</th>
<th>4 x 10¹⁴</th>
<th>5 x 10¹⁴</th>
<th>6 x 10¹⁴</th>
<th>7 x 10¹⁴</th>
<th>8 x 10¹⁴</th>
<th>9 x 10¹⁴</th>
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</thead>
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<td>0.9490395</td>
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<td>9 x 10¹⁷</td>
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<td>0.9519463</td>
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### TABLE 2.1
**PROGRAMMING RESULTS OF SELF-FOCUSING FOR N=0.1**

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<th>$8 \times 10^{18}$</th>
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<td>0.9727445</td>
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<td>0.9833742</td>
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66.
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<th>Laser Intensity (W/cm²)</th>
<th>Refractive Index</th>
<th>Power Threshold (Watts)</th>
<th>Laser Intensity (W/cm²)</th>
<th>Refractive Index</th>
<th>Power Threshold (Watts)</th>
<th>Laser Intensity (W/cm²)</th>
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<th>Power Threshold (Watts)</th>
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<td>4.4397E14</td>
<td>2 x 10⁶</td>
<td>4.309E-14</td>
<td>4.448E14</td>
<td>3 x 10⁶</td>
<td>4.309E-14</td>
<td>4.448E14</td>
<td>4 x 10⁶</td>
<td>4.309E-14</td>
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<td>4.448E14</td>
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<td>6 x 10⁶</td>
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<tr>
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Table 2.2: Programming Results of Self-Focusing for N<sub>0</sub>=0.9
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<th>7 × 10¹⁹</th>
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</thead>
<tbody>
<tr>
<td>Refractive Index</td>
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<td>0.8746149</td>
<td>0.8855511</td>
<td>0.8940823</td>
<td>0.9009743</td>
<td>0.9086905</td>
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<td>Iṣf/do</td>
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68.
### TABLE 2.3
PROGRAMMING RESULTS OF SELF-FOCUSBING FOR N=1.0

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<tbody>
<tr>
<td>Refractive Index</td>
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<tr>
<td>1st/2nd</td>
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<tr>
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<tbody>
<tr>
<td>Refractive Index</td>
<td>0.1157043</td>
<td>0.1620225</td>
<td>0.1965319</td>
<td>0.224807</td>
<td>0.2470369</td>
<td>0.2703588</td>
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<td>0.3067816</td>
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69.
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<th>Isld/do</th>
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**Note:** Table 2.3: Programming Results of Self-Focusing for N=1.0
TABLE 2.4
PROGRAMMING RESULTS OF SELF-FOCUSBING FOR N=1.5

<table>
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<th>Laser Intensity (W/cm²)</th>
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<th>$8 \times 10^{18}$</th>
<th>$9 \times 10^{18}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Refractive Index</td>
<td>0.5738888</td>
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<td>0.3346245</td>
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<td>Power Threshold (Watts)</td>
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<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>1.4</td>
<td>0.84</td>
</tr>
<tr>
<td>Laser Intensity (W/cm²)</td>
<td>$1 \times 10^{19}$</td>
<td>$2 \times 10^{19}$</td>
<td>$3 \times 10^{19}$</td>
<td>$4 \times 10^{19}$</td>
<td>$5 \times 10^{19}$</td>
<td>$6 \times 10^{19}$</td>
<td>$7 \times 10^{19}$</td>
<td>$8 \times 10^{19}$</td>
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</tr>
<tr>
<td>Refractive Index</td>
<td>0.4728239</td>
<td>0.5986005</td>
<td>0.7108983</td>
<td>0.7521301</td>
<td>0.7799053</td>
<td>0.8002091</td>
<td>0.8158688</td>
<td>0.8284068</td>
<td>0.8387368</td>
</tr>
<tr>
<td>lst/do</td>
<td>0.71</td>
<td>1.3</td>
<td>1.6</td>
<td>1.7</td>
<td>1.9</td>
<td>2.0</td>
<td>2.2</td>
<td>2.3</td>
<td>2.4</td>
</tr>
<tr>
<td>Laser Intensity (W/cm²)</td>
<td>$1 \times 10^{20}$</td>
<td>$2 \times 10^{20}$</td>
<td>$3 \times 10^{20}$</td>
<td>$4 \times 10^{20}$</td>
<td>$5 \times 10^{20}$</td>
<td>$6 \times 10^{20}$</td>
<td>$7 \times 10^{20}$</td>
<td>$8 \times 10^{20}$</td>
<td>$9 \times 10^{20}$</td>
</tr>
<tr>
<td>Refractive Index</td>
<td>0.8474341</td>
<td>0.8938206</td>
<td>0.913959</td>
<td>0.9258339</td>
<td>0.9338790</td>
<td>0.9397882</td>
<td>0.9443618</td>
<td>0.9480365</td>
<td>0.9510721</td>
</tr>
<tr>
<td>lst/do</td>
<td>2.5</td>
<td>2.9</td>
<td>3.5</td>
<td>3.7</td>
<td>4.0</td>
<td>4.1</td>
<td>4.4</td>
<td>4.5</td>
<td>4.7</td>
</tr>
</tbody>
</table>

The results marked * indicate that there does not exist a lst/do ratio, due to an insignificant result from the theoretical model.
TABLE 2.5
PROGRAMMING RESULTS OF SELF-FOCUSING FOR N=10

<table>
<thead>
<tr>
<th>Laser Intensity (W/cm²)</th>
<th>$1 \times 10^{20}$</th>
<th>$2 \times 10^{20}$</th>
<th>$3 \times 10^{20}$</th>
<th>$4 \times 10^{20}$</th>
<th>$5 \times 10^{20}$</th>
<th>$5.5 \times 10^{20}$</th>
<th>$6 \times 10^{20}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Refractive Index</td>
<td>0.9375688</td>
<td>0.5835789</td>
<td>0.3128250</td>
<td>0.2186095</td>
<td>0.3841150</td>
<td>0.4323734</td>
<td>0.470475</td>
</tr>
<tr>
<td>Power Threshold (Watts)</td>
<td>6.7502E15</td>
<td>2.2817E15</td>
<td>4.2914E14</td>
<td>1.5345E14</td>
<td>7.6008E14</td>
<td>1.048E15</td>
<td>1.3122E15</td>
</tr>
<tr>
<td>Ist/do</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>2.0138712</td>
<td>1.114519</td>
</tr>
<tr>
<td>Laser Intensity (W/cm²)</td>
<td>$6.5 \times 10^{20}$</td>
<td>$7 \times 10^{20}$</td>
<td>$7.5 \times 10^{20}$</td>
<td>$8 \times 10^{20}$</td>
<td>$8.5 \times 10^{20}$</td>
<td>$9 \times 10^{20}$</td>
<td>$9.5 \times 10^{20}$</td>
</tr>
<tr>
<td>Refractive Index</td>
<td>0.5017163</td>
<td>0.5280098</td>
<td>0.5505709*</td>
<td>0.5702232</td>
<td>0.5875508</td>
<td>0.6029840</td>
<td>0.6168440</td>
</tr>
<tr>
<td>Power Threshold (Watts)</td>
<td>1.5528E15</td>
<td>1.7716E15</td>
<td>1.9710E15</td>
<td>2.1533E15</td>
<td>2.3205E15</td>
<td>2.4744E15</td>
<td>2.6166E15</td>
</tr>
<tr>
<td>Ist/do</td>
<td>0.831231</td>
<td>0.6448637</td>
<td>0.6358506</td>
<td>0.7489099</td>
<td>0.8317591</td>
<td>0.9010296</td>
<td>0.9617598</td>
</tr>
</tbody>
</table>

The results marked * indicate that there does not exist a Ist/do ratio, due to an insignificant result from the theoretical model.
Illustrated in Figure (10) is the refractive index $|n|$ for various values of Neodymium glass laser intensity $I$ and density $N$ with $KT=10\text{eV}$ and $Z_i=1$, relatively cool hydrogen plasma. The dependence on $KT$ and $Z_i$ up to values of $10^3\text{eV}$ and 10 respectively, is very slight except for the lowest values of intensity. Several interesting features in Figure (10) should be pointed out. An increase in $N$, up to the cut-off value of unity, results in a steadily decreasing value of the refractive index, and intensity somewhat over $10^{16}\text{W/cm}^2$ Neodymium gives an $|n|\approx 0.05$ value. Since within the plasma $I_p=I_v/|n|$, where $I_v$ is the vacuum intensity value, this means that the local beam intensity is increased some twenty times upon plasma penetration. The electron number density is increased to a supercritical value, $N=1.5$, the minimum refractive index $|n|$ begins to increase. Cessation of the plots is due to numerical threshold intensity $I_v$, above which the self-focusing solution becomes predominant.
Figure 11: Power threshold as a function of laser intensity and density.

Illustrated in Figure (11) is the power threshold for self-focusing dependence on the neodymium glass laser intensity. From inspection of these results at different orders of laser intensities, there exists different power thresholds for self-focusing. This is notable due to the presence of the density ratio parameter. The use of the condition of equilibrium between the ponderomotive and hydrostatic forces is necessary to obtain the power threshold to produce self-focusing. It must be presumed that a plasma has been created, and secondarily self-focusing occurs.
Illustrated in Figure (12) is the $l_r/d_0$ ratio as a function of Neodymium glass laser intensity for various $N$ values of 0.10, 0.90, 0.95, 1.0, 1.5, 3.0, 10.0. Calculations were made at initially narrow intensity increments, to accurately determine the high gradient self-focusing dependency. It can be seen that as $N$ exceeds unity the $l_r$ limit is located further into the relativistic regime. In addition, the cusp-like distributions become more marked. It is obvious that extremely rapid self-focusing involving lengths of the same order as the magnitude of the initial width of the beam, between the half-irradiance maximum points is generated by relativistic effects.

The relativistic self-focusing has its maximum effects at the relativistic threshold. Its effect is lowering for higher intensities. This can be understood from the fact that, at these higher intensities there is an intensity dependent increase of the cut-off density. Hence the plasma

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becomes transparent for propagating waves at densities, where the nonrelativistic conditions would require evanescent waves.

It must be noted that both models mentioned in this chapter, for nonlinear force self-focusing and for relativistic self-focusing respectively, do not describe the complete process. The first model describes the threshold condition for a stationary case after sufficiently long interaction. The latter model describes relativistic self-focusing process in a homogeneous plasma, where the nonlinear forces disturb the homogeneity very quickly. The combination of these mechanisms is studied numerically in the next chapter of this thesis. Comparisons are made between both ponderomotive and relativistic self-focusing, to distinguish between the two mechanisms.

5.8 Conclusion

The phenomenon of self-focusing beams of waves in nonlinear media is one of the most interesting effects in nonlinear optics. In a nonlinear dispersive media, when the refractive index increases effectively with the wave intensity, a wave beam of finite amplitude is refracted to higher amplitude region as though making something like a convex lens, then self-focusing occurs.

Relativistic self-focusing results from the relativistic mass and energy dependency of the refracted index at higher laser intensities. This effect is also independent of the plasma temperature and becomes important when the laser beam is so intense that the directed component of electron velocity becomes comparable to the velocity of light. The process is virtually instantaneous then, due to the high intensity field and the short time scale. It dominates over all other mechanisms producing self-focusing in plasmas, such as the nonlinear force (ponderomotive) self-focusing.

Collisional-independent $\lambda_{e}/d_0$ calculations utilising a refractive index exhibit symmetric and convex downward distributions, with minimum values near the relativistic threshold. These are illustrated in Figure (12) of this thesis. These results in this thesis are produced by a new numerical code written in Borland C++ V.5, that incorporates both ponderomotive force in self-focusing mechanism and relativistic effects, at different plasma densities of various laser intensities.
The magnitude of the self-focusing length used in this thesis is inversely proportional to the square root of this ratio for $n_e/n_{\infty}^{NR} = N$, where $n_e$ is the electron number density and $n_{\infty}^{NR}$ is the non-relativistic cut-off density of the Neodymium laser ($10^{21}$ cm$^{-3}$).

Further new calculations are shown in Figure (12) for $N$ values and above, which exhibit extremely small $l_{sf}$ values, as well as limiting minimum intensity levels for non-singular $l_{sf}$ results. It is readily seen from Equation (5.77) that for $|n(n/2)| > |n(1)|$ the solution becomes singular and then imaginary at this threshold intensity $l_{s}$. This occurrence is crucial in the conditions illustrated in Figure (12) of this thesis. In particular, the collision frequency ($\nu$) magnitude is of maximum importance near this cut-off. Temperature, ion charge number, and qualitative allowance of field-induced collective effects on the total energy can all be influential in this region of Figure (12).

From Figure (12), $N$ values $> 1$ than unity produce the cusp-like distributions with the extremely small minimum self-focusing lengths. The minimum allowable intensities deeply found in the relativistic regime with $N >> 1$. It is seen that for $N$ values around unity a small change in $N$ produces a large change in both magnitude and shape of the $l_{sf}(l)$ distribution for $l > l_{cr}$. The distributions are mutually closer with $l > l_{cr}$. The influence of temperature, for a fixed $N$, is negligible for $N >> 1$. 

77.
CHAPTER 6
FOCUSBONG BALANCE DUE TO THE RELATIVISTIC EFFECTS

6.1 Introduction

Self-focusing and filamentation of laser beams have been experimentally observed in short wavelength laser-plasma interactions (Willi and Lee 120, 1985). It was shown that when the laser intensity was between $10^{13}$ to $10^{14}$ W/cm², for a short wavelength laser and high Z plasma, thermal self-focusing could initiate the self-focusing process. Although as self-focusing decreased the filament size and increased the intensity in the filament, the ponderomotive force effect was important and rapidly became the dominant self-focusing mechanism.

When the laser intensity is very high, ponderomotive self-focusing will be dominant. However at some point, when the oscillating velocity of the plasma electron becomes very large, relativistic effects will also play a role in self-focusing. In this chapter of the thesis, the nonlinear plasma dielectric function due to relativistic electron motion will be derived; from that one can obtain the nonlinear refractive index of the plasma and estimate the importance of relativistic self-focusing as compared to ponderomotive self-focusing at very high laser intensities.

6.2 Model of the Dielectric Function and Refractive Index

Self-focusing of optical beams in plasmas is due to an intensity dependent plasma dielectric function, given by:

$$\epsilon = \epsilon_o + \epsilon_2 |E|^2$$  \hspace{1cm} (6.1)

Or equivalently, an intensity dependent nonlinear refractive index:

$$n = n_o + n_2 |E|^2$$  \hspace{1cm} (6.2a)
Where $E$ is the electric field. Since $n = \sqrt{\varepsilon}$, $n_0$ is the linear part of the refractive index:

$$n_0 = \sqrt{\varepsilon_0}$$  \hspace{1cm} (6.2b)

and $n_2 |E|^2$ is the nonlinear part where:

$$n_2 = \frac{\varepsilon_2}{2n_0}$$  \hspace{1cm} (6.2c)

This is the coefficient of the first nonlinear term in the refractive index. The physics of self-focusing is contained in this coefficient and other coefficients of higher order nonlinear terms.

