CHAPTER 1

FROM CURVES TO FIELD THEORY 1

1.1 INTRODUCTION

In order to work with and understand the language of physics, there needs to be an appreciation of the assembled mathematical tools and methods. These take the form of equations, theorems and algebras. These algebras can describe efficiently the relationship of the quantities in theories such as Relativity – both special and general, and Quantum Field Theory.

Isaac Newton faced this problem in the 1600s when trying to express quantities, i.e. displacement of bodies in elliptical orbits, changing with time. It was his need to discuss quantitatively and continuously changing, the fluctuating quantities, that lead to the development of calculus.

There is a need to express changes of one quantity against another or, perhaps, several quantities varying in relationship to each other. These variations can be actual particles, or components of fields, i.e. oscillating electromagnetic waves, or the curves on a graph of experimental data. These are often synonymous occurrences.

What follows is a brief review of the 'forms' of mathematics (the mathematical methods) that will enable a dynamic description of the quantities, objects and events of the physics to be dealt with; plasma physics. Especially giving background to the parameters and types of equations needed for a description of plasma fusion.

By understanding, it is also meant, the ability to use all notations involved. Further, to be able to see the pictures these tools paint of the universe. This understanding has a rich and deep history.
There are mathematical expressions, equations, that allow values to be found, trends to be discovered, as well as methods which allow for manipulation of quantities. For example, vectors are those quantities having magnitude and direction; the tools or methods of linear algebra allow us to relate vectors, while vector analysis employs the methods of differential and integral calculus to examine trends amongst them. Out of these methods comes tools, such as Stokes' theorem, the defining tool for the classical theory of differential forms; Stokes' theorem being named after British analyst and physicist, Sir George Gabriel Stokes, 1819-1903 (Borowski, 1989).

It could be argued that all mathematics is a tool for physics. Hence, the distinction is somewhat arbitrary so as to distinguish methodologies as opposed to equations arising from theorems, etc.

Once such an understanding is obtained the problems and theory initiated fusion and its maintenance can be undertaken. The nature of the laser beam, the structure of the plasma and the dynamics of both can be undertaken by researchers. This stands out as a major applied area of the above mentioned physics/mathematics dialogue.

1.2 OBJECT OF DESCRIPTION: DEFINING A CURVE IN TYPES OF SPACE

The objects or quantities studied in physics are, for most part, vector in nature. Our quantities (shaped by definition) help us express models, formed by experiments, which we want to express in the most universal way. Our vector quantities, such as force, will be assigned to each point \( x \) of region \( R \) in space. These vectors \( F(x) \) will make up a vector field.

The space itself can be viewed as a set of points having structure. If it is in rectangular coordinates specifying position and the space having \( S \) dimension (one for each degree of freedom), the space is termed coordinate space (Borowski, 1989). If the \( S \) dimensions (again \( S \) degrees of freedom) specify momentum, the space is termed momentum space (Borowski, 1989). Should the coordinates have \( 2S \) dimensions (again for \( S \) degrees of freedom), then the coordinates represent position and momentum. The space is termed phase space. Indeed, this set of points can be
expressed in other coordinates, spherical or cylindrical. The space could be a geometrical construct or object in reality, or the imagination. When a collection of objects form a set, again either to express and actual object (more precisely, the surface of the shape), or one of the imagination, we have a manifold. The points here are defined quite rigidly, having 1-1 correspondence with a set of $S$ of $n$-tuples of real number $(u^1, u^2, \ldots, u^n)$; these are the coordinates. There are coordinate transformations to be defined, with their inverses, non-vanishing Jacobians, viz

$$\frac{\partial (u^1, \ldots, u^n)}{\partial (u^1, \ldots, u^n)} = \det \left( \frac{\partial u^i}{\partial u^j} \right) \neq 0$$

to be satisfied (Lipschutz, 1969). All in all, a very involved definition. This definition can be put in simple terms as, a manifold is a topological space which is locally Euclidean (Marsden & Tromba, 1981). At this point, one of our first objects emerges. We want to see how these quantities change, or are affected by being translated or rotated or reflected, simply moved about in space, or across a manifold. The line traced out by this makes up a 'curve'. The Euclidean group of rotation, say, is studied using vector calculus. Curves go from being lines between points to differential mapping (application between two sets). Tensors are tools used to describe problems involving curved objects, two dimension and up (Einstein, 1962; Flanders et al, 1970; Landau & Lipshitz, 1988; Lipschutz, 1969; Rindler, 1977)

$$dx : [a, b] \rightarrow R$$

$$dx : [a, b] = \int_{a}^{b} dx$$

From this, it can be stated that the curves spoken of are sets of points:

$$C = \{(x_1, x_2, \ldots, x_n) \in R^n / f(x_1, \ldots, x_n) = C\}$$

How do our quantities change? Change with respect to time, position, angle, etc; the rate of change, the derivative; the direction of our 'curve' takes and the speed with which it does it. Graphs can be drawn of these changes. Graphs can come from the derivative and be shown by a line segment at the point (see Fig 1.1).
These line segments are short and collectively resemble the way we draw a magnetic field formed by iron filings spread on paper over a magnet. Here, we can intuitively appreciate how curves will come to represent fields. If the lines are 'dense' enough, a surface shape, or manifold, will emerge.

These sets of functions are related if their meanings have the same direction i.e. $f'(x) = g'(x)[f(x) - g(x)]$, the difference function has zero derivatives

$$[f(x) - g(x)]' = f'(x) - g'(x) = 0 \quad \text{(see Fig 1.2)}$$

This can be further 'universalised' by parameterisation of the lines. Equations express geometric shapes (lines, planes in Euclidean space), so $l(t) = a + tv$, at $t = 0, l = a$; with increasing $t$, $l$ moves away from $a$. It is very useful to use time as a parameter, $r = r(t)$ in the Cartesian system as it forms a linear system. This also allows the introduction of other coordinate systems. Then, we need to know if the relationships expressed by our equations are invariant as the coordinates are changed, or the 'shape' transformed.
An understanding of the following is sought

\[ \Psi(x, y, z, t) \rightarrow MANIFOLD \]

\[ \psi \text{ set of numbers} \]

\[ \rightarrow x_1, x_2, \ldots, x_n \in \mathbb{R}^n \]

coordinated numbers correspond to a set of numbers

DIFFERENTIAL

MANIFOLD OR SPACE

Usually curves, as in \( \mathbb{R}^n \), are thought of as a set of values of a mapping (function) on an interval \([a, b]\) into \( \mathbb{R}^n \). This is often referred to as a path, a path on a plane is \( n=2 \), in space if \( n=3 \). The image of a path may be the line or curve sort. Paths may be trajectories; they are defined as quantities, such as velocity and acceleration, and so Newton's Laws are used to describe planetary motion (Fig 1.3). So, it can go back and forth from the image of the one-dimensional parametric axis in three-dimensional space, to the one-dimensional line.

Map \( \sigma \) is the path, it's image is the curve, that we 'see'.

Thus, there are three ways in the plane to represent curves

(i) parametrically \[ x(t) = (t, f(t)) \]
(ii) as graphs of function \[ y = f(x) \]
(iii) as the graphs of equation \[ F(x, y) = 0 \] (Flanders et al, 1970)

So, curves \( r : \mathbb{R} \rightarrow \mathbb{R} \) (Fig 1.4), parameterised using \( t \) (time) are for material to be described the most useful mode of description.
\[ r(t) = (x^1(t), x^2(t), x^3(t), \ldots, x^n(t)) \] \hspace{1cm} - 1.2a

Now if the derivative is found of position vectors, the velocity vectors result (Fig 1.5) (Goldstein, 1980; Harper, 1978; Landau & Lifschitz, 1988; Lipschutz, 1969; Martin, 1991).

![Fig - 1.4](image)

![Fig - 1.5](image)

The velocity vectors at

\[ [\sigma(t) \text{ is } v(t) = \sigma^1(t) = (x^1(t), y^1(t), z^1(t))] \] \hspace{1cm} - 1.2b

The speed is obtained from \( S(t) = \| \sigma'(t) \| \) which is the length of \( \sigma'(t) \) (Marsden & Tromba, 1981). If this is derived again

\[ a(t) = \sigma''(t) = (x''(t), y''(t), z''(t)) \] \hspace{1cm} - 1.2c

the quantity of acceleration results. Thus, Newton's second law becomes

\( F(\sigma(t)) = ma(t) \). A planet moving around the Sun would obey:

\[ mr'(t) = \frac{GmM}{\| r(t) \|^3} r(t) = -\frac{GmM}{r^3} r(t) \] \hspace{1cm} - 1.3

How much energy or work is used to move the path taken by the curve \( t \in I \subseteq R \)
\[ g(n) = f(r, \lambda) \] Initial conditions

\[ W = \int_{x}^{\delta} f(x,a)da \] - 1.4(a)

\[ = \int_{x}^{\delta} (r \cos a, r \sin a) rda \] - 1.4b

\[ r \rightarrow x^1, x^2, x^3 \]

So, it speaks about the speed with which the paths are transversed. The set of all these tangent vectors makes up the target space. The set of components of the directional derivative is a vector space (Fig 1.6).

\[ \frac{d}{dt} (f; r) = \frac{\partial}{\partial x^i} \frac{dx^i}{dt} \]  

Referring to the previous Fig 1.6, it gives the following definition of a vector:

Directional derivative around

the curve (this will later involve processes like Stokes' Theorem),

objects moving along the path

\[ \rightarrow \frac{d}{dt} = \frac{(dx^i)}{(dt)} \frac{\partial}{\partial x^i} \leftarrow \text{Basis of a vector} \]

\[ \uparrow \text{Component of a vector} \]

Vectors have directional derivatives, hence the application of mathematical analysis to investigating a scalar field \( U(M) \) makes it possible to describe its local properties, i.e. the variation of \( U(M) \) in passing from a given point \((M)\) to the point \(M'\) lying close to \(M\).