For a simple derivation, the plasma dielectric function due to relativistic electron motion will be found by noting that the dielectric function for a collisionless plasma is:

$$\varepsilon = 1 - (\omega_p/\omega)^2$$  \hspace{1cm} (6.3)

Where $\omega_p = (4\pi n_e e^2/m_0)^{1/2}$ is the plasma frequency and $\omega$ is the frequency of the incident laser, substituting for relativistic mass $m_v = \gamma m_0$, where $m_0$ is the electron rest mass:

$$\varepsilon = 1 - 1/\gamma (\omega_p/\omega)^2$$  \hspace{1cm} (6.4)

Where $\omega_p$ is now the plasma frequency with respect to the rest mass, $1/\gamma = (1 - \beta^2)^{1/2}$ and $\beta = V/c$, $V$ is the electron velocity and $c$ is the speed of light in vacuum. Noting that $\beta$ is due to the driving electric field in the laser, then for the small $\beta$, the familiar expression for the quiver velocity is obtained:

$$\beta = \frac{eE}{mc\omega}$$  \hspace{1cm} (6.5)

Using this in Equation (6.4) and retaining terms to second order, the following is achieved:

$$\varepsilon = 1 - (\omega_p/\omega)^2 \left[ 1 - \frac{1}{2} \left( \frac{e}{mc\omega} \right)^2 E^2 \right]$$  \hspace{1cm} (6.6)

This result compares very well with the plasma dielectric function obtained by (Kaw & Dawson 472, 1970).
In order to find out the range of validity of Eqn (6.6), i.e. how small must $\beta$ be for Eqn (6.6) to be valid, one requires the functional dependence of $\beta$ on the electric field, for an arbitrary large $E$. For this, the relativistic motion of electrons will have to be considered in detail.

### 6.3 Exact Solution of an Electron in an Electromagnetic Field

In this section, the exact solution for an electron in a linearly polarised electromagnetic wave field will be obtained. This will be done by starting with the energy and momentum equations:

$$\frac{d\gamma}{dt} = -\frac{e}{mc} (\beta \cdot E)$$  \hspace{1cm} (6.7)

$$\frac{d}{dt} (\gamma \beta_y) = -\frac{e}{mc} (E_y + \beta \times \beta)$$  \hspace{1cm} (6.8)

Where

$$\gamma = (1-\beta \cdot \beta)^{-1/2}$$  \hspace{1cm} (6.9)

The total energy of the electron is $\gamma m_o c^2$, $m_o$ is the rest mass.

For a linearly polarised electromagnetic wave propagating in the $x$-direction, $E = E_0 \cos(\omega t - \kappa x) e_y$, $e_y$ is the unit vector in the $y$-direction, which gives $|\beta_z| = |E_y| = |E|$. Equation (6.8) thus becomes:

$$\frac{d}{dt} (\gamma \beta_y) = -\frac{eE_0}{mc} \cos (\omega t - \kappa x) (1-\beta_x)$$  \hspace{1cm} (6.10)

and

$$\frac{d}{dt} (\gamma \beta_x) = -\frac{eE_0}{mc} \cos (\omega t - \kappa x) \beta_y$$  \hspace{1cm} (6.11)

For the $y$ and $x$ components; $\beta_z$ is a constant which can be assumed to be zero without any loss of generality. Thus Equation (6.9) becomes:

$$\gamma = (1-\beta_x^2-\beta_y^2)^{-1/2}$$  \hspace{1cm} (6.12)

The normalised quiver velocity $(v_{qu}/c)$ is given by $\beta_y$ and $\beta_x$ is the normalised component of the
electron velocity in the direction of the laser beam. Letting \( \theta = \omega t - \kappa x \), then:

\[
\frac{d\theta}{dt} = \omega (1 - \beta_x) \tag{6.13}
\]

Equations (6.10) and (6.11) can be transformed by a change of independent variable from \( t \) to \( \theta \), giving:

\[
\frac{d}{d\theta} (\gamma \beta_y) = -\frac{eE_y}{mc\omega} \cos \theta \tag{6.14}
\]

\[
\frac{d}{d\theta} (\gamma \beta_x) = -\frac{eE_y}{mc\omega} \cos \theta \left( \frac{\beta_y}{1 - \beta_x} \right) \tag{6.15}
\]

Equation (6.14) can be integrated directly to yield:

\[
(\gamma \beta_y) = -\frac{eE_y}{mc\omega} \sin \theta \tag{6.16}
\]

For the nonrelativistic case (\( \gamma \to 1 \)), the well known expression for quiver velocity is recovered. Note: The electron quivers in the opposite direction and is \( \pi/2 \) out of phase with respect to the driving electric field.

In order to solve Eqn (6.15), the energy equation is used. Transforming Eqn (6.7) with Eqn (6.13), this leads to:

\[
\frac{d\gamma}{d\theta} = -\frac{eE_y}{m_o c \omega} \cos \theta \left( \frac{\beta_y}{1 - \beta_x} \right) \tag{6.15b}
\]

Since the right hand sides of Eqn (6.15) and Eqn (6.15b) are identical, then:

\[
\frac{d (\gamma \beta_x)}{d\gamma} = 1 \tag{6.17}
\]

or

\[
\gamma d\beta_x + \beta_x d\gamma = d\gamma \tag{6.18}
\]
Upon integration, this leads to the identity:

\[ (1 - \beta_y) = \frac{1}{\gamma} \]  

(6.19)

Inserting Equation (6.19) into Eqn (6.15), using Eqn (6.16), this leads to:

\[ \frac{d}{d\theta} (\gamma \beta_y) = \left( \frac{eE_o}{m_0 c \omega} \right)^2 \cos \theta \sin \theta \]

(6.20)

This can be integrated immediately to yield:

\[ \gamma \beta_x = \frac{1}{2} \left( \frac{eE_o}{m_0 c \omega} \right)^2 \sin^2 \theta \]

(6.21)

Since \( \sin^2 \theta = \frac{1}{2} (1 - \cos 2\theta) \), it can be noted that for \( \gamma \to 1 \), \( \beta_x \) is a second harmonic oscillation superimposed on a constant drift, both in the direction of the laser beam.

From Equations (6.16), (6.21) and (6.12), one can find that:

\[ \gamma = \left( 1 + K^2 + \frac{1}{4} K^4 \right)^{1/2} \]

(6.22)

Where:

\[ K = \frac{eE_o}{m_0 c \omega} \sin \theta \]

(6.23)

Substituting Equation (6.22) into Eqns (6.16) and (6.21), finally this obtains both the quiver and directed velocity components in parametric form as functions of \( K \), i.e. as functions of electric field and laser frequency:

\[ \beta_y = -K' \left( 1 + K^2 + \frac{1}{4} K^4 \right)^{1/2} \]

(6.24)

\[ \beta_x = \frac{1}{2} K^2 \left( 1 + K^2 + \frac{1}{4} K^4 \right)^{1/2} \]

(6.25)

Equations (6.24) and (6.25) are the exact solutions for the velocity components of an electron driven by a linearly polarised electromagnetic wave field of arbitrary amplitude.
6.4 Nonlinear Dielectric Function and Nonlinear Refractive Index

The oscillating current in the direction of the electric field is:

\[ J_y = -n_e \varepsilon_0 c \beta_y \]  
(6.26)

Substituting Equation (6.16) into Eqn (6.27):

\[ J_y = \frac{1}{\gamma} n_e \varepsilon_0 c \mathbf{K} \]  
(6.27)

Inserting Equation (6.27) into the Maxwell equation, remembering that \( E_x = E_o \cos \theta \), then:

\[ (\nabla \times \beta) y = \frac{4\pi}{c} J_y + \frac{1}{c} \frac{\delta E}{\delta t} = -\frac{\omega}{c} \left[ 1 - \frac{1}{\gamma} (\omega_p/\omega)^2 \right] E_o \sin \theta \]  
(6.28)

However, since:

\[ \nabla \times \beta = \frac{1}{c} \frac{\delta D}{\delta t} = \frac{1}{c} \epsilon \frac{\delta E}{\delta t} = -\frac{\omega}{c} \epsilon E_o \sin \theta \]  
(6.29)

This obtains the plasma dielectric function as:

\[ \epsilon = 1 - \frac{1}{\gamma} (\omega_p/\omega)^2 \]  
(6.30)

This is exactly equation (6.4) derived earlier by simple substituting for relativistic mass in \( \omega_p \).

Note: In the above derivation, \( \beta_x \) did not play a role. Although \( J_x \) has a direct current as well as an oscillating current component, it does not contribute to the plasma dielectric function because the driving electric field has no component in the x-direction. Furthermore, while \( \beta_y \) can be expanded in even powers of \( E \) (when \( K \) is not too large) which causes harmonic generation.

For small values of \( K \), Equation (6.22) can be expanded to give:

\[ \frac{1}{\gamma} = 1 - \frac{1}{2} K^2 \]  
(6.31)

Substituting Equation (6.31) into Eqn (6.30) gives:
\[ \varepsilon = \left[ 1 - (\omega_p/\omega)^2 \right] + \frac{1}{2} \left( \frac{\omega_p}{\omega} \right)^2 \left( \frac{\theta}{m \omega} \right)^2 |E|^2 \]

\[ \varepsilon = \varepsilon_0 + \varepsilon_2 |E|^2 \]  

(6.32)

Using Equations (6.2b) and (6.2c), one thus obtains the coefficient \( n_2 \) to the quadratic nonlinear term of the refractive index (subscript 'r' denotes relativistic):

\[ n_{2r} = \frac{1}{4} \frac{n_e}{n_{ec}} \left( \frac{\theta}{m \omega} \right)^2 \left( 1 - \frac{n_e}{n_{ec}} \right)^{-1/2} \]  

(6.33)

Where \( n_e \) and \( n_{ec} \) are the electron and critical densities, respectively.

Numerical evaluation is obtained for various values of dielectric constant displayed in Tables (3.1), (3.2) and (3.3) for the exact equation, second order approximation and fourth order approximation respectively. These tables display the range of terms needed to approximate for \( 1/\gamma \). These results are illustrated in Figure (13.1).

A relationship is obtained for the numerical evaluation of the electron velocity \( \beta \), due to both the quiver velocity and directed velocity components displayed in Tables (4.1), (4.2) and (4.3) as a function of the dielectric constant. These results are illustrated in Figure (13.2).

**TABLE 3.1**  
**NUMERICAL EVALUATION OF THE EXACT EQUATION (6.22)**  
**FOR 1/\( \gamma \) AS A FUNCTION OF K**

<table>
<thead>
<tr>
<th>Dielectric Constant ( K )</th>
<th>Relativistic Factor ( 1/\gamma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0.1</td>
<td>0.9950248</td>
</tr>
<tr>
<td>0.3</td>
<td>0.9569378</td>
</tr>
<tr>
<td>0.5</td>
<td>0.8888888</td>
</tr>
<tr>
<td>0.7</td>
<td>0.8032128</td>
</tr>
<tr>
<td>0.9</td>
<td>0.7117437</td>
</tr>
<tr>
<td>1.1</td>
<td>0.6230529</td>
</tr>
<tr>
<td>1.3</td>
<td>0.5420054</td>
</tr>
<tr>
<td>1.5</td>
<td>0.4705882</td>
</tr>
</tbody>
</table>
### TABLE 3.2
NUMERICAL EVALUATION OF THE SECOND ORDER APPROXIMATION
EQUATION (6.31) FOR 1/\(\gamma\) AS A FUNCTION OF \(K\)

<table>
<thead>
<tr>
<th>Dielectric Constant (K)</th>
<th>Relativistic Factor 1/(\gamma)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0.1</td>
<td>0.995</td>
</tr>
<tr>
<td>0.3</td>
<td>0.955</td>
</tr>
<tr>
<td>0.5</td>
<td>0.875</td>
</tr>
<tr>
<td>0.7</td>
<td>0.755</td>
</tr>
<tr>
<td>0.9</td>
<td>0.595</td>
</tr>
<tr>
<td>1.1</td>
<td>0.395</td>
</tr>
<tr>
<td>1.3</td>
<td>0.155</td>
</tr>
</tbody>
</table>

### TABLE 3.3
NUMERICAL EVALUATION OF THE FOURTH ORDER APPROXIMATION
EQUATION (6.34) FOR 1/\(\gamma\) AS A FUNCTION OF \(K\)

<table>
<thead>
<tr>
<th>Dielectric Constant (K)</th>
<th>Relativistic Factor 1/(\gamma)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0.1</td>
<td>0.9949831</td>
</tr>
<tr>
<td>0.3</td>
<td>0.9536311</td>
</tr>
<tr>
<td>0.5</td>
<td>0.8644375</td>
</tr>
<tr>
<td>0.7</td>
<td>0.7144231</td>
</tr>
<tr>
<td>0.9</td>
<td>0.4841191</td>
</tr>
<tr>
<td>1.1</td>
<td>0.1475671</td>
</tr>
<tr>
<td>K</td>
<td>$\beta y$</td>
</tr>
<tr>
<td>----</td>
<td>----------</td>
</tr>
<tr>
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<tr>
<td>K</td>
<td>( \beta x )</td>
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<td>4.9</td>
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</table>
**TABLE 4.3**

NUMERICAL EVALUATION OF $\beta = (\beta x^2 + \beta y^2)^{1/2}$

AS A FUNCTION OF $K$

<table>
<thead>
<tr>
<th>$K$</th>
<th>$\beta$</th>
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<td>3.1</td>
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<td>3.3</td>
<td>0.9878894</td>
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<td>0.9965477</td>
</tr>
<tr>
<td>4.9</td>
<td>0.9970392</td>
</tr>
</tbody>
</table>
However from Equation (6.22), one can find the range of validity for Equation (6.31) and consequently Equation (6.33). In Figure (13.1), $1/y$ is plotted as a function of $K$. One finds that Equation (6.31) is a very good approximation up to $K=0.6$, the error being within 3%, see Figure (13.1).

The exact equation (6.22) for $1/y$ as a function of $K$ is represented by series 1. The second order approximation (equation 6.31), represented by series 2, is used to obtain $n_{2q}$, it is accurate up to $K=0.6$; the fourth order approximation (equation 6.34), represented by series 3, is used to obtain $n_{4q}$, it is accurate up to $K=1.1$.

**Figure 13.1**: The exact equation, second order approximation and fourth order approximation as a function of the dielectric constant.
The quiver velocity $|\beta y|$ as a function of $K$ (equation 6.24) is given by series 1; the directed velocity $\beta x$ (equation 6.25) is given by series 2. Series 3 represents $\beta = (\beta x^2 + |\beta y|)^{1/2}$. Note that for small $K$, $\beta = |\beta y|$, while for very large $K$, $\beta = |\beta x|$.

**Figure 13.2:** The quiver velocity and directed velocity as a function of the dielectric constant.

For values above $K = 0.6$, a higher order approximation is called for. This is found by a numerical fit giving:

$$\frac{1}{\gamma} = 1 - 0.5K^2 + 0.169K^4$$  \hspace{1cm} (6.34)

Equation (6.34) gives rise to an additional term in the plasma dielectric function:

$$\epsilon = \epsilon_0 + \epsilon_2 |E|^2 + \epsilon_4 |E|^4$$  \hspace{1cm} (6.35)

Where $\epsilon_0$ and $\epsilon_2$ are the same as those in Eqn (6.32), the new term is:

$$\epsilon_4 = -0.169 \left( \frac{\omega}{mc \omega} \right)^2 \left( \frac{\theta}{mc \omega} \right)^4$$  \hspace{1cm} (6.36)
which provides for a next higher order term in the nonlinear refractive index:

\[
n_{4(\lambda)} = -0.0845 \frac{n_e}{n_{ec}} \left( \frac{e}{m_o c^2} \right)^4 \left( 1 - \frac{n_e}{n_{ec}} \right)^{-1/2}
\]  

(6.37)

Both Equation (6.34) and (6.37) are valid up to \( K=1.1 \), being accurate to within 3% (see Figure 13.2). Since \( n_{4(\lambda)} \) is negative, it is a self-defocusing term, but the effect is smaller than the self-focusing caused by \( n_{2(\lambda)} \).

For values of \( K \) larger than one, more higher orders will be needed to approximate \( 1/\gamma \). For now it is arbitrary of approximating more higher orders, as from inspection so far, the trend is evident. Thus higher order expansions will provide alternating self-focusing and self-defocusing terms.

Although the refractive index can be expanded (Akhmanov et al 1151, 1972) as:

\[
n = n_o + n_2 |E|^2 + n_4 |E|^4 + \ldots
\]  

(6.38)

This will not be pursued any further, since the well known nonlinear Schrödinger equation (Ablowitz & Segur 1981), which governs self-focusing of a laser in a weakly nonlinear but strongly dispersive medium, will now have in addition to the 'cubic' term \( n_2 |E|^2 E \), many higher order terms (\( n_4 |E|^4 E, n_6 |E|^6 E, \ldots \)). It is not clear how or whether such an equation can be solved analytically.

### 6.5 Relativistic Effects on Self-focusing

The coefficient of the nonlinear refractive index due to relativistic electron motion can be written in another form. Since \( \omega = 2 \pi c / \lambda, \lambda \) being the laser wavelength, therefore from Equations (6.33) and (6.37) the following is obtained:

\[
n_{2(\lambda)} = \frac{1}{(4\pi)^2} \frac{n_e}{n_{ec}} \lambda^2 \left( \frac{e}{m_o c^2} \right)^2 \left( 1 - \frac{n_e}{n_{ec}} \right)^{-1/2}
\]  

(6.39)

\[
n_{4(\lambda)} = -\frac{0.0845}{(2\pi)^4} \frac{n_e}{n_{ec}} \lambda^4 \left( \frac{e}{m_o c^2} \right)^4 \left( 1 - \frac{n_e}{n_{ec}} \right)^{-1/2}
\]  

(6.40)
The nonlinear refractive index due to the ponderomotive force effect can be obtained by considering the density change caused by the ponderomotive force. Comparing the ponderomotive pressure against the plasma pressure and neglecting plasma heating, the electron density is:

$$n_{eo} = n_e \exp \left( - \frac{E^2}{8 \pi n_{ec} T} \right)$$  \hspace{1cm} (6.41)

Where $n_{ec}$ and $n_e$ are the modified and unperturbed electron densities, respectively, $T$, is the temperature in energy units. Expanding Equation (6.41) gives:

$$n_{eo} = n_e \left[ 1 - \frac{E^2}{8 \pi n_{ec} T} + \frac{1}{2} \left( \frac{1}{8 \pi n_{ec} T} \right)^2 E^4 + \ldots \right]$$  \hspace{1cm} (6.42)

Inserting this into the formula for refractive index:

$$n = \left( 1 - \frac{n_{eo}}{n_{ec}} \right)^{1/2} = n_o + n_2 |E|^2 + n_4 |E|^4 + \ldots$$  \hspace{1cm} (6.43)

Where $n_{ec} = \pi mc^2 / e^2 \lambda^2$:

$$n_o = \left( 1 - \frac{n_e}{n_{ec}} \right)^{1/2}$$  \hspace{1cm} (6.44)

$$n_{2(p)} = \frac{1}{(4\pi)^2} \frac{n_e \lambda^2}{n_{ec}} \frac{e^2}{m_o c^2} \frac{1}{T} \left( 1 - \frac{n_e}{n_{ec}} \right)^{-1/2}$$  \hspace{1cm} (6.45)

$$n_{4(p)} = - \frac{1}{(4\pi)^4} \frac{n_e \lambda^4}{n_{ec}} \left( \frac{e^2}{m_o c^2} \right)^2 \frac{1}{T^2} \left( 1 - \frac{n_e}{n_{ec}} \right)^{-1/2}$$  \hspace{1cm} (6.46)

The subscript 'p' refers to 'ponderomotive'. Notice that $n_{4(p)}$ is negative just like $n_{4(0)}$.

Comparing Equations (6.39) to (6.45) and Equations (6.40) to (6.46), one notes the striking similarity in scaling with respect to laser-plasma parameters. Both the ponderomotive force and the relativistic self-focusing mechanisms are most effective near the critical density, both mechanisms are less effective at shorter laser wavelengths. Due to such similarity in scaling, one can compare the effectiveness of each mechanism by examining $n_{2(p)}/n_{2(0)}$ and $n_{4(p)}/n_{4(0)}$.
Therefore:

\[ n_{2(\lambda)} / n_{2(\phi)} = T / m_c c^2 \]  
(6.47)

\[ n_{4(\lambda)} / n_{4(\phi)} = 1.352 \left( T / m_c c^2 \right)^2 \]  
(6.48)

Under normal circumstances, i.e. \( I < 10^{16} \text{ W/cm}^2 \), \( T \) is much smaller than the rest energy of the electron, the relativistic contribution is negligibly small. However, when \( T \) becomes large due to higher laser intensity, the relativistic contribution will become important.

A way to estimate \( T \) is to assume that the electron temperature is represented by the kinetic energy of the electron which is imposed by the oscillating electric field of the incident electromagnetic wave. With \( T \) as the kinetic energy, one can write:

\[ T = (\gamma - 1) m_c c^2 \]  
(6.49)

Or one can write the oscillation of electrons within the electromagnetic field that generates maximum kinetic energies \( \epsilon_{\text{kin}} \) of relativistic magnitude:

\[ \epsilon_{\text{kin}} = T = m_c c^2 \left[ \left( 1 + \frac{\epsilon E^2}{m_e^2 \omega^2 c^2} \right)^{1/2} \right] - 1 \]  
(6.50a)

Where:

\[ \epsilon_{\text{kin}} = m_e c^2 \frac{I}{I_{\text{rel}}} \text{ for } \epsilon_{\text{kin}} \ll m_c c^2 \]  
(6.50b)

Equations (6.46) and (6.48) thus become:

\[ n_{2(\lambda)} / n_{2(\phi)} = \gamma^{-1} \]  
(6.51)

\[ n_{4(\lambda)} / n_{4(\phi)} = 1.352 \left( \gamma - 1 \right)^2 \]  
(6.52)

To find out how effective relativistic self-focusing is, a few values of Eqn (6.51) and Eqn (6.52) are calculated at different intensities; noting that \( I = cE^2/8\pi \) and using Eqn (6.22) for calculating \( \gamma \), the results are given in Table 5.
TABLE 5
RELATIVISTIC SELF-FOCUSBING EFFECT COMPARED TO PONDEROMOTIVE SELF-FOCUSBING EFFECT FOR DIFFERENT INTENSITIES OF A NEODYMIUM GLASS LASER

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<th>I (W/cm²)</th>
<th>n₂₀/₀₂₀</th>
<th>n₄₀/₀₄₀</th>
</tr>
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<td>m𝑐ω</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>1.81 x 10^16</td>
<td>0.005</td>
<td>3.3 x 10⁻⁵</td>
</tr>
<tr>
<td>0.3</td>
<td>1.65 x 10^17</td>
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<td>0.0027</td>
</tr>
<tr>
<td>0.5</td>
<td>4.55 x 10^17</td>
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<td>0.021</td>
</tr>
<tr>
<td>0.7</td>
<td>9 x 10^17</td>
<td>0.25</td>
<td>0.082</td>
</tr>
<tr>
<td>0.9</td>
<td>1.48 x 10^18</td>
<td>0.40</td>
<td>0.22</td>
</tr>
<tr>
<td>1.1</td>
<td>2.21 x 10^18</td>
<td>0.60</td>
<td>0.49</td>
</tr>
</tbody>
</table>

One observes from Table 5, that for I < 10^17 W/cm² i.e., for ordinary laser plasmas with electron temperatures of no more than a few kilo-electron-volts, the relativistic contribution to ponderomotive self-focusing is entirely negligible. However, beyond ~ 4x10^17 W/cm², relativistic self-focusing becomes important very rapidly with increasing laser intensity.

A variety of nonlinear optical effects such as optical Kerr effects, self-focusing and optical solitons are related to a nonlinear refractive index. A phenomenon that refers to the intensity dependence of the refractive index. An optical soliton propagation occurs due to the nonlinear change in the refractive index of the media which is balanced by the group dispersion. It is known that when the Kerr effect due to nonlinear change in the refractive index of medium in the lower order is included, the propagation is governed by the nonlinear paraxial equation. The soliton formation and propagation can be observed and understood by using a numerical routine to solve the nonlinear paraxial equation. The characteristic of periodicity of the second order term is investigated by a numerical study of the semiclassical limit for the linear and nonlinear paraxial equation. Comparisons of both dimension and nonlinearity are identified in detail for the defocusing and focusing cases respectively. The observation of this phenomena is investigated in the next chapter of this thesis.

6.6 Conclusion

From the solutions of the fully relativistic equations of motion of an electron in a linearly-polarised plane wave field, the derivation of the nonlinear refractive index determined the
range of validity for each high order approximation, and also found that higher order expansions provide alternating self-focusing and self-defocusing terms.

A numerical example is provided to compare the effectiveness of relativistic self-focusing as compared to ponderomotive self-focusing. The relativistic effects tend to enhance the ponderomotive self-focusing process. The relativistic effects become important when the laser intensity exceeds $4 \times 10^{17}$ W/cm², for a longer wave-length laser, e.g. $\lambda=1\mu$m, the relativistic effect becomes important at a lower intensity when $I > 3 \times 10^{17}$ W/cm². The actual effect of relativistic self-focusing will probably be more severe than that described in this chapter of the thesis. That is because the mechanism of relativistic self-focusing is based on a change of optical constant, it occurs almost instantaneously, in the time of the order of $1/\omega$. While the ponderomotive self-focusing has a time delay due to electrons being expelled (i.e. plasma motion) from the high intensity region.

When self-focusing occurs, the filament constricts and the trapped intensity in the filament becomes larger. Therefore, when self-focusing is initiated at a lower intensity, the ponderomotive force mechanism has the potential of inducing relativistic self-focusing. On the other hand, since relativistic self-focusing occurs almost instantaneously, this mechanism is most effective for very short pulse (~1 picosecond) laser-plasma interactions.

For extremely large laser intensities, i.e. for $K >> 1$, $\gamma$ becomes very large, thus the refractive index approaches towards unity. This means that ultimately, the self-defocusing terms will balance out the self-focusing terms. The plasma will cease to behave as an optically refractive medium. When $K$ is large, the large axial ponderomotive force drives $\beta_z$ to approach unity, i.e. relativistic electrons in the direction of the laser beam will be generated.
CHAPTER 7
FOCUSBING AND DEFOCUSING OF THE NONLINEAR PARAXIAL EQUATION IN BOTH DIMENSION AND NONLINEARITY

7.1 Introduction

Wave propagation in materials with substantial dispersion or diffraction and a significant nonlinearity can be described by the nonlinear paraxial equation. The nonlinear paraxial equation has exact soliton solutions that correspond to a balance between nonlinearity and dispersion in the case of temporal solitons or between nonlinearity and diffraction in the case of spatial solitons.

The soliton concept is a sophisticated mathematical construct based on the integrability of a class of nonlinear differential equations. Integrable nonlinear differential equations have one feature in common, they are all conservative and thus derivable from a Hamiltonian. The integration is performed via the method of inverse scattering.

This chapter of this thesis will explore a numerical study of the semiclassical limit of linear and the nonlinear paraxial equation, in both defocusing and focusing cases. Along this, an analytical study of the semiclassical limit of the defocusing cases will also be investigated. Comparisons of both dimension and nonlinearity are observed for the defocusing and focusing cases respectively.

7.2 Nonlinear Paraxial Equation

One of the simplest nonlinear equations is the nonlinear paraxial equation for a complex-valued field \( \psi(x,t) \) over a spatial domain \( \Omega \subset \mathbb{R}^D \):

\[
\frac{1}{i} \frac{\partial u}{\partial \xi} + u + \frac{1}{2} \frac{\partial^2 u}{\partial t^2} - K'(|u|^2) u = 0 \tag{7.1}
\]
Where $K'$ is the first derivative of a twice differentiable nonlinear real-valued function and $\tau$ is a positive parameter. The parameter $\tau$ is analogous to Planck's constant, which in the quantum setting is usually very small when evaluated in the natural dimensional scales of the equation as determined by its initial and boundary conditions. For the moment, the precise specification of the domain $\Omega$ and the nature of the boundary conditions is left vague in order to make some general statements regarding the structure of Eqn (7.1).

The nonlinear function $u: R_+^* R$ is the potential energy density of the field and is clearly seen when the nonlinear paraxial equation Eqn (7.1) is recast as a Hamiltonian system in the form:

$$i \frac{\partial u}{\partial \xi} + u = \frac{\partial H}{\partial u}$$

$$H = \int_0^{\cdot} \frac{1}{2} \frac{\partial^2 u}{\partial \xi^2} + K(|u|^2) dx$$  \hspace{1cm} (7.2)

The associated Poisson bracket of any two functionals $F$ and $G$ is given by:

$$\{F, G\} = \frac{1}{i\tau} \int_0^{\cdot} \left( \frac{\partial F}{\partial u} \frac{\partial G}{\partial \bar{u}} - \frac{\partial F}{\partial \bar{u}} \frac{\partial G}{\partial u} \right) dx$$  \hspace{1cm} (7.3)

Where the variable $\bar{u}$ denotes complex conjugate. The evolution of any functional $F$ under the nonlinear paraxial equation flow in Eqn (7.2):

$$\frac{\partial F}{\partial t} = \{F, H\}$$  \hspace{1cm} (7.4)

This Hamiltonian structure plays a major role in our analysis.

Also associated with the nonlinear paraxial equation (7.1) are $D + 2$ local conservation laws corresponding to mass, momentum, and energy conservation. Their densities, $\rho$, $\mu$ and $\epsilon$ respectively, are given by:

$$\rho = |u|^2,$$

$$\mu = -i\frac{\tau}{2} \left( u \frac{\partial u}{\partial \tau} - \bar{u} \frac{\partial \bar{u}}{\partial \tau} \right),$$

$$\epsilon = \frac{\tau^2}{2} \left( \frac{\partial^2 u}{\partial \xi^2} + K(|u|^2) \right)$$  \hspace{1cm} (7.5)
The mass and momentum densities determine the field \( \psi \) up to a constant phase; the energy density can be written in terms of them as:

\[
\varepsilon = \frac{1}{2} \frac{\mu^2}{\rho} + \frac{\tau^2}{8} \frac{\nabla \mu^2}{\rho} + K(\rho)
\]  

(7.6)

The local conservation laws are then:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \mu = 0,
\]

\[
\frac{\partial \mu}{\partial t} + \nabla \left( \frac{\mu \otimes \mu}{\rho} \right) + \nabla P(\rho) = \frac{\tau^2}{4} \nabla \cdot \left[ \rho \nabla^2 \log \rho \right],
\]

\[
\frac{\partial \varepsilon}{\partial t} + \nabla \left( \frac{\mu}{\rho} \left( \varepsilon + P(\rho) \right) \right) = \frac{\tau^2}{4} \nabla \left[ \frac{\mu \Delta \rho}{\rho} - \frac{\nabla \cdot \mu \nabla P}{\rho} \right]
\]

(7.7)

Where \( P(\rho) = \rho K'(\rho) - K(\rho) \)

The first two of these are a closed system governing \( \rho \) and \( \mu \) that has the form of a perturbation of the compressible Euler equations of fluid dynamics with the pressure given by \( P(\rho) \). If the "Euler part" of these equations is to be hyperbolic then the pressure \( P(\rho) \) must be a strictly increasing function of \( \rho \); in that case:

\[
P'(\rho) = \rho K''(\rho) > 0
\]

This means that \( U \) must be a strictly convex function of \( \rho \) and corresponds to a defocusing nonlinear paraxial equation. In this context, a focusing nonlinear paraxial equation can be understood as a fluid whose pressure decreases when the mass density increases, a phenomenon leading to the development of mass concentrations.