A vector field \( A(M) \) is said to be a potential if it can be represented as a field of gradients on a scalar field \( U(M) \), i.e. \( A = \text{grad} \ U \) (Fig 1.7). This is useful as it is position and momentum that specify a physical system (Martin, 1991).
Tangent vector = derivative
\[ t = \frac{dr}{dt} \]
Direction of tangent vector is some different length.

\[ f(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]

Fig. 1.7

Now, two points of importance:

1. The need for physical systems to exist in a 4-dimensional system (Minkowskian space-time) (Einstein, 1962), and sometimes 10-13 dimensions (string theory). Can these dimensions reduce or "pullback" the system to a lesser set of dimensions without loss of generality or meaning? (Bressoud, 1991) (Fig 1.8)

2. Given this "pullback", does it alter the physical law? Indeed, does altering the frame of reference (or totally generalising it), or transforming curves alter the quantities? Are all the laws of physics invariant? (Bressoud, 1991; Marsden & Tromba, 1981)

Fig. 1.8
"The complete description of a system is the specification of all trajectories for all particles, together with a full description of all fields at all time".

For quantum mechanics, invariant means the probability amplitude is invariant (Martin, 1991).

\[ S \rightarrow S' \]

\[ \psi_o \psi_o^\dagger \Rightarrow |<\psi_o | \psi_o >|^2 = |<\psi_{o1} | \psi_{o1} >|^2 \]

- 1.6

\[ \psi_o^\dagger = u \psi_o > \quad u = \text{unitary} \]

In the tools, it is required that transformations, such as the Lorentz transformation, be chosen over the Galilean. Further, the parameterisation can not affect the 'curve' either. A vector could be tangent to an infinite number of different curves (Fig. 1.9). The set is called a "manifold". In this, there is no concern with geometry, as is commonly conceived. Point P in the manifold - mapping P set of numbers.

Plane is tangent to the curve.

\[ \frac{d}{dx} = \sum_i a \frac{dx^i}{d\eta} + b \frac{dx^i}{d\mu} \frac{\partial}{\partial x^i} \]

We will say that the tangent vector \[ \frac{d}{d\eta} \]

has components \[ \frac{(dx^i)}{d\eta} \] in the basis \[ \frac{\partial}{\partial x^i} \].

e.g - 4-D in Rel. phase space of Q,M, absolute space of Newtonian Mechanics.

Fig - 1.9
\{x'(P), x^2(P)...x^n(P)\} is associated to \(P\) in \(P^n\). This is called the coordinates of point \(P\). Sets of equations such as

\[
\begin{align*}
y'^i &= y'^i(x^1, x^2, x^3,...x^n) \\
y'^n &= n^n(x^1, x^2, x^3,...x^n)
\end{align*}
\]

- 1.7

define a coordinate transformation. If \(y \in C^k\) coordinates are called \(k\) related. (Set of all continuous derivatives till order \(k\).) If all the charts (Fig. 10) in the manifold are \(k\)-related, the manifold is called 'differentiable manifold' (Borowski, 1989; Dodson & Poston, 1997; Kay, 1998)(Fig. 11).

![Fig 1.10a](image-url)

**Fig 1.10a**

Several charts = the set of charts \(\Rightarrow\) atlas of the manifold

![Fig 1.10b](image-url)

**Fig 1.10b**

Basically, there are two types of space to work with (Fig.12), and as general a set of coordinates as possible (perhaps one free of coordinates, related to a specific frame of reference), and a set of mappings which keeps the physical entities invariant. Needed are mathematical tools to describe these spaces and transformations. Vector calculus is efficient for things like Maxwell's electromagnetic waves (till considered in 4-D space) and Euclidean geometry is nice for vectors, of the everyday. More general geometries and all continuous transformations, that leave invariant a physical law, more powerful tools are needed to handle the problem and this tool is provided by the language of tensors, as formulated by Einstein, Cartan, Christoffel and Poincaré, or
differential forms (Bressoud, 1991) (modern form). Perhaps needed is a tool that does not just deal with points, but looks at the functions, the mappings, "bundles" of curves – Lie Algebra.

Coordinate chart for every point bar the North Pole

Phase Space (usually 2-D)

Generalised Coordinate

Position $o/q_i$ particle. Union of Coordinate space and momentum space.

Configuration of space. Localising a particle.

Generalised Coordinate

Position of $t$ (time) the particle Vector space (S-D)

- Fig 1.11

- Fig 1.12
1.3 THE METHODS, THE TOOLS

"The general laws of nature are to be expressed by equations which hold good for all systems of coordinates, that are covariant with respect to any substitution whatever (generally covariant)"

- Albert Einstein

"Die Grundlagger algemeinan Relativitstheore"

(Bressoud, 1991)

The first thing needed is to find theorems. Theorems that talk about the sort of well-behaved curves that make up our fields. Three theorems come to mind:

\[
\text{Green's : } \left[ \int_{\partial \Omega} P \, dx + Q \, dy = \int_{\Omega} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy \right] \quad - \text{1.8}
\]

\[
\text{Stokes' : } \left[ \int_{\alpha} \text{curl } F \cdot ds = \int_{\partial \alpha} \nabla \times F \cdot ds = \int_{\alpha} \nabla \cdot F \cdot ds \right] \quad - \text{1.9}
\]

\[
\text{Gauss' : } \left[ \int_{\alpha} \nabla \cdot F \, dV = \int_{\partial \alpha} F \cdot ds \text{ or } \int_{\alpha} \text{div } F \, dv = \int_{\partial \alpha} (F \cdot n) \, ds \right] \quad - \text{1.10}
\]

Now there is a need to express these theorems in a more succinct, and general, way. General, as to be applied to \( k \) dimensions. The theory of differential forms provides such a tool.
1.4 DIFFERENTIAL FORMS

Differential forms make up part of the work on surface integration – straight away the connection to Stokes' theorems is apparent. A differential form of degree \( r \) in the \( k \) variables is a mapping from a domain in \( k \)-space into a set of \( r \)-covectors (Bressoud, 1991). This terminology is essentially equivalent to tensor analysis, but more of that later. They were first truly described in 1899 by Elie Cartan (Bressoud, 1991), while "tensor analysis" was Einstein's invention (Bressoud, 1991). It could be argued with Cartan's help, especially re absolute parallelism?

The differential forms are \( O \)-forms – real valued functions, a scalar fields:

\[
f : K \rightarrow \mathbb{R}
\]

upward to \( k \)-forms (hypervolumes). See Table 1 to quote p163, V.I. Arnold, *Mathematical Methods of Classical Mechanics*; 2nd Ed, Springer 1989, "Hamiltonian mechanics cannot be understood without differential forms".
### TABLE 1
Differential Forms
(Arnold, 89; Bressoud, 91; Marsden and Tomba, 81)

<table>
<thead>
<tr>
<th>FORM</th>
<th>DESCRIPTION</th>
<th>REMARKS</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>A real-valued function $f: K \rightarrow \mathbb{R}$, $K$ being an open set in $\mathbb{R}^3$. A scalar field. A point denoted by a 3-D vector.</td>
<td>A mapping from a point in the given space to a real number. $f(x) = x^4, f(x, y, z) = \frac{x + y - 2}{x^2 + y^2 + 2} - 2$</td>
</tr>
<tr>
<td>1</td>
<td>The expressions $dx$, $dy$, $dz$. A formal linear combination, the form $\omega$ on open set $K$; $\omega = P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz$, or $\omega = Pdx + Qdy + Rdz$. $P$, $Q$ and $R$ being real valued functions. A rule assigning a real number to each curve.</td>
<td>A mapping from a set of finite intervals to the reals. $G(x): [a, b] \rightarrow \int g(x)dx \in \mathbb{R}$</td>
</tr>
<tr>
<td>2</td>
<td>The expressions $dxdy$, $dydz$ and $dzdx$. Consider them as products. They reproduce in cyclic order $\begin{array}{c} \text{d}x \ \text{d}y \ \text{d}z \end{array}$</td>
<td>$\eta = Fdxdy + Gdydz + Hdzdx$</td>
</tr>
</tbody>
</table>

(see Fig. 13)
In synchronous order, the expressions $dxdydz$.