### 7.3 Semiclassical Limit

The "semiclassical limit" of the nonlinear paraxial equation can be described as follows. Consider the family, parametrised by \( \tau > 0 \), of solutions \( \psi^{(\tau)}(x,t) \) to the Cauchy problems:

\[
\frac{\partial u}{\partial \xi} + u^{(\tau)} + \frac{1}{2} \frac{\partial^2 u}{\partial \tau^2} - K' \left| u^{(\tau)} \right|^2 u^{(\tau)} = 0
\]

(7.8a)
\[ u^{(r)}(x,0) = A(x) \exp \left( \frac{i}{\tau} S(x) \right) \]  \hspace{1cm} (7.8b)

Where the (non-negative) amplitude \( A(x) \) and (real) phase \( S(x) \) are assumed to be smooth and independent of \( \tau \). The initial conserved densities are then:

\[ \rho^{(r)}(x,0) = |A(x)|^2, \]  \hspace{1cm} (7.9a)

\[ \mu^{(r)}(x,0) = |A(x)|^2 \nabla S(x) \]

\[ \epsilon^{(r)}(x,0) = \frac{1}{2} |A(x)|^2 |\nabla S(x)|^2 + \frac{\tau^2}{2} |\nabla A(x)|^2 + K(|A(x)|^2) \]  \hspace{1cm} (7.9b)

The general problem of the semiclassical limit is to determine the limiting behaviour of any function of the field \( \psi^{(r)} \) as \( \tau \to 0 \). In particular, to ascertain the existence (in some sense) of the limits of the conserved densities:

\[ \rho = \lim_{\tau \to 0} \rho^{(r)} \]  \hspace{1cm} (7.10a)

\[ \mu = \lim_{\tau \to 0} \mu^{(r)} \]  \hspace{1cm} (7.10b)

\[ \epsilon = \lim_{\tau \to 0} \epsilon^{(r)} \]  \hspace{1cm} (7.10c)

If the limits exist, to determine their dynamics.

Arguing formally, it is natural to conjecture for the defocusing case that the \( O(\tau^2) \) dispersive terms appearing in Eqn (7.7) are negligible as \( \tau \to 0 \), and that the limiting densities \( \rho \) and \( \mu \) satisfy the hyperbolic system (the Euler system):

\[ \frac{\partial \rho}{\partial t} + \nabla \mu = 0 \]  \hspace{1cm} (7.11a)

\[ \frac{\partial \mu}{\partial t} + \nabla \left( \frac{\mu}{\rho} \otimes \mu \right) + \nabla P(\rho) = 0 \]

With initial conditions inferred from Equation (7.9a) given by:

\[ \rho(x,0) = |A(x)|^2, \quad \mu(x,0) = |A(x)|^2 \nabla S(x) \]  \hspace{1cm} (7.11b)
This argument is self-consistent only so long as the solution of the Euler system Eqn (7.11) remains classical. In that case the limiting energy density will be given by:

$$\varepsilon = \frac{1}{2} \frac{\mu^2}{\rho} + K(\rho)$$  \hspace{1cm} (7.12)

And will satisfy:

$$\frac{\partial \varepsilon}{\partial t} + \nabla \left( \frac{\mu}{\rho} (\varepsilon + P(\rho)) \right) = 0$$  \hspace{1cm} (7.13)

Hence, playing the role of a Lax entropy for the Euler system in Eqn (7.11a).

The genuinely nonlinear nature of the Euler system in Eqn (7.11) will ensure that its classical solution will develop singular behaviour (an infinite derivative) for all but rarefaction initial data. At that instant, such a breaking occurs, the formally small dispersive terms on the right side of Eqn (7.7) will no longer be negligible, and the above characterisation of the semiclassical limit will break down. Since this small regularising term is dispersive, one expects that the impending singularity in $\rho$ and $\mu$ will be regularised by the development of small wavelength oscillations.

### 7.4 Weak Nonlinearity Before Breaking

Numerics is to be used in order to illustrate the breakdown of the "Euler" description of the semiclassical limit for the defocusing nonlinear paraxial equation. The problem is cast in one spatial dimension with a cubic nonlinearity:

$$\frac{\partial u}{\partial \xi} + u = -\frac{1}{2} \frac{\partial^2 u}{\partial x^2} + \gamma |u|^2 u$$  \hspace{1cm} (7.14a)

$$u(x,0) = A_{in}(x) \exp \left( \frac{i}{\tau} \sin(x) \right)$$  \hspace{1cm} (7.14b)
Where $\gamma$ here is a positive constant. For this one-dimensional case, the Euler system in Eqn (7.11), describing the formal semiclassical limit, reduces to the initial value problem:

\[
\frac{\partial p}{\partial t} + \frac{\partial \mu}{\partial x} = 0
\]

\[
\frac{\partial \mu}{\partial t} + \frac{\partial}{\partial x} \left( \frac{\mu^2}{\rho} + \gamma \frac{p^2}{2} \right) = 0
\]  

(7.15a)

\[p(x,0) = A_{in}^2(x), \quad \mu(x,0) = A_{in}^2(x) \frac{\partial}{\partial x} \sin(x)\]  

(7.15b)

Riemann invariants for the Euler system in Equation (7.15a) are given by:

\[r = \mu \frac{\rho}{2 \rho} = \sqrt{\gamma \rho} \]  

(7.16)

The system can be placed in the Riemann invariant form:

\[
\frac{\partial r^+}{\partial t} + \frac{1}{2} (3r^- + r^+) \frac{\partial r^+}{\partial x} = 0
\]  

(7.17a)

\[
\frac{\partial r^-}{\partial t} + \frac{1}{2} (r^- + 3r^-) \frac{\partial r^+}{\partial x} = 0
\]

With the initial conditions:

\[r^\pm(x,0) = \frac{1}{2} \frac{\partial}{\partial x} \sin(x) \pm \sqrt{\gamma} A_{in}(x)\]  

(7.17b)

### 7.5 Post Breaking Phenomena

Another method for deriving the reduced system in Equation (7.15a) is the classical WKB method. It considers:

\[
\frac{i}{\hbar} \frac{\partial u^{(i)}}{\partial \xi} + u^{(i)} = \frac{1}{2} \frac{\partial^2 u^{(i)}}{\partial x^2} + V(x) u^{(i)}
\]  

(7.18)

And begins with the Ansatz that $u^{(i)}$ is in the form:

\[u^{(i)}(x,\hbar) = A(x,\hbar) \exp \left( \frac{i}{\hbar} S(x,\hbar) \right) + o(\hbar)\]  

(7.19)
Inserting this Ansatz into the nonlinear paraxial equation (7.18), and balancing the leading two powers of $\tau$ yields:

$$\frac{\partial k}{\partial t} + k \frac{\partial k}{\partial x} + \frac{\partial v}{\partial x} = 0$$  \hspace{1cm} (7.20)

$$\frac{\partial A}{\partial t} + k \frac{\partial A}{\partial x} + \frac{1}{2} A \frac{\partial k}{\partial x} = 0$$

Where $k = \delta s/\delta x$, which is equivalent to the reduced system in Eqn (7.15a) upon making the identifications:

$$\rho = A^2, \quad \mu = A^2 u$$  \hspace{1cm} (7.21)

The development of a singularity in this reduced system must then be interpreted as a breakdown in the Ansatz Eqn (7.19). In other words, after the break time the wave form no longer continues to resemble that of the Ansatz.

In linear theories such as quantum mechanics and classical electromagnetism, the presence of singularities in reduced systems and their consequences for the full system are well understood. For example, in quantum mechanics the characteristics of the reduced hyperbolic system Eqn (7.20) are the paths of classical particles in the conservation force field $F(x) = -\delta V(x)$, which is defined in terms of the prescribed potential energy function $V(x)$. In this manner classical mechanics arises as the semiclassical limit of quantum mechanics.

Singularities in the reduced semiclassical equations result from foci and envelopes of families of these classical paths. These envelopes separate regions in the $(x,t)$-plane that consists of points lying on only one classical path from regions consisting of points that lie on multiple paths. Along these envelopes (called caustics), neighbouring rays coalesce and geometric conservation law properties of the transport equation for the amplitude $A(x,t)$ force $|A|$, to diverge along these caustics.

More mathematically, in the linear case of quantum mechanics, the reduced hyperbolic system is degenerate in the sense that the eikonal equation for $K$ and the eikonal equation for $A$ have identical characteristics. Since the eikonal equation does not depend upon the amplitude $A$, it can be solved first and its characteristics are the classical paths. The transport equation for the amplitude $A$ is then integrated along these classical paths, and geometrical considerations
force $|A|$ to diverge along caustic envelopes. This divergence of the amplitude $A$ is a
direct consequence of the degeneracy of system in Eqn (7.20).

This caustic behaviour can be present even in the trivial case of the free paraxial equation
(with $V(x)=0$) provided one begins with compressional initial data such as:

$$u(x,0) = \partial_x S_h(x) = - \tan h(x)$$  \hspace{1cm} (7.22)

In this $V(x)=0$ case, the solution $u^{(i)}(x,t)$ of initial value problem Equation (7.14) can be
represented exactly as a Fourier integral:

$$u^{(i)}(x,t) = \frac{1}{\sqrt{2\pi i \tau t}} \int_{-\infty}^{\infty} \exp \left( \frac{i}{\tau} \left( \frac{(x-y)^2}{2t} + \sin(y) \right) \right) A_{\psi}(y)dy$$  \hspace{1cm} (7.23)

The behaviour of $\psi^{(i)}(x,t)$ as $\tau \to 0$, uniformly in $(x,t)$, can be obtained from an asymptotic
(stationary phase) evaluation of this integral. The result shows that, away from the
characteristic envelopes, $\psi^{(i)}(x,t)$ behaves asymptotically as the linear superposition:

$$u^{(i)}(x,t) \sim \sum_j \exp \left( \frac{i}{\tau} S^{(j)}(x,t) \right) A^{(j)}(x,t)$$  \hspace{1cm} (7.24)

Where the index $j$ in the sum runs over the different classical paths through the point $(x,t)$, $S^{(j)}$
is the classical path. $A^{(j)}$ is computed by integrating the transport equation along the $j^{th}$
classical path, with adjustments by phase shifts of the form $\exp (i n^{(j)} \pi / 4)$. The integers $n^{(j)}$ are
computed from the number of times the $j^{th}$ path touches the caustic envelope.

In the linear case for general $V(x)$, the classical paths are not straight lines and the asymptotic
behaviour cannot be calculated with Fourier theory. Nevertheless, the asymptotic behaviour
is still given by the superposition of Eqn (7.24). Using the method of characteristics for the
eikonal equation, one constructs a surface $u=u(x,t)$ over the $(x,t)$-plane. With this surface, one
integrates the transport equation for $A(x,t)$ and assembles formula in Eqn (7.24).

Arguments from the theory of uniform asymptotic expansions then show that formula in Eqn
(7.24) is asymptotically valid except at the focus and along the characteristic envelopes, the
only effect of which is phase shifts of integer multiples of $\pi / 4$.

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In the linear case, the qualitative consequence of formula in Eqn (7.24) is striking. Before the caustic, only one classical path passes through each space time point \((x,t)\); only one term appears in the sum in Eqn (7.24); the intensity \(|A(x,t)|^2\) is slowly varying on the \(\tau\) scale. After the caustic, three classical paths pass through each point \((x,t)\); three terms appear in the sum; the intensity \(|A(x,t)|^2\):

\[
|A(x,t)|^2 = \left| \sum_j \exp \left( \frac{i}{\tau} S^j(x,t) \right) A^j(x,t) \right|^2
\]  

(7.25)

This has rapid oscillations on the \(\tau\) scale due to phase interference between the three terms in the sum. Mathematically, these rapid oscillations prevent strong convergence after the caustic. In summary, in this linear case the weak limit can be constructed simply by summing over the contributions from each classical path, with interference between different terms in this sum causing rapid oscillations.

7.6 Nonlinear Simulations of Defocusing and Focusing Cases

Numerical results which illustrate the contrast formations of oscillations in solutions of Eqn (7.14) for the linear \((\gamma=0)\) the nonlinear defocusing \((\gamma>0)\) and the nonlinear focusing \((\gamma<0)\) cases are shown in this section.

The linear case \((\gamma=0)\) is considered, in order to illustrate the semiclassical linear theory just described. Figure (14.1) shows \(|u(x,t)|\) as a surface over the \((x,t)\) plane, clearly contains one focus of that theory, from which two caustics emanate.
Figure 14.1: The amplitude $|u'|$ for the linear ($\gamma=0$) equation as a surface over the $(x,t)$ plane, for $\tau=0.1$.

Figure 14.2a shows that the defocusing case is treated here with a repulsive nonlinearity ($\gamma>0$). As in the linear case, oscillations intensity form at specific points in space and time, and then persist.

Figure 14.2a: The amplitude $|u'|$ for the defocusing nonlinear paraxial equation as a surface over the $(x,t)$-plane, for $\tau=0.1$.

In the defocusing case these oscillations form two packets, one travelling to the right and the other to the left. The central region of the spatial profile that separates the two oscillatory regions is a quiescent plateau at times beyond the focus.
However when compared with the linear case, several distinguishing features arise from the nonlinearity. First, the amplitude $|u(x,t)|$ is much less intense at the focus than in the linear case, which is depicted from the same perspective, in Figure (14.2b).

For comparison, the result for the linear case ($y=0$) for the identical initial data and $r$ is depicted here. Note the change in vertical scale.

**Figure 14.2b:** The comparison result for the linear paraxial equation.

The response in the focusing case is treated here with an attractive nonlinearity ($y<0$), Figure (14.3a). Indeed, the defocusing case had the mildest response, followed the linear case, with the most extreme behaviour found with focusing nonlinearity. It is the focusing nonlinearity which supports solitons in one spatial dimension and which blows up in finite time in two or more dimensions.
The amplitude $|u'|$ for the focusing ($\gamma<0$) nonlinear paraxial equation as a surface over the $(x,t)$-plane, for $r=0.1$.

**Figure 14.3a:** The amplitude $|u'|$ for the focusing nonlinear paraxial equation.

The blow up of a portion of the oscillatory region shown in Figure (14.3b) admits the intriguing interpretation of these oscillations as a dense "sea of solitons", located near the centre of the spatial profile, with a sharp boundary separating the oscillatory from the quiescent regions of space.

**Figure 14.3b:** An enlargement for the focusing nonlinear paraxial equation.
The next chapter of this thesis will consider a generalised equation which will extend further from the nonlinear paraxial equation, to take care of higher order terms in soliton formation. This is derived by starting from the Maxwell equation for the electric field in a nonlinear media with an inhomogenous dielectric constant, to derive the generalised nonlinear paraxial equation with higher-order terms (as the perturbation terms) in an appropriate asymptotic sense. The method of developing a third order term in the generalised form is based on the asymptotic perturbation technique and gives a consistent scheme for the derivation of the generalised equation and the higher order corrections, which plays an important role to maintain structure of the soliton.

7.7 Conclusion

In conclusion, this chapter investigated a numerical and theoretical study of the generation, and propagation, of oscillations in the semiclassical limit (\(\tau \to 0\)) of the nonlinear paraxial equation.

In a general setting of both dimension and nonlinearity, the essential differences between the defocusing and focusing cases were investigated. Numerical comparisons of the oscillations are made between the solutions of linear and the nonlinear paraxial equation.

This chapter has presented several numerical simulations which illustrate and contrast the formation of oscillations in solutions of Equation (7.14), for the linear (\(\gamma = 0\)), the nonlinear defocusing (\(\gamma > 0\)), and the nonlinear focusing (\(\gamma < 0\)) cases.

In the first numerical simulation of Figure (14.1), the linear case of (\(\gamma = 0\)) was considered, in order to illustrate the semiclassical linear theory. Figure (14.1) illustrates \(|u(x,t)|\) as a surface over the \((x,t)\) plane, clearly contains one focus, from which two caustics emanate. Notice that \(O(\tau)\) wavelength oscillations in the intensity form at the focus and persist for later times. The spatial region containing these oscillations is bounded by the two caustics.

Figure (14.2a) treats the defocusing case with a repulsive nonlinearity (\(\gamma > 0\)). Figure (14.2a) shows that, as in the linear case, oscillations in the intensity form at specific points in space and time, and then persist. However, when compared with linear case, several distinguishing features arise from the nonlinearity. First, the amplitude \(|u(x,t)|\) is much less intense at the focus than in the linear case, which is depicted from the perspective, but with a different
vertical scale shown in Figure (14.2b). Second, while the oscillations are given to be $O(1)$ in amplitude and $O(\tau)$ in wavelength, this can be compared for the defocusing case.

In the defocusing case these oscillations form two packets, one travelling to the right and the other to the left. The central region of the spatial profile that separates the two oscillatory regions is a quiescent plateau at times beyond the focus.

Figure (14.3a) treats the focusing case with an attractive nonlinearity ($\gamma < 0$). The response in this focusing case is the most violent of the three graphical forms presented. Indeed, the defocusing case had the mildest response, followed by the linear case, with the most extreme behaviour found with focusing nonlinearity. It is the focusing nonlinearity which supports solitons in one spatial dimension and which blows up in finite time in two or more dimensions. The blow up of a portion of the oscillatory region illustrated in Figure (14.3b), admits the intriguing interpretation of these oscillations as a dense "sea of solitons" located near the spatial profile, with a sharp boundary separating the oscillatory from the quiescent regions of space.
CHAPTER 8
PROGRAMMING OF THE GENERALISED NONLINEAR PARAXIAL EQUATION FOR THE FORMATION OF SOLITONS (WITH MATHEMATICA)

8.1 Introduction

The field of nonlinear optics has developed in recent years, as nonlinear material has become available, and widespread applications have become apparent. This is particularly true for solitons and other types of nonlinear pulse transmission in optical fibres.

The purpose of this chapter is to describe the use of a very powerful tool to solve the generalised nonlinear paraxial equation, that has stable solutions called solitons. The solitary wave (or soliton) is a wave that consists of a single symmetrical hump that propagates at uniform velocity without changing its form. The physical origin of the solitons is the Kerr effect, which relies on a nonlinear dielectric constant, that can balance the group dispersion in the optical propagation medium. The resulting effect of this balance is the propagation of solitons, which has the form of a hyperbolic secant.

This chapter shows that the formation and propagation of solitons can be observed and understood by using a numerical routine which is implemented with the use of mathematica.

8.2 Nonlinear Optical Effects

The electric field $E$ in an optical fibre with the dielectric constant $\varepsilon$ satisfies the Maxwell equation:

$$\nabla \times \nabla \times E = -\frac{1}{c^2} \frac{\partial^2 \varepsilon}{\partial t^2} D$$

(8.1)
Where $c$ is the speed of light, and the displacement $D=\varepsilon * E$ may be given in the following form:

$$
(\varepsilon * E)(t) = \int_{-\infty}^{t} dt_1 \epsilon^{(0)}(t-t_1)E(t_1) + \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2 \int_{-\infty}^{t} dt_3 \epsilon^{(2)}(t-t_1,t-t_2,t-t_3) \\
\times [E(t_1), E(t_2), E(t_3)] + (\text{higher nonlinear terms})
$$

(8.2)

Here the second term indicates the Kerr effect, and the coefficients $\epsilon^{(0)}$, $\epsilon^{(2)}$, depend also on the spatial coordinates implying the inhomogeneity in dielectric constant. By virtue of the formula of the vector calculus, Eqn (8.1) can be written in the form:

$$
\nabla \cdot E - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} D = \nabla (\nabla \cdot E)
$$

(8.3)

It should be noted that $\nabla \cdot E$ in Eqn (8.2) is not zero, since $\nabla \cdot D = 0$ (the constraint for $D$ in Maxwell's equation) implies that:

$$
\varepsilon * (\nabla \cdot E) = -(\nabla \varepsilon *) . E \neq 0
$$

(8.4)

This enables that Eqn (8.3) cannot be reduced simply to a scalar equation. For the purpose of this chapter, Eqn (8.3) can be reduced in the sense of asymptotic perturbation method. This is done by writing Eqn (8.3) in the following form:

$$
L \cdot E = 0
$$

(8.5)

Where $E$ expresses a column vector, i.e. $E = (E_x, E_y, E_z)^t$, and in the cylindrical coordinates with the z-axis as the axial direction of the fibre, the matrix $L$ consisting of the three parts $L = L_a + L_{b} + L_{c}$ is defined by:

$$
L_a = \begin{bmatrix}
\nabla^2_\perp & \frac{1}{r^2} & -\frac{2}{r^2} \frac{\partial}{\partial \theta} & 0 \\
\frac{2}{r^2} \frac{\partial}{\partial \theta} & \nabla^2_\parallel & -\frac{1}{r^2} & 0 \\
0 & 0 & \nabla^2_\perp
\end{bmatrix}
$$

(8.6a)
\[
L_b = \begin{bmatrix}
\frac{\partial^2}{\partial z^2} & 1 & \frac{\partial^2}{c^2 \partial t^2} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]  
(8.6b)

\[
L_c = \begin{bmatrix}
\frac{\partial}{\partial r} & 1 & \frac{\partial}{r \partial \theta} \\
\frac{\partial}{\partial r} & 1 & \frac{\partial}{r \partial \theta} \\
\frac{\partial}{r} & 1 & \frac{\partial}{r \partial \theta} \\
\end{bmatrix}
\]  
(8.6c)

It should be noted that these matrices imply that:

\[
L_a E = \nabla_\perp^2 E = \left(\frac{1}{r} \frac{\partial}{\partial r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}\right) E
\]  
(8.7a)

\[
L_b E = \left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2 \partial t^2} \epsilon^*\right) E
\]  
(8.7b)

and

\[
L_c E = \nabla.(\nabla E)
\]  
(8.7c)

The electric field is considered as a monochromatic wave propagating along the z-axes with the wave number \( k \) and angular frequency \( \omega \), that is, the field \( E \) is assumed to be in the expansion form:

\[
E(r, \theta, x, t) = \sum_{l=\infty}^\infty E_l(r, \theta, \xi, \tau; \epsilon) \exp[il(kx - \omega t)]
\]  
(8.8)

With \( E_l = E_l^\star \) (complex conjugate) where \( k = \text{Im} \), \( \omega = \text{Re} \) and the summation is taken over all harmonics generated by the nonlinearity due to the Kerr effect and \( E_l \) \( (r, \theta, \xi, \tau; \epsilon) \) is the envelope of the lth harmonic changing slowly in \( x \) and \( t \). The slow variables \( \xi \) and \( \tau \) are defined by:

\[
\xi = \epsilon^2 x : \tau = \epsilon \left( t - \frac{X}{V_g} \right)
\]  
(8.9)
Where the small parameter $\epsilon (|\epsilon | << 1)$ expresses the order of nonlinearity (i.e., the order of the electric field) and $V_0$ is the group velocity of the wave.

In this scale of the coordinates, Eqn (8.9) is looking at a behaviour of the field in the balance between the nonlinearity and the dispersion which results in the forming of optical solitons confined in the transverse direction. From Eqn (8.8) and Eqn (8.9), the displacement is found by:

$$ D = \epsilon * E = \Sigma D_i \exp [k_i x - \omega_i t] $$

(8.10)

Equation (8.10) is given by:

$$ D_i (r, \theta, \xi, \tau; \epsilon) = \epsilon_i^{(0)} E_i + \epsilon_i^{(0)} \frac{\partial E_i}{\partial \tau} - \epsilon_i^{(1)} \frac{\partial^2 E_i}{\partial \tau^2} + \epsilon_i^{(2)} \left( \frac{\partial^3 E_i}{\partial \tau^3} + \frac{\partial^2 E_i}{\partial \tau^2} \right) + \epsilon_i^{(2)} \left( E_i \cdot \frac{\partial E_i}{\partial \tau} \right) + \epsilon_i^{(2)} \left( E_i \cdot \frac{\partial E_i}{\partial \tau} \right) + \epsilon_i^{(2)} \left( E_i \cdot \frac{\partial E_i}{\partial \tau} \right) + \epsilon_i^{(2)} \left( E_i \cdot \frac{\partial E_i}{\partial \tau} \right) + \epsilon_i^{(2)} \left( E_i \cdot \frac{\partial E_i}{\partial \tau} \right) $$

(8.11)

Where $\epsilon_i^{(0)}$ is the Fourier coefficient $\epsilon^{(0)} (\Omega)$ of $\epsilon^{(0)}(t)$ at $\Omega = \omega$, i.e. $\epsilon_i^{(0)} = \epsilon^{(0)} (\omega)$, and

$$ \frac{\partial \epsilon_i^{(0)}}{\partial \omega} = \epsilon_i^{(0)} (\Omega) = \epsilon_i^{(0)} (\omega) $$

(8.11a)

$\epsilon_{n \omega 3}^{(2)}$ is the Fourier coefficient $\epsilon^{(2)} (\Omega_1, \Omega_2, \Omega_3)$ of $\epsilon^{(2)}(t_1, t_2, t_3)$ at $\Omega = \omega_1$, $\Omega_2 = \omega_2$, $\Omega_3 = \omega_3$ and

$$ \frac{\partial \epsilon_{n \omega 3}^{(2)}}{\partial \omega} = \epsilon_{n \omega 3}^{(2)} (\Omega_1, \Omega_2, \Omega_3) $$

(8.11b)

and so on.

Therefore $E_i (r, \theta, \xi, \tau; \epsilon)$ can be expanded in terms of $\epsilon$:

$$ E_i (r, \theta, \xi, \tau; \epsilon) = \Sigma n \epsilon^n E_i^{(n)} (r, \theta, \xi, \tau) $$

(8.12)
Then from Eqns (8.5), (8.8), (8.9) and (8.11) at order $\epsilon$,

$$L_{i}E_{i}^{(1)} = 0$$  \hspace{1cm} (8.13)

Where $L_{i}$ is $L$ with the replacements:

$$\frac{\partial}{\partial x} = iK_{i}, \quad \frac{\partial}{\partial t} = -i\omega_{i}, \quad \epsilon^{*} = \epsilon_{i}^{(0)}$$  \hspace{1cm} (8.13a)

Note that the operator $L_{i}$ is self-adjoint, $L_{i}^{+} = L_{i}$, in the sense of the following inner product:

$$(U,V) = \int_{D} U^{*} V \, ds$$  \hspace{1cm} (8.14)

Where $ds = rdrd\theta$, $D$ is the cross section of the fibre, and $A^{+}$ is the adjoint of $A = (A_{ij})^{*} = (A_{ji})^{*}$. 

In Equation (8.13), there is only one bound state with the eigen value $K_{i}^{2}$ (i.e. $l = \pm 1$) and the eigen function $U = U(r,\theta)$ called the mode function describing the confinement of the pulse in the transverse direction and, in general, consists of two parts corresponding to the right and left polarisations. Then, the solution to Eqn (8.13) may be written as:

$$E_{i}^{(1)}(r,\theta,\xi,\tau) = \begin{pmatrix} u_{i}^{(1)}(\xi,\tau) U(r,\theta) \\ 0 \end{pmatrix} \quad \text{for } l = 1$$

$$\begin{pmatrix} 0 \end{pmatrix} \quad \text{for } l \neq \pm 1$$  \hspace{1cm} (8.15)

Here the coefficient $u_{i}^{(1)}(\xi,\tau)$ with $u_{i}^{(1)*} = u_{i}^{(1)}$ is a complex scalar function satisfying certain equations given in the higher-order equation of (8.5). From the equation $L_{i}U = 0$, the inner product $(U, L_{i}U) = 0$ give the linear dispersion relation $K_{i} = K_{i}(\omega_{i})$.

$$K_{i}^{2} = \frac{\omega^{2}}{c^{2}} (U, N_{0}^{2} U) + (U, L_{0} U)$$  \hspace{1cm} (8.16)

Where $n_{0} = (\epsilon_{i}^{(0)})^{1/2}$ is the index of refraction, and have assumed the normalisation for $U$ by $U_{x}^{2} + U_{y}^{2} = 1$.

At order $\epsilon^{2}$,

$$L_{i}E_{i}^{(2)} = i \left[ \frac{\partial L_{1}}{\partial \omega_{i}} + \left( \frac{1}{V_{g}} - \frac{\partial K_{i}}{\partial \omega_{i}} \right) \frac{\partial}{\partial \omega_{i}} \left( L_{1} \frac{\omega_{1}^{2}}{c^{2}} \epsilon_{i}^{(0)} \right) \right] \frac{\partial E_{i}^{(1)}}{\partial \tau}$$  \hspace{1cm} (8.17)
From which $E^{(2)}_l=0$ is obtained if $l=\pm 1$. In the case $l=1$, it is required that the inhomogeneous Equation (8.17) satisfies the compatibility (or integrability) condition:

$$ (U, L_1 E^{(2)}_l) = 0 $$ \hfill (8.18)

This gives the group velocity $V_g$ in terms of the linear dispersion relation Eqn (8.16):

$$ \frac{1}{V_g} = \frac{\partial k_1}{\partial \omega_1} $$ \hfill (8.19)

For $l=1$, Eqn (8.17) becomes:

$$ L_1 E^{(2)}_l = -i \frac{\partial L_1}{\partial \omega_1} \frac{\partial E^{(1)}_l}{\partial \tau} = i \frac{\partial L_1}{\partial \omega_1} \frac{\partial u^{(1)}_l}{\partial \tau} . U $$ \hfill (8.20)

From Eqn (8.13) for $l=1$, the solution of Eqn (8.20) may be found in the form:

$$ E^{(2)} = i \frac{\partial u^{(1)}_l}{\partial \tau} \cdot \frac{\partial U}{\partial \omega_1} + u^{(2)}_l U $$ \hfill (8.21)

Where $u^{(2)}_l = u^{(2)}_l (\xi, \tau)$ with $u^{(2)}_l = u^{(2)*}_l$ is a scalar function to be determined in the higher-order equation.