Rule assigning a real number to each elementary sub-region. Real numbers to curves, surfaces and regions, contained in formal expressions defined

$d x_1, d x_2, ... d x_k$, a mapping form oriented $k$-simplices. A simplex being the most elementary geometric figure of a given dimension.

point = O-simplex
line segment = 1-simplex
triangle = 2-simplex
tetrahedron = 3-simplex

oriented $k$-simplex - $k$-D, defined by $kH$ ordered points; to their signed hypervolumes.

Hypervolumes of the standard, $k$-simplex is $\frac{1}{k!}$ = $\frac{1}{1.2...k}$

$v = f(x, y, z)dxdydz$

$v$ being a 3-form on an open set $KCR^3$.

$f$=real valued function on $K$.

$\int_T dxdydz = \frac{1}{6} (b-d)(c-a)x(d-a)$

Can be evaluated by "pullbacks" in 4-D.
Can be evaluated by "pullback".
Can be multiplied by a constant.

$\int_T d x_1, d x_2 = \int_u D(a_1, -a_2, ... a_k - a_0)dt
= \frac{1}{k!} D(a_1, a_2, ... a_k - a_0)$

where $D(a_1, a_2) = \Sigma sgn(\sigma) \Pi a_{i0(k)}$

Can be multiplied by a constant.

---

The technique best suited to simplify differential forms and to 'pull down' the number of dimensions is the "pullback". The technique involves parameterisation. It allows us to take, or pull, a differential form in $x, y, z$ space (or more) and take it back (move) to a differential form in $t$ space. Each form has its own version of the technique; summarised by the "Remarks" column for $k$-forms. A "pullback" then is a transforming of different coordinate planes.
Differential forms constitute an algebra, the rules that govern their relationships. These rules are matters such as, distributivity, associativity, how and what you can add or multiply together. For example, a $k$-form and a $j$-form (if $k \neq j$) can never be added. New forms arise from their multiplication operation or exterior product, sometimes termed "wedge product", vis, $dx \wedge dy = dxdy$, the "wedge" emphasises (Bressoud, 1991) the anticommutativity (foreshadowing their relationship to Lie Algebra)

$$\omega \wedge \eta = (-1)^{k_1}(\eta \wedge \omega)$$

But, there is still a need to describe curves, bundles of curves, being transformed objects with multiple actions (like forces) on them, or the forces themselves, where there are more than three degrees of freedom for each facet.

With differential forms, the three theorems mentioned earlier look like:

Green's Theorem : $\int_{\partial \omega} \omega = \int_{\omega} d\omega$ \hspace{1cm} - 1.12

which means

$$\int_{\partial \omega} Pdx + Qdy = \int_{\omega} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

$d\omega$ being a 2-form

Stokes' Theorem : $\int_{\partial s} \omega = \int_{s} d\omega$ \hspace{1cm} - 1.13

Gauss' Theorem : $\int_{\partial \Omega} \eta = \int_{\Omega} d\eta$ \hspace{1cm} - 1.14

Going from the curl for regions in $R^2$ for Green's theorem, to surfaces in $R^3$ for Stokes' theorem, to the divergence for regions in $R^3$ for Gauss' theorem.
Highlighting Stokes' theorem:

Let $S$ be an orientated surface in $\mathbb{R}^3$ with a boundary consisting of a simple closed curve $\partial S$ (Fig. 14) orientated as the boundary of $S$ (Fig. 15). Suppose that $\omega$ is a $|k-1|$-form on some open set $K$ that contains $S$. Then:

$$\int_{\partial S} \omega = \int_S d\omega$$

- 1.15

This comes from a unifying of the three theorems, by using an orientated $k$-manifold with boundary in $\mathbb{R}^3$, the surface (Marsden et al, 1981).

The tools of vectors and differential forms deal with linear functions, three degrees of freedom. There is a need to deal with multi-linear functions (indeed, non-linear as well). The natural extension, then, is the tensor.
1.5 Tensors

Tensor – the word derives from elasticity theory, "tensor", that which exerts tension (Kay 1988, Simmonds, 1994). They are bilinear forms, multi-linear forms having real-valued functions of several vector variables, being restricted to linearity conditions (Gerretsen, 1962).

A multi-linear form \( \gamma = \gamma(x, y, z, ..., \omega, ...) \), the value of \( \gamma \) depends only on the values of its vector arguments. The aim of being free, a particular coordinate system is realised (Abivis et al, 1977). An \( n^{th} \) rank can be stated simply as a quantity having \( 3^n \) components \( T_{ijk...} (N \text{ indices}) \) (Goldstein, 1980); thus, scalars are tensors of rank zero, vectors – first rank.

To restart and state the above more in terms of chapter, forms and vectors – let \( M \) be a manifold. A tensor of the type \( \begin{pmatrix} M \\ n \end{pmatrix} \) at \( P \) is a multi-linear function of \( m \) forms, and \( n \) vectors is \( F[\omega \sigma + \beta \sigma, \gamma \nu + \delta \omega] \) is a \( \begin{pmatrix} 2 \\ 2 \end{pmatrix} \) tensor. Vectors then are \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) tensors, forms are \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) tensors, while scalars are \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) tensors.

While by linear, it is meant:

\[
F[d + \beta \sigma, \gamma \nu + \delta \omega] = dF[\omega, \sigma] + \beta F[\sigma, \nu] + \gamma F[\nu, \omega] + \delta F[\omega, \sigma]
\]

- 1.16

The parts are 'related', or have an isomorphism through the concept of a dual basis: consider at a point \( P \) of the manifold the coordinate system \( \{x^1, x^2, ..., x^n\} \). The conical (natural) basis associated to that point is \( e_i \frac{\partial}{\partial x^i} \), then every vector \( \mathbf{v} \) can be written as:

\( \mathbf{v} = v^i \frac{\partial}{\partial x^i} \). Reciprocally, the 1-form in \( P \) can be written as: \( \omega = \omega_i dx^i \), the isomorphism being:
\[
\frac{\partial}{\partial x^i} \quad dx^i \\
\frac{\partial}{\partial x'^i} \\
\frac{\partial}{\partial x'^n} \quad dx'^n
\]

- 1.17a

\[\text{hence, } \dim \left[ \frac{\partial}{\partial x^i} \right] = \dim [dx^i] \]  
- 1.17b

Keeping with the idea of independence from a coordinate system, a frame of reference, the tensor can be now contracted without altering the basis used, and produce a new tensor. This is done by cancelling an index.

\[\omega = \omega_i dx^i \]  
- 1.18a

\[T[\omega, \nu] = \omega_i \nu^i \] are the components of a \[\begin{bmatrix} 1 \\ 1 \end{bmatrix} \] tensor.

\[\omega \cdot \nu = \omega_i dx^i \nu^i \frac{\partial}{\partial x^i} \]  
- 1.19

\[\text{then } \omega \cdot \nu = \omega_i \nu^i \frac{\partial x^i}{\partial x'^i} = \omega_i \nu'^i \delta^i_j = \Sigma \omega_j \nu'^j = \omega_j \nu'^j \]  
- 1.20

\[
\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
\text{or } \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

Being independent of a coordinate system, there is still a need to see how a tensor transforms from one coordinate system to another. Here is the 'test' of the ideal of Einstein's, expressed at the beginning in 2 basis \[\{e_i\}\] and \[\{e'_i\}\], then \[e_{j'} = A'j e_i\]. The vectors are independent, then \[A'^i_j\] is non-singular:

\[\begin{bmatrix} A'^i_j \end{bmatrix}^{-1} = A'^i_j \]  
- 1.21
as $L = A^t A$ - 1.22

Then $A^k W_i (e_j) = W^{k1} (e_j) = \delta^k_{j1}$ - 1.23a

$W^{k1} = A^i W^i$ - 1.23b

$E^{j1} = A^i_1 e_i$ - 1.23c

The forms and vectors transform in the fashion (co-variant), while vectors and form components transform in like fashion (contravariant), like

$y^{j1} = f^{j1}[x^1, x^2, \ldots x^n]$ - 1.24

$$\frac{\partial}{\partial y^i} = \frac{\partial x^i}{\partial x^{j1}} \frac{\partial}{\partial x^j}$$

thus, $A_j^i \int_i^j = \frac{\partial x^i}{\partial x^{j1}}$ or inverse $A_j^i = \frac{\partial x^{j1}}{\partial x^i}$ - 1.25

The curves are traced out by calculating the tangents to them. Newton needed a measuring stick or gauge length; on our manifold (Fig. 1.16).
From this diagram can be seen the differential (absolute):

\[ DA^i = dA^i - \delta A^i = (A^i + dA^i) - (A^i + \delta A^i) \quad - 1.26a \]

\[ \delta A^i = T_{xe}^i A^k dx^i \quad - 1.26b \]

With the metric:

\[ ds^2 = g_{ik} dy^i dy^k \quad - 1.27 \]

\((g_{ij}\) - co-variant components of the metric tensor, this gives the distribution of matter in the universe). 

Here, the Christoffel symbols have been introduced. Defined (Simmonds, 1991)

\[ \Gamma^k_{ij} = \frac{1}{2} g^{kp} (g_{ip,j} + g_{jp,i} - g_{ij,p}) \]

\[ N.B. \ g^{ij} = g_{ij} \cdot g^{ij} \quad - 1.28 \]

Contravariant and co-variant components of identity tensor 1. Using in 4-vector, Greek symbols are used (Rindler, 1977).

These coefficients help us 'see' how curves twist and rotate in a multi-dimensional space, especially when the need to transform one quantity from one coordinate system to another. Minkowskian space has 4-vectors, i.e. vectors expressed by 4-dimensions, each quantity (e.g. acceleration) is multi-variant and has to stay a tensor. The law it is a part of must be invariant (a Lorentz Transformation situation), the coordinates general with the origin being naturally defined, where the absolute derivative reduces to the ordinary derivative. Here also all "twisting" should cease \( \Gamma = 0 \) (Calkin, 1996; Dodson & Poston, 1997; Harper, 1978; Rindler, 1977).
\( (D/ds)(x''/ds) \Rightarrow d^2 x'' / ds^2 \); this would also give the origin of a local inertial frame, hence the acceleration example should be zero, thus (Kay, 1998)

\[
\frac{D}{ds} \left[ \frac{dx^m}{ds} \right] \frac{d^2 x''^m}{ds^2} + \Gamma^m_{\sigma \rho} \frac{dx^\sigma}{ds} \frac{dx^\rho}{ds} = 0
\] - 1.29

This is the geodesic (the shortest metric possible on a curved surface (Harper, 1978; Rindler, 1977)), equation, and is analogous to \( a = \Sigma f \) in a classical non-inertial reference frame. The \( T \) is analog of forces such as the Coriolis.

If looking at a 4-vectors \( (x^1, x^2, x^3, ct) \) and what forces operate out of the vector field, the options can be 'narrowed down' to the type of curvature the field oppresses. The key gravitational space is the Riemann, a distortion of the Euclidean geometry. A manifold that has a metric tensor, Riemannian curvature \( R_{ijkl} \) is thus defined (Kay, 1998; Rindler, 1977).

\[
R_{ijkl} = g_{is} R^s_{jk} = g_{is} \frac{\partial \Gamma^s_{jk}}{\partial x^i} - g_{is} \frac{\partial \Gamma^j_{ki}}{\partial x^l} + g_{is} \Gamma^s_{jl} - g_{is} \Gamma^r_{rk} \Gamma^s_{jr} = \frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} + \frac{\partial g_{ij}}{\partial x^l} \Gamma^s_{jr} - \frac{\partial g_{ij}}{\partial x^r} \Gamma^s_{rl} - \Gamma^s_{jk} \Gamma^r_{rl}
\] - 1.30

as \( \frac{\partial g_{ik}}{\partial x^j} = \Gamma_{jk} + \Gamma_{jk} \)

\[
\frac{\partial g_{i}^{\ell}}{\partial x^i} - \Gamma_{i\ell} = \Gamma_{i\ell}
\] - 1.31

By substitution,

\[
R^i_{\ jkl} = \frac{\partial \Gamma_{jkl}}{\partial x^k} - \frac{\partial \Gamma_{jkl}}{\partial x^l} + \Gamma_{i\ell} \Gamma^r_{jk} \Gamma^r_{j\ell} - \Gamma^r_{j\ell} \Gamma^r_{jk} \] - 1.32a

or \( R^i_{\ jkl} = \Gamma^i_{j\ell,k} - \Gamma^i_{j\ell,k} + \Gamma^j_{ik\ell} - \Gamma^j_{ik\ell} \) - 1.32b

(comma denotes the partial)
It has 256 components (each Christoffel symbol in 4-D has 40) but, luckily, has only 20 independent ones due to symmetries \[ n^2 (n^2 - 1) / 12 = \frac{16 + 15}{12} = 20 \]. This space is known to be flat space when there is no Riemann curvature, i.e. \( R_{ijkl} = 0 \); the space then being known as pseudo-Euclidean (Denisova, 1998; Dodson and Poston, 1997; Kay, 1998, Rindler, 1977).

From the curvatures and Minkowski metric is that for flat space giving three interesting properties for:

\[
R_{\alpha\nu\sigma\mu} = \partial_{\alpha} \Gamma^\beta_{\nu\sigma} + \Gamma^\beta_{\alpha\sigma} - \Gamma^\beta_{\alpha\nu} \Gamma^{\rho\sigma\mu} - 1.33
\]

(i) \( R_{iklm} = -R_{klmi} \) - 1.34a

(ii) \( R_{iklm} = R_{ikml} \) - 1.34b

(iii) \( R_{iklm}^{m} + R_{inkj}^{m} + R_{ilmk}^{m} = 0 \) - 1.34c

The last being the Bianchi identity (Kay, 1998) which is derived in a local inertial frame, i.e. \( T_{\alpha\beta} = 0 \). So the gradients are \( \varphi(x^i,..,x^n) \)

\[
\nabla_i \varphi = \frac{\partial \varphi}{\partial x^i} = \varphi_x
\] - 1.35a

Curl \( x = \{x_i\} \)

\[
R_{ij} = \frac{\partial x_i}{\partial x_j} - \frac{\partial x_j}{\partial x_i}
\] - 1.35b

\[
R_{ij} = \frac{\partial \varphi}{\partial x_j} + \frac{\partial \varphi}{\partial x_i} - \frac{\partial \varphi}{\partial x_j} \Gamma_{ij}^{mk} U_k
\] - 1.35c

\[
Div\, u_{ij}' = \partial \mu' + g^h_{ik} \Gamma_{jk}^{h} U_s
\] - 1.35d

The usual operations on vectors.
The Reimann tensor determines the variation or shape of the vector \( A_\nu \) during parallel displacement. So now Stokes' theorem involvement looks like:

\[
\nabla A_\nu = \int_\Sigma \delta A_\nu - \int \Gamma^\rho_{\nu\lambda} A_\rho dx^\lambda \quad - 1.36a
\]

\[
\int_\Gamma A d\Omega = \left[ \nabla \times A \right] dx \quad - 1.36b
\]

\( \partial \Sigma = \Gamma \quad = \partial = \text{boundary operator} \)

\[
\int_\Gamma ^n A_\nu dx^n = \int ds^\mu \partial_\mu A_\nu \quad - 1.36c
\]

\[
= \frac{1}{2} \int ds^\mu \left[ - \partial_\mu A_\nu - \partial_\nu A_\mu \right] \quad - 1.36d
\]

If the change is along a geodesic:

\[
\nabla A_\nu = \int_\Sigma \delta A_\nu + \int \Gamma^\rho_{\nu\lambda} A_\rho dx^\lambda
\]

\[
= \frac{1}{2} \int ds^\rho \left[ \delta_\rho \left( \Gamma^\beta_{\nu\lambda} - \partial_\nu \Gamma^\beta_{\lambda\rho} - \partial_\lambda \Gamma^\beta_{\nu\rho} \right) \right]
\]

\[
= \frac{1}{2} \int ds^\rho \left\{ \partial_\rho \Gamma^\beta_{\nu\lambda} - \partial_\nu \Gamma^\beta_{\lambda\rho} - \Gamma^\rho_{\nu\lambda} \Gamma^\beta_{\lambda\rho} + \Gamma^\rho_{\nu\lambda} \Gamma^\beta_{\lambda\rho} - \Gamma^\rho_{\nu\lambda} \Gamma^\beta_{\rho\lambda} - \Gamma^\rho_{\nu\lambda} \Gamma^\beta_{\rho\lambda} \right\}
\]

- 1.37

The absolute derivative

\[
DA_\mu = 0 \Leftrightarrow D_\mu A_\nu = \Gamma^\alpha_{\mu\nu} A_\alpha \quad - 1.38
\]

normal derivative \( \rightarrow \) parallel derivative

\[
\nabla A_\nu = \frac{1}{2} \int ds^\rho \left[ \partial_\rho \Gamma^\beta_{\nu\lambda} - \partial_\nu \Gamma^\beta_{\lambda\rho} + \Gamma^\rho_{\nu\lambda} \Gamma^\beta_{\lambda\rho} - \Gamma^\rho_{\nu\lambda} \Gamma^\beta_{\rho\lambda} \right]
\]

- 1.39

If \( R^\beta_{\alpha\mu\nu} = \partial_\alpha \Gamma^\beta_{\nu\mu} - \partial_\nu \Gamma^\beta_{\alpha\mu} + \Gamma^\alpha_{\mu\delta} \Gamma^\beta_{\nu\delta} - \Gamma^\alpha_{\nu\delta} \Gamma^\beta_{\mu\delta} \)

\[
gives \text{us} \quad \nabla A_\mu + \frac{1}{2} \int ds^\rho \Gamma^\rho_{\alpha\mu\nu} A_\beta
\]

- 1.40

which is the variation of \( A \) along a geodesic line due to the shape of the manifold.
1.6 WHAT CAN BE CONCLUDED?

There is the means of tracing a curve on a manifold, especially on a Riemann manifold. Specially noted is the Minkowskian space, because it is the 4-dimension situation of specific interest to this work.

The arsenal of mathematical tools has been assembled to describe the multiplex nature of the internal workings of a plasma when exposed to a laser beam. Many parameters have to be established for this description. Many forces, fields and graphs of these, and the parameters, have to be discussed in order to describe the interaction of plasma and laser beam.