At order $\epsilon^3$,

$$ L_i E^{(3)} = \begin{cases} 0 & \text{if } l \neq \pm 1, \pm 3 \\ -\frac{9 \omega_1^2}{c^2} \epsilon^{(2)}_3 U^{(1)3}_1 (U, U) U & \text{if } l = 3 \\ -i \frac{\partial L_1}{\partial \omega_1} \frac{\partial E^{(2)}_l}{\partial \tau} + \frac{1}{2} \frac{\partial L_1}{\partial \omega_1^2} \frac{\partial^2 E^{(1)}_l}{\partial \tau^2} + \left( i \frac{\partial u^{(1)}_l}{\partial \tau} - \frac{1}{2} \frac{\partial^2 k_1}{\partial \omega_1^2} \frac{\partial u^{(1)}_l}{\partial \tau^2} \right) \\ \times \left[ \frac{\partial}{\partial K_1} \left( L_1 - \frac{\omega_1^2 n_0^2}{c^2} \right) \right] U \\ -\left| u^{(1)}_l \right|^2 \frac{u^{(1)}_l}{c^2} \frac{\omega_1^2 \epsilon^{(2)}_3}{c^2} (U, U) U, & \text{if } l = 1 \end{cases} $$ \hfill (8.22)

Where $\epsilon^{(2)}_l = \Sigma_{i1+i2+i3=l} \epsilon^{(2)}_{i1i2i3} \ (l_i = \pm 1)$. 

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Note that $\epsilon_1^{(s)}$ is a positive real number for the Kerr effect. From Eqn (8.22), one can obtain the solutions, $E_1^{(s)} = 0$ for $l = \pm 1$ or $\pm 3$, and since $L_0$ does not have the eigenmode:

\[ E_3^{(s)} = -\frac{9w_1^2}{c^2} u_1^{(1)} L_0^{-1} \epsilon_3^{(2)} (U_U) U \]  \hspace{1cm} (8.23)

which is a harmonic generated by the nonlinearity. For $L=1$ the compatibility condition is required:

\[ (U, L_1 E_1^{(s)}) = 0 \]  \hspace{1cm} (8.24)

from which the generalised nonlinear paraxial equation for $u_i^{(1)} (\xi, \tau)$ is obtained, (Kruskal: Private Communication):

\[ i \frac{\partial u}{\partial \xi} + \frac{1}{2} \frac{\partial^2 u}{\partial \tau^2} + u |u|^2 = -i \Gamma u + \frac{\partial^3 u}{\partial \tau^3} \]  \hspace{1cm} (8.25)

Now the new variables and constants are introduced:

\[ \xi = x : \tau = t - \frac{x}{V_g} \]  \hspace{1cm} (8.26)

\[ u = \frac{\omega_0}{2k_0} \sqrt{\frac{u_0 c_0 \psi}{V_g}} : V_g^{-1} = \frac{\partial k}{\partial \omega} = k' \]  \hspace{1cm} (8.27)

\[ \Gamma = \frac{u_0 c_0}{\sqrt{u_0 c_0}} \psi : \gamma = \frac{\sqrt{\omega_0 c_0 k'''} 6k'' \sqrt{2k_0 k''}}{6k'''} \]  \hspace{1cm} (8.28)

\[ k'' = \frac{\partial^3 k}{\partial \omega^2} : k''' = \frac{\partial^3 k}{\partial \omega^3} \]  \hspace{1cm} (8.29)

The importance of Equation (8.25) is that it can be solved into normalised reference coordinates. A clear view of the evolution of the envelope along the normalised propagation path results. This will also allow us to study the different cases, such as the classical situation, where $\Gamma = \gamma = 0$, which results in the standard nonlinear Schrodinger equation.
8.3 Initial Conditions

The solution of the nonlinear paraxial equation can be solved exactly by the inverse scattering method. A planar stationary light beam in a medium with a nonlinear refractive index can be described as a dimensionless form:

\[
\frac{i}{\xi} \frac{\partial u}{\partial \xi} + \frac{\partial^2 u}{\partial \tau^2} + k \, |u|^2 \, u = 0 \tag{8.30}
\]

The method used to solve the exact inverse scattering method is applicable to equations of the type:

\[
\frac{\partial u}{\partial \xi} = s^*[u] \tag{8.31}
\]

Where \( s^* \) is a nonlinear operator differential in \( x \), which can be represented in the form

\[
\frac{\partial L^*}{\partial \xi} = i \,[L^*,A^*] \tag{8.32}
\]

Here \( L^* \) and \( A^* \) are linear differential operators containing the sought function \( u(x,t) \) in the form of a coefficient.

If condition in Equation (8.32) is satisfied, then the spectrum of the operator \( L \) does not depend on the initial values with respect to time. The reconstruction of the function \( u(x,t) \) at any arbitrary instant of time is realised by solving the inverse scattering problem for the operator \( L^* \). The result in Equation (8.30) can be verified in Equation (8.32) with the operators \( L^* \) and \( A^* \) taking the form:

\[
L^* = i \begin{bmatrix} 1+p & 0 \\ 0 & 1-p \end{bmatrix} \frac{\partial}{\partial x} + \begin{bmatrix} 0 & \dot{u} \\ u & 0 \end{bmatrix}, \quad k = \frac{2}{1-p^2} \tag{8.33a}
\]

\[
A^* = \rho \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{\partial^2}{\partial x^2} + \begin{bmatrix} |u|^2 & iu_x \\ 1+p & -|u|^2 \\ -iu_x & -1-p \end{bmatrix} \tag{8.33b}
\]

Without the loss of generality, the assumption \( k > 2 \) and \( p^2 > 0 \) can be made on the above equation. As \( t \rightarrow \pm \infty \) of any initial condition, a finite set of solitons becomes present. In this
problem, an analogous role is played by the particular solutions of the paraxial equation:

\[ u(x,t) = 2\eta \tanh[2\eta (x-x_0) + 8\eta \xi \phi \exp(-2\xi x - 4(\xi^2 - \eta^2)t + \phi)] \]  

(8.34)

Where \( \eta, \xi, \phi, x_0 \) are scaling parameters. This form of the solution can also be known as a soliton that has a stable formation.

The soliton Equation (8.34) is the simplest representative of an extensive family of exact solutions of Equation (8.30), this can be shown in an explicit form.

In the general case such a solution can also be called a N-soliton solution, which depends on 4N arbitrary constants, \( \eta_i, \xi_i, \phi_i, x_{\eta_i} \). However for the non-coinciding \( \xi_i \) this solution breaks into individual solitons if \( t \rightarrow \pm \infty \). Using this solution and beginning at the origin \( x=0 \), a wave formation can be acknowledged by:

\[ u(0,t) = \tanh[t - t_0] \]  

(8.35)

### 8.4 Finite Difference Solution

In order to compute a valid solution, Equation (8.25) is converted to a finite difference equation using \( x = \xi \) and \( t = \tau \). The time discretisation will be indicated by an \( n \) superscript and the spatial position will have an associated integer subscript \( i \). Thus \( u(x,t) \) is denoted by \( u^n_i \). The various \( x \) values become \( i \Delta x \) where \( \Delta x \) is the mesh width and \( i=0,1,2,\ldots,l \). Similarly, the time variable becomes \( n \Delta t \) where \( \Delta t \) is the time step \( n=0,1,2,\ldots,N \).

Following a standard explicit procedure:

\[ \frac{\partial u}{\partial \xi} = \frac{u_i^{n+1} - u_i^{n-1}}{2\Delta x} \]  

(8.36a)

\[ \frac{1}{2} \frac{\partial^2 u}{\partial \tau^2} = \frac{u_i^{n-1} - 2u_i^n + u_i^{n+1}}{2(\Delta \eta^2)} \]  

(8.36b)

\[ \frac{|u_i^n|^2}{u_i^{n-1}} u_i^{n-1} \]  

(8.36c)

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\[ -i \Gamma u = -i \Gamma u_i^{n-1} \]  
\[ (8.36d) \]

\[ \frac{\partial^2 u}{\partial t^2} = \frac{u_i^{n-2} - 2u_i^{n-1} + 2u_i^{n-1} - u_i^{n-2}}{2(\Delta t)^3} \]  
\[ (8.36e) \]

The finite difference version for Equation (8.25) is found by:

\[ i \frac{u_i^{n+1} - u_i^{n-1}}{2\Delta x} + \frac{u_i^{n-1} - 2u_i^{n-1} + u_i^{n+1}}{2(\Delta t)^2} + \left| u_i^{n-1} \right|^2 u_i^{n-1} \]

\[ = -i \Gamma u_i^{n-1} - i \delta \frac{u_i^{n+2} - 2u_i^{n+1} + 2u_i^{n-1} - u_i^{n-2}}{2(\Delta t)^3} \]  
\[ (8.37) \]

The computer software Mathematica V.3 has all the tools to code this finite difference equation, to solve it and to produce the output graphics in two and three dimensions.

### 8.5 Numerical Simulation of Solitons

A numerical program in mathematica is implemented to investigate the formation and propagation of solitons, in which can be observed and understood by using a numerical routine to solve the nonlinear paraxial equation. A sequence of code has been developed to assist and explore in depth several features of the soliton’s solution. The sequence of code developed for mathematica is displayed in Appendix B of this thesis.

The solutions obtained by the numerical program of mathematica are all associated with the soliton that propagates along the x-coordinate with, and without losses respectively. These losses determine an attenuation change that causes a change of height and a change of width, which is illustrated in Figures (15.1) and (15.2) of this thesis. To have results that can be compared with recent publications, the following cases were selected for the analysis (Kodama and Hasegawa 463, 1994).
(1) **Propagation of a Soliton**

- **Without Losses ($\tau=0.0$):**

In this case, an initial hyperbolic secant pulse of unitary amplitude has the form $u(0,t) = \text{sech}(t-t_d)$ and follows the evolution of this pulse along the $x$ coordinate. This can be observed in Figure (15.1). The initial launched pulse propagates as a fundamental soliton without perturbation.

![Soliton propagation and formation with an initial launched pulse without perturbation.](image)

*Figure 15.1: Propagation of a fundamental soliton along the $x$-coordinate without attenuation.*

- **With Losses ($\tau=0.1$):**

Now we introduce an attenuation factor $\tau=0.1$ and use the same initial pulse shape as before. Figure (15.2) shows the effect of the attenuation on a fundamental soliton and suggests the use of amplification for long distance soliton transmissions.
Soliton propagation and formation with an initial launched pulse with an attenuation factor $\tau = 0.1$.

**Figure 15.2:** Propagation of a fundamental soliton with an attenuation factor $\tau = 0.1$.

(2) **Propagation of Solitons of Higher Order**

To observe the evolution of a second order soliton along the propagation path, a wave packet of the form $u(0,t) = 2\text{sech}\{t-t_0\}$ was launched. In Figure (15.3) the evolution of the second order soliton up to two locations along the propagation path can be seen.

In Figure (15.3a) the very beginning of the propagation of this launched soliton can be observed. Furthermore, Figure (15.3b) shows the same second order soliton up to a more remote location from the initial launching point. The characteristic periodicity of any higher order soliton propagations can be clearly observed as predicted by theory.

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Second order soliton evolution starting with a launched pulse \( u(0,t) = 2 \text{sech}(t-t_0) \) at the beginning of the soliton's propagation.

**Figure 15.3a:** Second order soliton propagation and formation at its beginning stage.

Soliton at a remote location from the initial launching point. Note the characteristic periodicity of the second order soliton.

**Figure 15.3b:** Second order soliton propagation and formation at a remote location from the initial launching point.
8.6 Conclusion

The formation and propagation of solitons has been observed and understood in this chapter, by implementing a numerical routine to solve the generalised nonlinear paraxial equation. A sequence of code has been developed to explore in depth several features of the soliton’s formation and propagation. The method of developing a higher order term in the generalised nonlinear paraxial equation, is based on the asymptotic perturbation technique, to give the soliton a stable nonlinear pulse, for which plays an important role to maintain the structure of the soliton.

Nonlinearity and dispersion effects involved in the propagation of optical solitons have been presented in graphical form, using a numerical routine for mathematica. This program is displayed in Appendix B of this thesis, and can be used extensively to study other combined effects of pulse propagation in nonlinear dispersive media. Results displayed in this chapter give a very clear idea of this interesting and important practical phenomenon.
CHAPTER 9
CONCLUSION

In conclusion, this thesis presented an investigation of the behaviour of a laser beam of finite diameter in a plasma with respect to forces and optical properties, which leads to self-focusing of the beam. The transient setting of ponderomotive nonlinearity in a collisionless plasma has been studied here in this thesis and consequently the self-focusing of the pulse and the focusing of the plasma wave occurs.

Evaluation of laser beam relativistic self-focusing lengths is dependent upon the accurate specification of the plasma refractive index, as well as the optical wave distortion process. Collisional-independent $l_p/d_o$ calculations utilising a refractive index exhibit roughly symmetric and convex downward distributions, with minimum values near the relativistic threshold of value $3.66 \times 10^{16}$ W/cm$^2$ for a Neodymium glass laser intensity.

Illustrated in Figure (12) of this thesis is the $l_p/d_o$ ratio as a function of Neodymium glass laser intensity for various $N$ values of 0.1, 0.9, 0.95, 1.0, 1.5, 3.0, 10.0. Calculations were made at initially narrow intensity increments, to accurately determine the high gradient self-focusing dependency. It can be seen from Figure (12) that as $N$ exceeds unity the $l_p$ limit is located further into the relativistic regime. In addition, the cusp-like distributions become more marked. It is obvious that extremely rapid self-focusing involving lengths of the same order, as the magnitude of the initial width of the beam between the half-irradiance maximum points, is generated by relativistic effects.

These results illustrated in this thesis are produced by a new numerical code written in C++, that incorporates both ponderomotive force in self-focusing mechanism and relativistic effects, at different plasma densities of various laser intensities.

From the solutions of the fully relativistic equations of motion of an electron in a linearly polarised plane wave field, the derivation of the nonlinear refractive found that higher order expressions provide alternating self-focusing and self-defocusing terms.

A numerical example is provided to compare the effectiveness of relativistic self-focusing as
compared to ponderomotive self-focusing process. The relativistic effect becomes important when the laser intensity exceeds $4 \times 10^{17}$ W/cm$^2$, for a longer wavelength laser, e.g. $\lambda = 1 \mu$m, the relativistic effect becomes important at a lower intensity.

The actual effect of relativistic self-focusing will probably be more severe than that described in this thesis. That is because the mechanism of relativistic self-focusing is based on a change of optical constant, it occurs almost instantaneously in the time of the order of $1/\omega$, while the ponderomotive self-focusing has a time delay due to electrons being expelled (i.e., plasma motion) from the high intensity region. When self-focusing occurs, the filament constricts and the trapped intensity in the filament becomes larger. Therefore, when self-focusing is initiated at a lower intensity, the ponderomotive force mechanism has the potential of inducing relativistic self-focusing. On the other hand, since relativistic self-focusing occurs almost instantaneously, this mechanism is most effective for very short pulse (~1 picosecond) laser-plasma interactions.

For extremely larger laser intensities, i.e. for $K >> 1$, $\gamma$ becomes very large, thus the refractive index tends to unity. This means that ultimately, the self-defocusing terms will balance out the self-focusing terms. The plasma will cease to behave as an optically refractive medium. When $K$ is large, the large axial ponderomotive force drives $\beta_x$ to approach unity, i.e. relativistic electrons in the direction of the laser beam will be generated.

In addition, to the results of this thesis, a numerical study of the semiclassical limit of linear and the nonlinear paraxial equation, in both the defocusing and focusing cases has been investigated. In a general setting of both dimension and nonlinearity, the essential differences between the defocusing and focusing cases has been presented, by several numerical simulations, which illustrate and contrast the formation of oscillation of the nonlinear paraxial equation.

The formation and propagation of solitons has been observed and understood in this thesis by implementing a numerical routine to solve the generalised nonlinear paraxial equation. Nonlinearity and dispersion effects involved in the propagation of optical solitons have been presented in graphical form, using a numerical routine for mathematica. This program can be used extensively to study other combined effects of pulse propagation in nonlinear dispersive media. Results displayed in this thesis are compared with recent publications (Kodama and Hasegawa 463, 1994) and give a very clear idea of this interesting and important practical phenomenon.
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127.


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APPENDIX A

This Appendix shows a numerical program in C++ for the behaviour of a laser beam of finite
diameter in a plasma with respect to forces and optical properties which lead to self-focusing.

/** Numerical Programming of Self-Focusing at Laser Plasma Interaction***/
/** Written by Frederick Osman , 1997**/

/* Include Libraries */
#include <stdio.h>
#include <math.h>
#include <conio.h>
#include <stdlib.h>

#include "tables.h"

/* Define Constants */
#define pi 3.141592654  /* Constant*/
#define c 2.997925E8    /* Vacuum Velocity of Light */
#define m 9.109E-31     /* Mass of Electron */
#define ec 4.803E-10    /* Electron Charge */
#define cl 1.63E-5      /* Constant for the Laser Power Threshold */
#define Relativistic_Threshold 3.66E18  /* Relativistic Threshold for Nd Glass */

/*****************************/
/* This Module Calculates the Refractive Index */
float RI (float I, float N)
{
    float n;  /* Floating point for the Refractive Index */

    n = (sqrt(fabs(((1-N*sqrt(1/I)+Relativistic_Threshold)))));

    return (n);
}
/*****************************/
/* This Module Calculates the Power Threshold */

float PowerTh (float n)
{
    float P;               /* Floating point for the Power Threshold */

    P = pow(pi*c, 2)*n*n*n*n/m/(c*ec* sqrtl (2/e)*c1*c1*(1+n*n));

    return(P);
}

/* This Module Calculates the Self-Focusing Lengths */

float LSF (float I, float N)
{
    float n, nI, nl2, lsf=0, lsfSquared;

    nI = sqrt(fabs(1-(N/sqrt(1+I/Relativistic_Threshold))));
    nl2 = sqrt(fabs(1-(N/sqrt(1+I/Relativistic_Threshold*2))));
    lsfSquared = (nI+nl2) / (nI-nl2);                /* Expression for the Self-Focusing equation */
    if (lsfSquared >= 0 )                           /* Positive Condition attained for the Self-focusing length */
        lsf = (sqrt(fabs(lsfSquared)))/2;

    return (lwf);
}

void main (void)
{
    menu ();

    /* This File includes the graphic modules that are called by THESIS.C */

    /* Include Libraries */

    #include <stdio.h>
    #include <conio.h>
    #include <stdlib.h>
    #include <math.h>

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float RI (float I, float N);
float PowerTh (float n);
float LSF (float I, float N);

/**************************************************************************/  
/**************************************************************************/

/* This Module Draws The Table of Results */
void DrawTable (int x)
{
  gotoxy(x,1); printf("x");
  gotoxy(x,2); printf("y");
  gotoxy(x,3); printf("R");
  gotoxy(x,4); printf("Th");
  gotoxy(x,5); printf("LSF");
  gotoxy(x,6); printf("PowerTh");
  gotoxy(x,7); printf("I");
  gotoxy(x,8); printf("N");
  gotoxy(x,9); printf("RI");
  gotoxy(x,10); printf("PowerTh");
  gotoxy(x,11); printf("LSF");
  gotoxy(x,12); printf("x");
  gotoxy(x,13); printf("y");
  gotoxy(x,14); printf("R");
  gotoxy(x,15); printf("Th");
  gotoxy(x,16); printf("I");
  gotoxy(x,17); printf("N");
  gotoxy(x,18); printf("RI");
  gotoxy(x,19); printf("PowerTh");
  gotoxy(x,20); printf("LSF");
  gotoxy(x,21); printf("x");
  gotoxy(x,22); printf("y");
  gotoxy(x,23); printf("R");
  gotoxy(x,24); printf("Th");
  gotoxy(x,25); printf("I");
}

/******************************************************************************/
/** This Module Lists out a New Page of Results */

void NewPage (int choice)
{

    int x;

clscr ( );
DrawTable ( 1 );
DrawTable ( 28 );
DrawTable ( 55 );

    for (x=3; x<=59; x=x+28 )
    {
        gotoxy ( x, 2 );
        printf ( "W/cm^2" );
    }
    switch ( choice )
    {
        case `1`
        {
            for (x=18; x<=70; x=x+26)
            {
                gotoxy ( x, 2 );
                printf ( "n" );
            }
            break;
        }
        case `2`
        {
            for (x=18; x<=70; x=x+26)
            {
                gotoxy ( x, 2 );
                printf ( "P" );
            }
            break;
        }
        case `3`
        {
            for (x=18; x<=70; x=x+26)
            {
                gotoxy ( x, 2 );
                printf ( "LSF" );
            }
            break;
        }
    }
    gotoxy ( 1,1 );

}
void RITable (int choice)
{
    int counter, x=0, y, lower=0, power=0, inc=0;
    float N;
    clrscr();
    gotoxy(10, 10);
    printf("Enter N:");
    scanf("%g", &N);
    gotoxy(10, 12);
    printf("Enter the lower limit of exponent:");
    scanf("%d", &lower);
    inc=1;
    power=lower;
    NewPage (choice);
    for (y=4; y<=25; y++)
    {
        gotoxy (x+15, y);
        printf("%.2g", inc*pow(10, power));
        gotoxy (x+15, y);
        switch (choice) {
        case '1'
            printf("%.4g", RI(inc*pow(10, power), N));
            break;
        case '2'
            printf("%.4g", PowerTh (RI (inc*pow (10, power), N)));
            break;
        case '3'
            {
                if ((LSF (inc*pow (10, power), N)) == 0)
                    printf("**");
                else
                    if (power > 30)
                        {
                            printf("-------------");
                            getch();
                            exit (0);
                        }
else
    printf("%.4g", LSF(inc*pow(10, power), N));
    break;
}
}
inc++;
}
if (((wherey() >=70) && (wherey() >=24))
{
    getch();
    if (power >36) exit(0);
x=0;
y=3;
}

}/*%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%*/

/*%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%*/
/* This function displays the menu */
int menu (void)
{
int choice = 0;
while (choice != 3)
{
    clrscr();
gotoxy(5, 5);
    printf("1 : Calculate Refractive Index");
gotoxy(5, 7);
    printf("2 : Calculate the power threshold");
gotoxy(5, 9);
    printf("3 : Calculate LSF");
gotoxy(5, 11);
    printf("4 : Quit");
    choice = getch();
    if (choice == '4') exit(0);
    RTable(choice);
}
return (choice);
}

/*%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%*/
APPENDIX B

This Appendix shows a numerical program in mathematica that investigates the formation and propagation of solitons.

(** Written by Frederick Osman, 1997 **)  

(*## Initialisation Routine ##*)

(*# Initial Data #*)

```mathematica
paso=40;p=20;dt=0.2;dx=0.002; (*Signal Amplitude*)
amp=1.0 (*Attenuation Factor*)
gamma=0.0; (*Time Division*)
max=64; (*Initial Pulse Definition*)
pulse=amp Sech[j-max/2]dt;
```

(*########################################################*)

```
der2=D[puls. {j,2}]; (*Second Derivative*)
der3=D[puls. {j,3}]; (*Third Derivative*)
Deriv2=Table[der2, {j,1,max}];
Deriv3=Table[der3, {j,1,max}];
Old=Table[N[puls. {j,1,max}];
Act=Old;uNew=Act;
Do[uAct[[k]]=Old[[k]]+2 I dx (I gamma uOld[[k]]+Abs[uOld[[k]]]^2 uOld[[k]]+0.5 uDeriv2[[k]]-I beta3 uDeriv3[[k]]), {k,1,max}];
```

(*########End of the Initialization Routine########*)

(* Numerical Routine *)

Do[
    Do[
        UNew[[k]] = uOld[[k]] + 2 I dx (I gamma uAct[[k]] + Abs[uAct[[k]]]^2 uAct[[k]] + 0.5 (uAct[[k + 1]] - 2 uAct[[k]] + uAct[[k - 1]] + 2 UAct[[k - 1]] - UAct[[k - 2]])/2 dt^3), {k, 3, max - 2}];
        uOld = uAct;
        uAct = UNew;
        If [Mod[n, paso] == 0, b[1 + n/paso] = Abs[UNew], {n, 0, (p - 1)}];

    (* End of the Numerical Routine *)

(* 3D Plotting Routine *)

(* Ticks Must be Adjusted for Each Case *)

c = ListPlot3D[Array[b, p], PlotRange -> All, ViewPoint -> {1.450, 1.750, 1.160},
    LightSources -> {{1, 0, 1}, RGBColor -> {1, 0, 8, 0}},
    {{0, 1, 0}, RGBColor -> {1, 0, 8, 0}},
    {{0, 0, 1}, RGBColor -> {1, 0, 8, 0}}],

    Boxed -> False,
    AxesLabel -> {"x", "x", "u(x, t)"},
    Ticks -> {{0, "-6.4"}, {20, "-2.4"}, {32, "0.0"}, {40, "1.6"}, {60, "-5.6"},
        {1, "0"}, {10, "0.8"}, {20, "1.6"}}, Automatic,
    PlotLabel -> FontForm["*Adjust for each case !!! *"]

U(0,t) = Sech(t - T0), Attenuation Factor = 0.0", {"Courier-Bold", 16}]

(* End of the 3D Plotting Routine *)

(* *)
APPENDIX C

This Appendix shows the calculation of the exact equation, second order approximation and fourth order approximation for $1/\gamma$ as a function of $K$.

The exact equation given by the derivation from Eqns (6.16), (6.21) and (6.12) is given to be:

$$\gamma = \left(1 + K^2 + \frac{1}{4} K^4\right)^{1/2} \quad (C1)$$

Where $K$ is given to be:

$$K = \frac{eE_0}{mc\omega} \sin \theta \quad (C2)$$

Known as the dielectric constant equation.

Consider the dielectric constant to be 0.1. Substituting this value in Eqn (C1):

$$\gamma = \left(1 + (0.1)^2 + \frac{1}{4} (0.1)^4\right)^{1/2} \quad (C3)$$

$\gamma = 1.005$

Therefore $1/\gamma = 0.9950248$.

The second order approximation is given from Eqn (6.22) for small values of $K$, which can be expanded to give:

$$\frac{1}{\gamma} = 1 - \frac{1}{2} K^2 \quad (C4)$$

Consider the dielectric constant to be 0.1. Substituting this value in Eqn (C4):

$$\frac{1}{\gamma} = 1 - \frac{1}{2} (0.1)^2 \quad (C5)$$

Therefore $1/\gamma = 0.995$.

The fourth order approximation is given for when a higher order approximation is called for. This is found by a numerical fit giving:
\[ \frac{1}{\gamma} = 1 - 0.5K^2 + 0.169K^4 \]  

(C6)

Consider the dielectric constant to be 0.1. Substituting this value in Eqn (C6):

\[ \frac{1}{\gamma} = 1 - 0.5 \times (0.1)^2 + 0.169 \times (0.1)^4 \]  

(C7)

Therefore \(1/\gamma = 0.9949831\).
APPENDIX D

This Appendix shows the calculation of the quiver velocity $|\beta y|$ and directed velocity $\beta x$ as a function of $K$.

Both the quiver and directed velocity components are given by substituting Eqn (6.22) into Eqns (6.16) and (6.21). The two equations are given in parametric form as functions of $K$. The quiver velocity is given to be as:

$$\beta y = -K / (1 + K^2 + \frac{1}{4} K^4)^{1/2} \quad (D1)$$

Consider the dielectric constant to be 0.1. Substituting this value in Eqn (D1):

$$\beta y = -(0.1) / (1 + (0.1)^2 + \frac{1}{4} (0.1)^4)^{1/2} \quad (D2)$$

Therefore $\beta y = 0.0995024$.

The directed velocity is given to be as:

$$\beta x = \frac{1}{2} K^2 / (1 + K^2 + \frac{1}{4} K^4)^{1/2} \quad (D3)$$

Consider the dielectric constant to be 0.1. Substituting this value in Eqn (D3):

$$\beta x = \frac{1}{2} (0.1)^2 / (1 + (0.1)^2 + \frac{1}{4} (0.1)^4)^{1/2} \quad (D4)$$

Therefore $\beta x = 0.0049751$. 

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APPENDIX E

Symbols, Notation, Fundamental Equations and Numerical Values

The SI units used throughout this thesis are employed from the following set of symbols and conventions for notation wherever possible.

Vector Notation
vector quantities: bold face italic
tensor quantities: bold face italic
orthonormal unit vectors $\hat{e}_x, \hat{e}_y, \hat{e}_z$
generalised spatial vector $x = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$
scalar product: raised bold face dot $\mathbf{a} \cdot \mathbf{b}$
vector product: bold face cross $\mathbf{a} \times \mathbf{b}$
gradient operator: $\nabla$
divergence: $\nabla \cdot$
curl: $\nabla \times$
length element $dl$
surface elements $dS$
volume element $d^3x$
velocity space element $d^3v$

Length Scales
Debye length $\lambda_D$
electron Debye length $\lambda_{De}$

Frequencies and Related Quantities
plasma frequency $\omega_p$
electron plasma frequency $\omega_{pe}$
ion plasma frequency $\omega_{pi}$
collision frequency $\nu_c$
electron-ion collision frequency $\nu_{ei}$
Coulomb logarithm $\lambda_c = ln \Lambda$
**Velocities**
velocity $v$
perpendicular velocity $v_\perp$
parallel velocity $v_\parallel$
thermal velocity $v_T$
electron thermal velocity $v_\text{te}$
phase velocity $v_{ph}$

**Distributions and Energy**
Maxwellian $\sim \exp \left(-v^2/v_T^2\right)$, hence $v_T = (2k_B T/m)^{1/2}$
distribution function $f(x,v,t)$
electron distribution function $f_e(x,v,t)$
temperature $T$
electron temperature $T_e$
Boltzmann's constant $k_B$

**Density and Mass**
number density $n$
electron number density $n_e$
ion number density $n_i$
mass density $\rho$
electron mass $m$
proton mass $m_p$
ion mass $M$

**Electric and Magnetic Quantities**
charge density $\rho_e$
electron charge $-e$
proton charge $e$
ion charge $Ze$
electric and magnetic fields $E, B$
unit vector along magnetic field $\mathbf{b}$
magnetic moment $\mu$
current density $\mathbf{J}$

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Fundamental Equations of Electricity and Magnetism

The Coulomb field:

\[ E = \frac{1}{4 \pi \varepsilon_0} \int \rho \frac{r}{r^3} \, d^3 \mathbf{x} \]

Lorentz force on an electron with charge \(-e\):

\[ F = -e \left( E + v \times B \right) \]

Maxwell's equations 'in vacuo':

\[ \nabla \cdot E = \frac{\rho}{\varepsilon_0}, \]

\[ \nabla \cdot B = 0, \]

\[ \nabla \times E = -\frac{\partial B}{\partial t}, \]

\[ \nabla \times B = \mu_0 \left( J + \varepsilon_0 \frac{\partial E}{\partial t} \right) \]

Here \( \mu_0 \varepsilon_0 = c^{-2} \), and numerical values are given below.