The definition of plasma, the fourth state of matter, has changed since the middle of the ninth century. Essentially, it is referred to as "a high temperature gas consisting of free electrons and ions exhibiting collective behaviour, such as waves, is a manifestation of a true plasma. A plasma must have many particles within a sphere whose radius is the Debye length" (p45, Fusion Energy, Robert A. Gross, John Wiley & Sons, 1984). Gross further points out the role of many time scales, such as particle-collision times.


"Plasma is a physical state of high electrical conductivity and mostly gaseous mechanical properties at high temperature".
The equations that will be used to describe the parameters derive from the areas of electrodynamics, hydrodynamics and magnetohydrodynamics. They are often field-theoretical generalisations of well-known areas in physics. For example, Euler's equation of motion, 'the fluid version based on Newton's single-particle equation

\[ M \mathbf{a} = \mathbf{F} = -\nabla \phi \]  

- 1.42

The force is expressed here as a gradient of a potential, \( \phi \) (Hora, 2000). The fluid equation requires the velocity of a field \( \mathbf{v}(x,y,z,t) \); a mass density field \( f(x,y,z,t) \); the composition of the fluid for a plasma is that of ions, subscript \( i \) and electrons, subscript \( e \)

\[ f(x,y,z,t) = m_i n_i(x,y,z,t) + m_e n_e(x,y,z,t) \]  

- 1.43

Here can be seen spatial coordinate \((x,y,z)\) and the temporal coordinate \((t)\). Four dimensions, Minkowski space-time, originally developed for most physics through relativity theory (Borowski, Borwein, 1989). This situation further defines and simplifies the relationships to be explored because no Christoffel symbols will appear; the space encountered is flat, Riemann. The Maxwellian equations can, thus, be taken in the tensor form, set in Minkowski space. Thus (after Harper, 1976)

Position: \( x^\nu = (x,y,z,ict) \)  

'ict' being the more usual complex for fourth vector time, \( t \)

Current density: \( J_n = (J_x, J_y, J_z + icp) \)

Potential: \( A_\mu = (A_x, A_y, A_z, \frac{i \Phi}{e}) \)

Later in this work, this tensor form will be developed when dealing with Maxwell's equations. The question needs to be asked here; is it necessary in the laser-plasma physics to express things in a tensor form? Why must it even be considered?
The answer to this is, yes! In this new type of understanding of non-linear equations, details could be missed when analysing some result. For example, quiver drift, which is the result of the phase shift that exists between the $E$ and $H$, experienced by a laser wave field in an inhomogenous plasma. The electrons in the laser field are affected by a non-linear force, which drives them out of the beam. In looking for an explanation of this, one could simply take the gradient of the laser electric field $E^2$ as it is in the radial direction. The force on a single electron using (Hora, 2000) a time averaged result of the non-linear force

$$f_{e, nL} = i_x \frac{e^2}{4 \omega^2} \frac{E^2}{n^2} \frac{\partial n}{\partial x} = -i_x \frac{w_p^2}{16\pi\omega^2} \frac{\partial E^2}{\partial xn}$$  

so using nabla or del operator in cylindrical coordinates in place of the Cartesian coordinates, i.e. $x$ in

$$\bar{f}_{e, nL} = -e_x \frac{1}{8\pi} \frac{d}{dx} (E^2 + H^2) = -e_x \frac{E^2}{16\pi} \frac{d}{dx} \left( \frac{1}{n} + n \right)$$

$$= -e_x \frac{1}{16\pi} \frac{\omega^3 p}{\omega^2} \frac{d}{dx} \frac{E^2}{n}$$

$$f_{e, nL} = \frac{e^2}{2m\omega^2} \frac{\partial}{\partial r} E^2$$  

where time averaged $E^2 = E^2 \sqrt{2}$.

from centre out to a large radial distance, gives the translational energy of the effected electron

$$E_{e}^{\text{trans}} = \frac{e^2 E^2}{4m\omega^2}$$  

This quiver drift in radii considered by experiments such as the Boreham (Boreham et al, 1979), shows that following the driftant on $H$-vector direction, the energy is zero, not as equation 1.47, or half the average kinetic energy of the electron. A fuller examination of Maxwell's equations for a finite beam will turn up a longitudinal component. It was this very small component that took the quiver motion along the $H$-vector (zero) to that measured for $E$-direction.
It was Cicchitelli (Cicchitelli et al, 1990) using a series of Bessel functions that arrived at values which agreed with polarisation independent of a repeated Boreham experiment. If the Maxwellian exact longitudinal field is included in the calculations, the force density turns out to be independent of the polarisation (Hora, 2000). The same calculations using transversal laser field (in vacuo) showed obvious polarisation dependence by the radial force density.

To get the correct solution, with all the parts Cicchitelli had to use, the tensor formulation of the non-linear force with all relevant parts including Maxwell's stress tensor

\[ f_{NL} = \frac{1}{4\pi} \nabla \cdot \left[ (EE+HH) \frac{1}{2} (E^2+H^2) (1+n^2-1) EE \right] \frac{1}{4\pi c} \frac{\partial}{\partial t} E \times H \]  

- 1.48

The stress tensor being

\[ T = \frac{1}{4\pi} \left[ (EE + HH) - \frac{1}{2} (E^2 + H^2) I \right] \]

\[ I = i_x j_x + i_y j_y + i_z j_z \]

(unit tensor in Cartesian coordinates)

\[
4\pi T = \begin{pmatrix}
\frac{1}{2} (E_x^2 - E_y^2 - E_z^2 + H_x^2 - H_y^2 - H_z^2) & E_x E_y + H_x H_y & E_x E_z + H_x H_z \\
E_x E_y + H_x H_y & \frac{1}{2} (-E_x^2 + E_y^2 - E_z^2 - H_x^2 + H_y^2 + H_z^2) & E_y E_z + H_y H_z \\
E_x E_z + H_x H_z & E_y E_z + H_y H_z & \frac{1}{2} (-E_x^2 + E_y^2 + E_z^2 - H_x^2 - H_y^2 + H_z^2)
\end{pmatrix}
\]

- 1.49b

which may be included in the expression for the non-linear force (Hora, 2000; Osman et al, 1999)

\[ f_{NL} = \nabla \cdot \left[ T + \frac{n^2 - 1}{4\pi} EE \right] - \frac{1}{4\pi c} \frac{\partial}{\partial t} E \times H \]

- 1.50
This example shows the need in non-linear physics to avoid "naive" assumptions (Hora, 2000). The problem arising from the components $ij\ij$ etc., show the dominating vector components $E_y^2 - H_z^2$ are associated with minus in the Maxwellian stress tensor, hence amount to zero, whereas in equation 1.45 there was an addition sign. This equation will be re-examined in more detail in Chapter 2.

A term that was employed above was "non-linear". The non-linear forces could be summarised to arise from laser-plasma interaction; be related to electrodynamic forces.

Functions, equations and resulting formulae are termed 'linear' if the ordinary differential equation

$$F(x,y,y',\ldots,y^{(n)}) = 0 \quad - 1.51$$

if $F$ is a linear function of variables $y, y', \ldots, y^{(n)}$ (Boyce, et al. 1992). This holds for partial differential equations as well. They would be of the general form $a_0(x)y'' + a_1(x)y'(x-1) + \ldots + a_n(x)y = g(x)$. A non-linear example would be $y' + x^2y'' = 0$ or $(1 + y^2)y'' + x^2y' + y = e^x$. A further example of the nonlinear equation is taken from the simple pendulum

non-linear

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin \theta = 0 \quad - 1.52a$$

is made linear if $\sin \theta \approx \theta$, i.e. for angles smaller than $10^\circ$ for a first approximation.

$$\frac{d^2\theta}{dt^2} + \frac{g}{l} = 0 \quad - 1.52b$$
Equation 1.52a results in chaotic behaviour, while 1.52b is "well behaved" (Boyce & Diprima, 1992; Edwards & Penny, 1993).

An ordinary differential equation that does not contain any products (including powers >1) of the derivative and dependent variable (Borowski & Borwein, 1989). Looking at equations 1.48, 1.49, 1.49a and 1.49b, it can be seen that they are quadratic; $E$ and $H$ as binary products making the force expressions.

These equations are Lorentz and gauge invariant (Hora, 2000), just to comment on that aspect.