**Numerical values of constants**

\[ c = \text{speed of light in a vacuum} = 2.998 \times 10^8 \text{ m s}^{-1} \]

\[ \varepsilon_0 = \text{permittivity of a vacuum} = 8.854 \times 10^{-12} \text{ s}^2 \text{ c}^2 \text{ kg}^{-1} \text{ m}^3 \]

\[ \mu_0 = \text{permeability of a vacuum} = 4\pi \times 10^{-7} \text{ kg m c}^{-2} \]

\[ e = \text{charge of a proton} = 1.602 \times 10^{-19} \text{ C} \]

\[ m = \text{mass of an electron} = 9.109 \times 10^{-31} \text{ kg} \]

\[ m_p = \text{mass of a proton} = 1.673 \times 10^{-27} \text{ kg} \]

\[ m_p/m = \text{proton-electron mass ratio} = 1.836 \times 10^3 \]

\[ k_B = \text{Boltzmann's constant} = 1.381 \times 10^{-23} \text{ J K}^{-1} \]

\[ eV = \text{electron-volt} = 1.602 \times 10^{-19} \text{ J} \]

\[ eV/k_B = \text{temperature equivalent to 1eV} = 1.160 \times 10^4 \text{ K} \]

\[ G = \text{gravitational constant} = 6.673 \times 10^{-11} \text{ N m}^2\text{kg}^{-2} \]
Numerical Values of Plasma Quantities

The plasma parameters in the following expressions are in SI units; hence, for example, \( n_e \) is in units of \( \text{m}^3 \), \( B \) is in units of tesla, \( v_\perp \) is in units of \( \text{m s}^{-1} \), and \( T_e \) is in units of kelvin. Fundamental frequencies are:

\[
\omega_{pe} = \text{angular electron plasma frequency} = 5.65 \times 10^7 n_e^{1/2} \text{ rad s}^{-1},
\]
\[
\omega_{pp} = \text{angular proton plasma frequency} = 1.32 n_p^{1/2} \text{ rad s}^{-1}
\]

Note that frequencies in hertz are obtained by dividing the above expressions by \( 2\pi \). To convert the expressions above which apply to protons into expressions for ions of charge \( Z e \) and mass \( M \):

\[\omega_i \text{ multiply } \omega_{pp} \text{ by } Z (m_p/M)^{1/2}\]

Fundamental velocities and length scales:

\[v_{Te} = \text{thermal electron velocity}, \ (2k_B T_e/m)^{1/2} = 5.51 \times 10^3 T_e^{1/2} \text{ m s}^{-1}\]
\[v_{Ti} = \text{thermal ion velocity}, \ (2k_B T_i/M)^{1/2} = 1.29 \times 10^2 (m_p/M)^{1/2} T_i^{1/2} \text{ m s}^{-1}\]
\[\lambda_{De} = \text{electron Debye length}, \ (k_B T_e/m)^{1/2}/\omega_{pe} = 6.90 \times 10^1 T_e^{1/2} n_e^{1/2} m\]

Key vector identities

\[A + B = B + A\]
\[A \cdot B = B \cdot A\]
\[A \times B = - B \times A\]
\[A \cdot (B \times C) = C \cdot (A \times B) = B \cdot (C \times A)\]
\[A \times (B \times C) = B(A \cdot C) - C(A \cdot B)\]
\[(A \times B) \cdot (C \times D) = (A \cdot C)(B \cdot D) - (A \cdot D)(B \cdot C)\]

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Key results from vector calculus

**Gradient:** \( \nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \)

For a vector field \( A(x,y,z) \),

**Divergence:** \( \nabla \cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \)

**Curl:** \( \nabla \times A = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \)

\( \nabla \cdot (\nabla \times A) = 0 \)

\( \nabla \times (\nabla \times A) = \nabla (\nabla \cdot A) - \nabla^2 A \)

For a scalar function \( \phi(x,y,z) \),

\( \nabla \times \nabla \phi = 0 \)

\( \nabla \cdot \nabla \phi = \nabla^2 \phi \)

For a scalar function \( \phi \) in combination with a vector field \( A \),

\( \nabla \cdot (\phi A) = \phi \nabla \cdot A + A \cdot \nabla \phi \)

\( \nabla \times (\phi A) = \phi \nabla \times A + (\nabla \phi) \times A \)

For two vector fields \( A(x,y,z) \) and \( B(x,y,z) \),

\( \nabla \cdot (A \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B) \)

\( \nabla \times (A \times B) = A(\nabla B) - B(\nabla A) + (B \cdot \nabla) A - (A \cdot \nabla) B \)

**Divergence theorem:**

\( \int_V \nabla \cdot A \, d^3 x = \int_S A \cdot dS \)

**Stoke's theorem:**

\( \int_S (\nabla \times A) \cdot dS = \oint_c A \cdot dl \)
APPENDIX F

This Appendix shows some glossary terms which are helpful for the understanding of this thesis.

Ablation: Blow off of highly ionised material from a target irradiated by an incident laser or particle beam.

Absorption: The loss of light as it passes through a material, generally due to its conversion to other energy forms.

Attenuation: The phenomenon of the loss of average optical power.

Bernoulli Terms: Mathematical definition for the changes that occur between potential and kinetic energy. These formulas express wave motion.

Boltzmann’s Constant: A constant equal to the universal gas constant divided by the Avogadro number. It is approximately equal to $1.381 \times 10^{-23}$ J/K and is commonly expressed by the symbol $K_B$.

Bremsstrahlung: X-ray emission from electrons colliding with ions, for instance, produced by slowing down of electrons in an X-ray tube or from thermal motion in a plasma (which produces a continuous spectrum).

Caustic: A surface that envelopes a bundle of rays or bundle of normals to the wave surface. It may be observed as a hollow, luminous cusp in a turbid medium.

Dielectric Constant: A number that indicates the magnitude of the shift in a solid of positive and negative charges in opposite directions when a voltage is applied across the solid.

Dielectric Cylindrical Waveguide: Waveguide made up of a dielectric material, such as plastic of mica, in a cylindrical form, through which the waves travel.

Electromagnetic Wave: Wave of radiation identified by individual fluctuations of electric and magnetic fields.
Envelope: A rapidly oscillating wave that propagates with a characteristic constant shape, and can be pictured as cut off by a smoothly modulating envelope.

Fusion: 1. The combination of the effects of two or more stimuli in any given sense to form a single sensation. With respect to vision, the perception of continuous illumination formed by the rapid successive presentation of light flashes at a specified rate. 2. The transition of matter from solid to liquid form. 3. With respect to atomic or nuclear fusion, the combination of atomic nuclei, under extreme heat, to form a heavier nucleus.

Gaussian Beam: A beam of light whose electrical field amplitude distribution is Gaussian. When such a beam is circular in cross-section, the amplitude is $E(r) = E_0 \exp \left(-\frac{(r/\omega)^2}{2}\right)$, where $r$ is the distance from beam centre and $\omega$ is the radius at which the amplitude is $1/e$ of its value on the axis; $\omega$ is called the beam width.

Hydrodynamic Equations: Three equations which express the net acceleration of a unit water particle as the sum of the partial accelerations due to pressure gradient force, frictional force, gravitational force, and other factors.

Index of Refraction: The ratio of the velocity of light in a vacuum to the velocity of light in a refractive material for a given wavelength.

Intensity: Flux per unit solid angle.

Inverse Scattering Theory: The discipline that determines the nature of the scattering object, or an interaction potential energy, in a scattering process or collision, from knowledge of the amplitudes of the scattered fields.

Kerr Effect: A quadratic nonlinear electro-optic effect found in particular liquids and crystals which are capable of advancing or retarding the phase of the induced ordinary ray relative to the extraordinary ray when an electric current is applied. It varies as the square of the voltage.

Kinetic Theory: A theory which attempts to explain the behaviour of physical systems on the assumption that they are composed of large numbers of atoms or molecules in vigorous motion; it is further assumed that energy and momentum are conserved in collisions of these particles, and that statistical methods can be applied to deduce the particles' average
behaviour. Also known as molecular theory.

**Laser**: An acronym of Light Amplification by Stimulated Emission of Radiation. A laser is a cavity, with plane or spherical mirrors at the ends, that is filled with lasable material. This is any material, crystal, glass, liquid, dye or gas, the atoms of which are capable of being excited to a semistable state by light or an electric discharge. The light emitted by an atom as it drops back to the ground state releases other nearby, excited atoms, the light being thus continually increased in intensity as it oscillates back and forth between the mirrors. If one mirror is made to transmit 1 or 2 percent of the light, a brilliant beam of highly monochromatic, coherent radiation is emitted through the mirror. If plane mirrors are used, the beam is highly collimated. With concave mirrors the beam appears to emerge from a point source near one end of the cavity.

**Laser Fusion**: Irradiation of a target, principally a pellet containing deuterium and tritium, already compressed by pressures of tens of hundreds of atmospheres to increase its density, with terawatts of laser power to implode the target and release its energy in the form of heat. Carbon-dioxide lasers are the favoured means in current experiments, but excimers and free-electron lasers are also candidates.

**Laser Plasma**: A plasma produced by the interaction of an intense laser pulse with a material surface.

**Lawson Criterion**: Defines the minimum operational standards for a self-sustaining fusion reactor as equivalence between energy released per volume unit and kinetic energy per unit volume.

**Macroscopic Theory**: A theory concerning only phenomena observable with the naked eye or with an ordinary light microscope, and not with the behaviour of atoms, molecules, or their constituents which may underlie these phenomena.

**Magnetohydrodynamics (MHD)**: In astrophysics, the study of the interactions between a magnetic field and an electrically conducting fluid. The highly conductive interplanetary plasma is permeated by a weak magnetic field.
Maxwell's Equation: The mathematical equation showing that an oscillating electric charge gives rise to oscillating electric and magnetic fields that must travel outward in space at a given velocity, equal to the observed velocity of light. It is expressed as:

\[ c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \]

Where \( \mu_0 \) equals the magnetic permeability and \( \varepsilon_0 \) equals the electric permeability.

Microscopic Theory: A theory concerned with the interactions of atoms, molecules, or their constituents, involving distances of the order of \( 10^{-10} \) metre or less, which underlie observable phenomena.

Neodymium Glass Laser: An amorphous solid laser in which glass is doped with neodymium; characteristics are comparable with those of a pulsed ruby laser, but the wavelength of radiation is outside the visible range.

Nonlinear Force: Force in a plasma produced by the high-frequency field of an intense electromagnetic wave due to nonthermal direct dielectric interaction; generalisation of the ponderomotive or Lorentz force.

Permeability: The ratio of magnetic induction to magnetising force in a given medium.

Phase Velocity: The velocity of a point that moves with a wave at constant phase. Also known as celerity; phase speed; wave celerity; wave speed; wave velocity.

Plasma: A gas made up of electrons and ions.

Plasma Frequency: Electrostatic or Langmuir oscillation frequency of plasma electrons between ions; its value is \( \omega_p = (4\pi e^2 n_e/m)^{1/2} \) with charge \( e \), density \( n_e \), and mass \( m \) of the electrons.

Plasma Laser: Operates with light collectively emitted by the recombination of free electrons and ions in the plasma state.
**Plasma Physics:** The study of highly ionised gases. Many phenomena not exhibited by uncharged gases are associated with plasma physics.

**Relativistic Threshold:** Laser intensity at which the oscillation energy of the electrons moving coherently in the field reaches $mc^2$, where $m$ is the electron mass, and $c$ the speed of light.

**Self-defocusing:** The action of a medium whose index of refraction decreases with increasing optical intensity on a laser beam that is more intense in the centre than at the edges, whereby the profile of the refractive index corresponds to that of a negative lens, causing the beam to defocus.

**Self-focusing:** The action of a medium whose index of refraction increases with increasing optical intensity on a laser beam that is more intense in the centre than at the edges, whereby the profile of the refractive index corresponds to that of a positive lens, causing the beam to focus.

**Soliton:** Any isolated wave that propagates without dispersion of energy. Specifically to photonics, an ultrashort pulse of laser light that propagates through a waveguide without characteristic chromatic dispersion.

**Soliton Laser:** A colour centre laser whose output is coupled to an external control laser cavity and then fed back to the main laser cavity to provide enhanced pulse shaping, leading to ultrashort pulse generation.

**Threshold:** 1. In visual perception, the minimum value of stimulus that can be perceived on the average. 2. In optical detection systems, that signal level at which the probability of detection is 50 percent.

**Transparent:** Capable of transmitting light with little absorption and no appreciable scattering or diffusion.

**Waveguide:** A system or material designed to confine and direct electromagnetic waves in a direction determined by its physical boundaries.

**Waveguide Laser:** Gas laser in which the tube acts as a channel for the laser beam.
**Wavelength:** Electromagnetic energy is transmitted in the form of a sinusoidal wave. The wavelength is the physical distance covered by one cycle of this wave; it is inversely proportional to frequency.
NONLINEAR PARAXIAL EQUATION
AT
LASER PLASMA INTERACTION

by

Frederick Osman

A thesis submitted as part of the requirements
for the degree of Doctor of Philosophy

Faculty of Business and Technology
University of Western Sydney, Macarthur

March, 1998

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PLEASE NOTE

The greatest amount of care has been taken while scanning this thesis,

and the best possible result has been obtained.
ACKNOWLEDGEMENT

My sincere thanks go to my supervisor Dr Reynaldo Castillo for his constant supervision, which not only encouraged, but motivated me, to complete my research project.

Dr Reynaldo Castillo has been a positive guiding influence since the earliest undergraduate years.
STATEMENT OF AUTHENTICATION

The work presented in this thesis is, to the best of my knowledge and belief, original except as acknowledged in the text. I hereby declare that I have not submitted this material, either in whole or in part, for a degree at this or any other institution.

Frederick Osman
March 1998
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ABSTRACT

This thesis presents an investigation into the behaviour of a laser beam of finite diameter in a plasma with respect to forces and optical properties, which lead to self-focusing of the beam. The transient setting of ponderomotive nonlinearity in a collisionless plasma has been studied, and consequently the self-focusing of the pulse, and the focusing of the plasma wave occurs.

The description of a self-focusing mechanism of laser radiation in the plasma due to nonlinear forces acting on the plasma in the lateral direction, relative to the laser has been investigated in the non-relativistic regime. The behaviour of the laser beams in plasma, which is the domain of self-focusing at high or moderate intensity, is dominated by the nonlinear force. The investigation of self-focusing processes of laser beams in plasma result from the relativistic mass and energy dependency of the refractive index at high laser intensities. Here the relativistic effects are considered to evaluate the relativistic self-focusing lengths for the neodymium glass radiation, at different plasma densities of various laser intensities. A sequence of code in c++ has been developed to explore in depth self-focusing over a wide range of parameters.

The nonlinear plasma dielectric function due to relativistic electron motion will be derived in the latter part of this thesis. From that, one can obtain the nonlinear refractive index of the plasma and estimate the importance of relativistic self-focusing as compared to ponderomotive non-relativistic self-focusing, at very high laser intensities. When the laser intensity is very high, ponderomotive self-focusing will be dominant. But at some point, when the oscillating velocity of the plasma electron becomes very large, relativistic effects will also play a role in self-focusing.

A numerical and theoretical study of the generation and propagation of oscillation in the semiclassical limit $\tau \to 0$ of the nonlinear paraxial equation is presented in this thesis. In a general setting of both dimension and nonlinearity, the essential differences between the "defocusing" and "focusing" cases hence is identified.
Presented in this thesis are the nonlinearity and dispersion effects involved in the propagation of solitons which can be understood by using a numerical routine to solve the generalised nonlinear paraxial equation. A sequence of code has been developed to explore the depth of the soliton formation. These numerical routines were implemented through the use of the mathematica program, and results give a very clear idea of this interesting phenomena.