One final mathematical aspect that requires comment, however brief, is that of dealing with collections of vectors, all moving in the same direction, then the direction is given by 'rays'. What then are the methods for dealing with 2-vector fields, interacting on the same manifold? A vector field is a mapping

\[ M \rightarrow \text{a smooth manifold} \]
\[ A \rightarrow \text{a smooth vector field} \]
\[ A(x) \in TM_x \rightarrow \text{tangent to the smooth manifold} \]
The one-parameter group of the differ-morphisms of flow $A^G$ (Fig. 1.18B)

$A^t : M \rightarrow M$

\[
\frac{d}{dt}
\bigg|_{t=0}
A^t(x) = A(x)
\]

The flow is the whole group. $-1.53$

Any two vector fields on a manifold determine a new vector field, the Lie or Poisson bracket (which the new one is called), the Poisson bracket extends the structure of the vector space of infinitely differentiable vector field on the manifold into a Lie Algebra (see Figs. 1.19a, 1.19b, and 1.19c).
The Poisson Bracket or commutator of 2-vector fields $A$ and $B$ on the manifold $M$ is the vector field $C$ for which

$$L_c = L_B L_A - L_A L_B \quad \text{expressed } C = [A, B] \quad A^i B^j = C^{ij}$$

The first order differential operator, $L_A$ such as $I$

$$\phi IM \rightarrow R$$

$$\left( L_A \phi \right)_x = \left. \frac{d}{dx} \phi(A^i x) \right|_{x=0}$$ \hspace{1cm} \text{(Fig 1.9c)} \hspace{2cm} - 1.54$$

$$L_A [\eta_1 \phi_1 + \eta_2 \phi_2] = \eta_1 L_A \phi_1 + \eta_2 L_A \phi_2$$ \hspace{1cm} - 1.55a

$$L_A [\phi_1, \phi_2] = \phi_1 L_A \phi_2 + \phi_2 L_A \phi_1$$ \hspace{1cm} - 1.55b

(Fig. 1.20) Poisson bracket can be used to see how non-commutative the flows are. One can start with a group of flows, associate a vector space (tangent bundle). Using the element of tangent bundle, we look at non-commutativity of the flows. This allows us to define "a vector product" or commutator that transforms or extends the vector flow to an algebra. Flows being the important objects now. There is a need to be able to define the origin; also to define a 'structure constant' to know the group.
Tangent vectors have been used to show a flow \( R^3 = \{\Omega = (x, y, z)\} \). Affine, to be homogenous, there is a need to be able to move the origin. The 0, goes to where curves start

\[
[\alpha, \beta] = [\alpha \beta - \beta \alpha]
\]

For closure

\[
[\alpha_\mu, \beta_\mu] = \sum_{n=1}^{\mu} C_{\mu}^{k} \alpha_k
\]

\( C_{\mu} \Rightarrow \) structure constant. Every Poisson bracket is a combination of tangent vectors that make up the algebra. There is a need to study the symmetries and invariants in the fields. Symmetries translate in a certain way. What has been briefly developed here is a Lie algebra. A 'general' algebra for examining how flows behave, the functions themselves, not the specific points along the curves. In Minkowskian space (Arnold, 1989; Bressoud, 1991) using infinitesimal, it is the Lorentz group, where transformations play the role of basis to generate the Lie algebra.

\[
X = (\delta_\mu^\nu + \epsilon_\mu^\nu)^\mu
\]
Transformation in Lorentz space involves tensor fields

\[ x^r = a^r_\mu x^\mu \quad \text{transforms like} \]

\[ x^r = a^r_\mu x^\mu + c^r \]

The following chapters will build on this selective mathematical background, occasionally there will be occasion to use such tools as Stokes Law, Gauss theorem, and deal with forces resulting from the gradient of a field. Most equations will be symmetric, whereas the above dealt with non-commutative material, but two flows commute if, and only if, the Poisson bracket of the related vector fields is equal to zero (Arnold, 1989). The following will be vector analysis, tensor orientated. There is a whole research area dealing with just the mathematical tools that could be brought into play, that is, beyond the charter of this work.
CHAPTER 2

KINETIC THEORY OF PLASMA

The use of Boltzmann equation describing kinetic theory allows description of non-equilibrium and many of the nonlinear properties in a more detailed way. As it deals with N – particles the computations need large, efficient computers; especially dealing with plasma instabilities. The one area is still undergoing detailed research is that of collision terms.

2.1 DISTRIBUTION FUNCTIONS AND AVERAGES (Dolan, 1982, Hora, 1991)

The use of 'phase space' and its size dimensional space plus velocity terms, is abbreviated into vector terms, \((x, y)\). Hence \(f(x, y, z, v_x, v_y, v_z, f)dx dy dz dv_x dv_y dv_z\) gives the number of particles between \(x\) and \(x + dx\), and so on, at time \(t\); plus the velocities of them between \(v_x\) and \(v_x + dv_x\) at time \(t\). Thus the distribution function \(f\), as the particles per unit volume, as if it were two cubes, one spatial, one for velocities.

The particle density becomes the integral of \(f\) over all velocities

\[
n(x, y, z, t) = \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_z f(x, y, z, v_x, v_y, v_z, t)\]

- 2.1

The \(x_1, x_2\) etc term could have been used, vis:-

\[
f(x_1, x_2, x_3, v_1, v_2, v_3, t)d^3 xd^3 v
\]

Vectorially

\[
n(x, t) = \int dv
\]

- 2.1a

Equation 2.1a, being integration over 3 dimensions.
The mean or average value is sort for this distribution

$$<g(u)> \equiv \frac{\int f(u)g(u)du}{\int f(u)du}$$  \hspace{1cm} -2.2

All possible values being allowed for. If the description is for ions and electrons, \(i\) or \(e\), the density of particles per cubic metre could be written as

$$n_{i,e}(r,t) = \int_{-\infty}^{\infty} \int f(r,v,t)d^3v$$  \hspace{1cm} -2.3

\(r\) being a radius vector description of a point where the particle is located having velocity \(v\), at time \(t\).

The average velocity is then :

$$<v> \equiv \frac{\int f(r,v,t)v \ d^3v}{\int f(r,v,t)d^3v}$$  \hspace{1cm} -2.4

The denominator equalling the particle density \(n(r,t)\), \(i.e\) ions and electrons combined.

The average kinetic energy \(W = \frac{1}{2}mv^2\) (non relativistic);

$$<w> \equiv \frac{\int f(r,v,t)mv^2\ d^3v}{\int f(r,v,t)dv}$$  \hspace{1cm} -2.5

At equilibrium this will give a Maxwellian distribution:

$$f_m(r,v,t) = n(r,t)(\beta / \pi)^{3/2} \exp(-\beta v^2)$$  \hspace{1cm} -2.6

\(\beta = m/2kT\) \hspace{1cm} m\(=\) particle mass, \(h\) = Boltzmann constant \((1.38066\times10^{-23} \text{ J/K})\),

\(T\) = temperature, \(v^2 = v_x^2 + v_y^2 + v_z^2\)

Such Maxwellian distribution functions can be solved using standard integral tables.

Evaluating any of the \(<mv_i^2/2>\) results in \(kt/2\). Therefore

$$<w> = <mv_x^2 + v_y^2 + v_z^2>/2 = <mv_x^2/2 + mv_y^2/2 + mv_z^2/2>$$

$$= 3/2 \ kT$$  \hspace{1cm} -2.6
Such a result agrees with equation 2.5. The integrals \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r) \, dx \, dy \, dz \) where \( r = (x^2 + y^2 + z^2)^{1/2} \) can be given in spherical coordinates as \( \int_{0}^{\infty} 4\pi r^2 f(r) \, dr \).

Transforming in velocity space gives:

\[
< W > = \frac{\int_{0}^{\infty} 4\pi v^2 n(r,t)(\beta / \pi)^{3/2} e^{-\beta v^2} \, dv \, m v^{3/2}}{\int_{0}^{\infty} 4\pi v^2 n(r,t)(\beta / \pi)^{3/2} e^{-\beta v^2} \, dv}
\]

\[
= \frac{(m/2) \int_{0}^{\infty} e^{-\beta v^2} \, dv \, v^4}{\int_{0}^{\infty} e^{-\beta v^2} \, dv \, v^2}
\]

\[
= \frac{3}{2} kT
\]

While in terms of the kinetic energy:

\[
F_\text{M}(r,w,dt) = f_\text{M}(r,v,t) 4\pi v^2 \, dv
\]

Since \( v = (2w/m)^{1/2} \), \( dv = (2/m)^{1/2} \, dw / 2w^{1/2} \), and 2.8 becomes

\[
F_\text{M} = (r,w,t) \, dw = 2\pi^{-1/2} (kT)^{-3/2} n(r,t) w^{1/2} \exp(-w/kT) \, dw
\]

The average value of particle speed \( v = (2w/m)^{1/2} \) is

\[
< v > = \frac{\int_{0}^{\infty} 2\pi^{-1/2} (kT)^{-3/2} n(Br,t) w^{1/2} \exp(-w/kT)(2w/m)^{1/2} \, dw}{\int_{0}^{\infty} 2\pi^{-1/2} (kT)^{-3/2} n(r,t) w^{1/2} \exp(-w/kT) \, dw}
\]

\[
= (8kT/\pi m)^{1/2}
\]

Temperature having meaning only for average energy of a population of particles. Temperature can be in degrees Kelvin (K), \( kT \) in Joules (J) and \( kT/e \), electron volts (eV). Note: \( 1eV \approx e/k = 11604.9K \), temperature and \( 1keV = 1000eV = 1.160 \times 10^7 K \). These sorts of distributions can be used for nuclear reaction rates, averaged over Maxwellian distribution of fuel ions.
As the fuel temperature remains relatively low, the interparticle distances become of the order of de Broglie wavelengths of the electrons. This is the degenerate electron gas, degenerate matter terms are used to describe plasmas especially in stellar literature. Here Fermi-Dirac statistics are preferred by some over Maxwellian-Boltzmann (Duderstadt and Moses, 1982). This is because as electrons are compressed, their density will be limited by the number of quantum states available. Similar remarks can be made for ions, hence

\[ n_e = \int_0^{p_F} \frac{8\pi p^2}{h^3} dp \] 

-2.11

giving a maximum energy of \( \varepsilon_F \), the Fermi energy:

\[ \varepsilon_F = \frac{p_F^2}{2m_e} \]

\[ = \frac{1}{8} \left( \frac{3}{\pi} \right)^{2/3} \frac{h^2}{m_e} n_e^{2/3} = 2.19 \times 10^{-15} n_e^{2/3} (eV) \] 

-2.12a

Corresponding pressure from a degenerate gas

\[ p_F = \frac{1}{20} \left( \frac{3}{\pi} \right)^{2/3} \frac{h^2}{m_e} n_e^{5/3} = 3.3 p \left( \frac{G}{m^3} \right)^{5/3} (Mbar) \] 

-2.12b

Other models consider Monte Carlo methods, or extend the above to Thomas-Fermi or even Thomas-Fermi-Dirac models. These allow for Coulomb force between ions and electrons, as \( Ze^2/\epsilon_{el} > \varepsilon_F \), \( \epsilon_{el} \) being the electron-ion separation distance which scales as \( p^{-1/2} \). D-T fuel for example \( n \sim 4.5 \times 10^{22} \), \( T = 5eV \), and \( Z=1 \), the Fermi and Coulomb energy are comparable, hence both must be allowed for. The Thomas-Fermi model gives electron energy as a sum of kinetic and Coulomb terms

\[ E_e = \frac{p_F^2}{2m_e} - e\phi(r) \] 

-2.13

\( n_e(r) = \) self consistent electron density

\( \phi(r) = \) potential
Poisson’s equation is solved to find these.

The Thomas-Fermi-Dirac model is more complex and allows for exchange effects. The later being effective interactions being electrons with parallel and antiparallel spins allowing for the Pauli exclusion principle. (Duderstadt and Moses, 1982).

2.2 KINETIC THEORY

The main equation involved in the Boltzmann equation:

\[
\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial t} + \frac{\partial f}{\partial v_x} \frac{\partial v_x}{\partial t} + \frac{\partial f}{\partial v_y} \frac{\partial v_y}{\partial t} + \frac{\partial f}{\partial v_z} \frac{\partial v_z}{\partial t}
\]

\[
= \frac{\partial f}{\partial t} \text{ collisions}
\]

Or collectively

\[
\frac{df}{dt} + \sum_i \frac{\partial x_i}{\partial t} \frac{\partial f}{\partial x_i} + \sum_i \frac{\partial v_i}{\partial t} \frac{\partial f}{\partial v_i} = \frac{df}{dt} \text{ collisions}
\]

as \( \frac{\partial v_i}{\partial t} \) is acceleration = \( \frac{F_i}{m} \) (non-relativistic), 2.14b becomes

\[
\frac{df}{dt} + v_i \left( \frac{\partial f}{\partial x_i} \right) + \frac{F_i}{m} = \frac{df}{dt} \text{ collisions}
\]

\[
\frac{df}{dt} + v \cdot \frac{\partial f}{\partial x} + \frac{F}{m} \cdot \frac{\partial f}{\partial v} = \frac{df}{dt} \text{ collisions}
\]

and

or

\[
\frac{df}{dt} + v \cdot \nabla f + \frac{F}{m} \cdot \nabla_v f = 0 \quad \text{(Vlasov equation)}
\]

\[\text{F is usually the Lorentz force; pressure gradient effects are} \ \frac{\partial f}{\partial x_i} \text{ and frictional drag effects by} \ \text{the collision term. In effect the above expresses the rate of change of} \ f \ \text{due to} \ \frac{df}{dt} \ \text{collisions. To allow for the thousands of particles in a plasma involved in binary collisions, or Coulomb collisions; all also interacting with thousands of surrounding particles, the Fokker-Planck (Dolan, 1982, Hora, 1991) is employed.}\]
\[
\frac{df}{dt} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial v_1 \partial v_2 \cdots \partial v_n} \left( \alpha_{(n)} f \right) \quad (i_a = 1, 2, 3 \ldots n)
\]

\[\hat{\alpha}\] being the Fokker-Planck coefficients.

This allows the plasma to be discussed having reached thermal equilibrium. Normalising \(f\) at each point of a given particle density \(n (r, t)\), the factor \(f_n\) from

\[
f(r, v, t) = n(r, t) \hat{f}_m(r, v, t)
\]

Being the Maxwellian-Boltzmann distribution (see appendix B, Hora, 1991)

\[
\hat{f}_m = \left( \frac{m}{2\pi kT} \right)^{3/2} \exp\left(-\frac{v^2}{v_{th}^2}\right)
\]

\[
v_{th} = (2kT / m)^{1/2} \text{ the average velocity}
\]

Using scalar values of velocity \(v\), another distribution arises

\[
\int_0^\infty g(v) dv = \int_0^\infty f(v) d^3v
\]

where

\[
g(v) = 4\pi m(r, t) \left( \frac{m}{2\pi kT} \right)^{3/2} v^3 \exp\left(-\frac{v^2}{v_{th}^2}\right)
\]

a term compatible with the kinetic energies discussed earlier.

Using a Hamiltonian approach for the complete system, collisionless Boltzmann equation can be found, vis:-

\[
\frac{\partial f}{\partial t} + v \cdot \nabla f + \frac{F}{m} \cdot \nabla_v f = 0 \quad \text{as opposed to collisional}
\]

\[
\frac{\partial f}{\partial t} + v \cdot \nabla f + \frac{E}{m} \nabla_v f = \left( \frac{\partial f}{\partial t} \right)_c
\]

Here the use of forces involving electric and magnetic fields in the form

\[
F = e\left(\mathbf{E} + \mathbf{v} \times \mathbf{H}\right)
\]


\[
\frac{\partial f}{\partial t} + v \cdot \nabla f + \frac{e}{m} (E + \gamma v \times H) \cdot \nabla \cdot f = 0
\]  

-2.22

Renders 2.20, specifically.

Further, it is noted that many authors make many assumptions based on symmetry, use of the Vlasov equation (Dolan, 1982, Hora, 1991), the state of motion, of ions or electrons, equilibrium distributions; to names a few, to simplify the situation to be discussed.

2.3 BOLTZMANN EQUATION TO MACROSCOPIC EQUATIONS

Integrating 2.20, term by term over velocity space (as was done in the general case), assuming no velocity dependant force. Integrate from \(-\infty\) to \(\infty\), first term

\[
\int \frac{\partial f}{\partial t} d^3v = \frac{\partial}{\partial t} \int f d^3v = \frac{\partial}{\partial t} n
\]  

-2.23

Integrating the second term:

\[
\int v \cdot \nabla f d\nu = \nabla \cdot (n \overline{v}) = \nabla \cdot \overline{n v} \quad \text{"—" denoting average}
\]  

-2.24

The velocity here being the sum of drift velocity and thermal velocity.

Integrating the third term:

\[
\int \frac{F}{m} \cdot \nabla \cdot f d^3v = -\int \nabla \cdot F d^3v = -\overline{n v} \cdot F = 0
\]  

-2.25

The forces being velocity independent, as per assumption.

The R.H.S :-

\[
\int_{-\infty}^{\infty} \left( \frac{\partial f}{\partial t} \right) d^3v = 0
\]  

-2.26

This being the hydrodynamic equation of continuity or conservation of mass.
If the Boltzmann equation is first multiplied by $mv$ (momentum), the hydrodynamic equation of motion, or conservation of momentum, is gained:

$$\int m \mathbf{v} \frac{\partial}{\partial t} f d^3v + \int m \mathbf{v} \cdot \nabla f d^3v + \int m \mathbf{v} \cdot \nabla f d^3v = \int m \left( \frac{\partial f}{\partial t} \right) d^3v$$

-2.27

Taking a distribution based on average velocity of $N$ particles at $r$ and $t$:

$$< \mathbf{v}(\mathbf{r}, t) > = \frac{\int \int \int f(r, \mathbf{v}, t) d^3v}{\int \int \int f(r, \mathbf{v}, t) d^3v} \text{ as equation 2.4}$$

The first term, $A$, is

$$A = \int m \mathbf{v} \frac{\partial}{\partial t} f d^3v = \frac{\partial}{\partial t} < nm \mathbf{v} >$$

-2.28

Braking the velocity into drift velocity $V$, and random velocity, or thermal, $V_T$:

$$A = \int m \mathbf{v} \frac{\partial}{\partial t} \mathbf{v} + \frac{\partial}{\partial t} mn$$

-2.29

as $<V_T> = 0$

$$A = mn \frac{\partial}{\partial t} V + \mathbf{v}_v \frac{\partial}{\partial t} mn$$

-2.30

Using the same terms and logic applied to the second term

$$B = \int m \mathbf{v} \cdot \nabla f d^3v$$

$$= \nabla \cdot < nm \mathbf{v} \cdot \nabla f > = \nabla \cdot nm < (\mathbf{v}_v + V_T)(\mathbf{v}_v + V_T) >$$

$$= \nabla \cdot nm \left[ < \mathbf{v}_v \mathbf{v}_v > + < V_T \mathbf{v}_v > + < \mathbf{v}_v V_T > + < V_T \mathbf{v}_v > \right]$$

-2.31

The last two dyadic terms, $<V_T> = 0$

$$B = < \mathbf{v}_v \mathbf{v}_v > \cdot mn + mn < \mathbf{v}_v \cdot \nabla \mathbf{v}_v > + \nabla \cdot mn < \mathbf{v}_v V_T >$$

-2.32

Provided the second terms of 2.29 and the first term of 2.32 are zero

$$\mathbf{v}_v \frac{\partial}{\partial t} mn + \mathbf{v}_v \cdot \nabla mn = \mathbf{v}_v \frac{d}{dt} m = 0$$

-2.33
This allows for conservation of mass within a volume element. The third term on R.H.S of 2.32

\[
mnv \nabla \cdot \nu = -v \frac{\partial}{\partial t} mn - mn v \nabla \cdot \nabla
\]  

-2.34

This equation describes the adiabatic heating after compensating for the first term of 2.32 against the last term of 2.34

The fourth term in 2.32

\[
\nabla \cdot mn \nu = \nabla \cdot 2mn \frac{1}{2} \nabla^2 = \nabla nkT
\]  

-2.35

Here the unity tensor is used. This equation gives the inner energy from the random motion energy.

There is a third term from equation 2.27.

\[
C = \frac{1}{m} \nu \cdot \nabla \cdot Fv = -nF \cdot \nabla - nv \cdot F
\]  

-2.36

Second term of which goes to zero, as \( F \) depends on \( \nu \) of 2.21 where \( \nabla \times \nu = 0 \).

\[
\nabla \nu \cdot F = 0
\]  

-2.37

Using the tensor of the first term 2.36

\[
\nabla \nu = i_1i_1 \frac{\partial}{\partial \nu_1} + i_1i_2 \frac{\partial}{\partial \nu_1} V_2 + i_1i_3 \frac{\partial}{\partial \nu_1} V_3 + i_2i_1 \frac{\partial}{\partial \nu_2} V_1 + i_2i_2 \frac{\partial}{\partial \nu_2} V_2 + i_2i_3 \frac{\partial}{\partial \nu_2} V_3 + i_3i_1 \frac{\partial}{\partial \nu_3} V_1 + i_3i_2 \frac{\partial}{\partial \nu_3} V_2 + i_3i_3 \frac{\partial}{\partial \nu_3} V_3
\]  

-2.38

A diagonal tensor, with non diagonal being zero, therefore

\[
\nabla \nu = 1
\]  

-2.39

\[
\therefore \ C = -nF \cdot 1 = -nF
\]
where \( \mathbf{F} \) in plasma's is

\[
\mathbf{F} = Z e \mathbf{E} + \frac{Z e}{c} \mathbf{V} \times \mathbf{H} + \mathbf{F}_g
\]

-2.40

\( Z = \) number of charges for ions, \( Z = 1 \) for electrons, \( \mathbf{F}_g = \) gravitation, Coriolis forces etc.

Last term \( D \), from 2.27

\[
D = \int \mathbf{v} \left( \frac{\partial \mathbf{f}}{\partial t} \right)_e d^3V
\]

-2.41

being momentum per volume given to ions by collisions with electrons

\[
D = P_{ie}
\]

-2.42

For no asymmetrics in velocity distribution

\[
P_{ie} = 0
\]

-2.43

However this can not be the case for an electron injected into a plasma.

Combining these four terms

\[
m_n \frac{\partial}{\partial t} \mathbf{v} + m_n \mathbf{v} \cdot \nabla \mathbf{v} = \nabla (n k T) + n m Z e \left[ \frac{\mathbf{E}}{c} + \frac{1}{c} \mathbf{v} \times \mathbf{H} \right] + m_n \mathbf{F}_g
\]

-2.44

Noting second term of \( A \) and first term of \( B \) cancel due to 2.33.

Now the hydrodynamic (macroscopic) equation of motion becomes

\[
m_n \frac{\partial}{\partial t} \mathbf{v} + m_n \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla n k T + n m Z e \left[ \frac{\mathbf{E}}{c} + \frac{1}{c} \mathbf{v} \times \mathbf{H} \right] + m_n \mathbf{E}_g
\]

-2.45

simply

\[
m_n \frac{d}{dt} \mathbf{v} = \mathbf{f} \quad \text{(force density)}
\]

-2.46

The Bernoulli equation for a vortex-free stationary state with external \( \mathbf{F}_g \) and \( \mathbf{E} \) and \( \mathbf{H} \) is found from 2.45

\[
\nabla \left( n k T + \frac{m n}{2} \mathbf{v}^2 \right) = 0
\]

-2.47

If static pressure \( p = n k T \) and density \( p = n m \) are used, and 2.47 is integrated:
Now that the macroscopic picture is formed from the microscopic one, the model allows for work on inhomogeneous plasma. This N-particle model deals with interpenetration of plasma particles where hot and cold plasma meet. It helped develop the ideas of dynamic internal electric fields; as will be discussed in chapter three. Added to this is the model handling of harmonics from low-density plasma surfaces, as well as the concept of surface tension for a plasma.
2.4 MAXWELL'S EQUATIONS

The laser beam is intense, coherent, in phase electromagnetic radiation in the visible frequencies. The plasma is a degenerate (i.e. electrons stripped off their nuclei) matter, full of electrically charged particles. Due to thermokinetic effects the electrons are moving about relative to their parent ions; moving charges produce magnet fields. The two together, or taken separately produce a complex situation of electric fields interacting, magnetic fields interacting and each field type producing the other! To start to unravel this situation the basic rules covering each field needs to be briefly explored first. This was done by James Clerk Maxwell in the 1860's from the work of Charles Coulomb, Hans Oested, Michael Faraday and Andre Charles Ampere. Landau and Lifshitz (1993), Hora (2000, 1991) begin there. Questions of conductivity of mediums when distribution functions are Maxwellian or governed by Maxwellian tensors (Novak, 1980, and the many referred to in it.)

Einstein said “the purpose of theoretical physics is to reduce our understanding of the world to the least number of logically independent axioms.” The following four equations do that.

\[ \nabla \cdot \mathbf{E} = \rho / \varepsilon_0 \quad \text{Gauss' Law (Charges create E-field)} \quad -2.49a \]

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{Faraday's Law (Changing B-field creates E-field)} \quad -2.49b \]

\[ \nabla \cdot \mathbf{B} = 0 \quad \text{Gauss' law for magnetic field (no magnetic charges)} \quad -2.49c \]

\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \quad \text{Maxwell's modified form of Ampere's Law} \quad -2.49d \]

These are supplemented by the Lorentz force law:

\[ \mathbf{E} = \mathbf{Q} \quad (\mathbf{E} + \mathbf{V} \times \mathbf{B}) \quad \text{or} \quad d\mathbf{F} = (\rho \mathbf{E} + \mathbf{J} \times \mathbf{B}) d\tau \quad -2.50 \]
Modern Reviews of Maxwell’s equation add to them by the use of the scalar and vector potentials

\[ E = - \nabla V - \frac{\partial A}{\partial t} \]  
-2.51a

\[ B = \nabla \times A \]  
-2.51b

Using these concepts the general solutions appear as differential equations

\[ \nabla^2 A - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = -\frac{J}{\varepsilon_0 c^2} \]  
-2.52a

\[ \nabla^2 V - \frac{1}{c^2} \frac{\partial^2 V}{\partial t'} = -\frac{\rho}{\varepsilon_0} \]  
-2.52b

with solutions

\[ V(P,t) = \int_{r_s} \frac{\rho(s,t-R_{ps}/c)}{4\pi\varepsilon_0 c^2 R_{ps}} d\tau_s \]  
-2.53a

\[ A(P,t) = \int_{r_s} \frac{J(s,t-R_{ps}/c)}{4\pi\varepsilon_0 c^2 R_{ps}} d\tau_s \]  
-2.53b

\[ \varepsilon_0 \text{ being the permittivity electric constant, permittivity of vacuum} \]

\[ F \cdot m^{-1} = A \cdot s \cdot V^{-1} \cdot m^{-1} \]

\[ \varepsilon = D/E \text{ in an isotropic medium} \]

\[ \varepsilon_0 = 1/(\mu_0 c_0^2) = 8.85418782 \text{ pF m}^{-1} \]

\[ c = \text{speed of light in a vacuum} \]

There are specialised versions of this constant, where \( D=\varepsilon_0 E \) for external sources in vacuum, \( D \) being the electric displacement. \( D=\varepsilon_0 E + p \). It is interesting for future reference to note, \( E \) and \( B \) deal with microscopic descriptions. While \( D \) and \( H \) deal with the macroscopic. (Ogrim and Vaughan, 1977; Jackson, 1998).
$V(p,t)$ and $A(p,t)$ being the potential’s evaluated at point $p$ and time $t$. This gives charge density and current density at $S$ and at a previous time, $t-R_{ps}/c$. $R_{ps}$ is the distance from the source $S$ to the point $p$. Note these are retarded potential’s, taking this simple form only in the Lorentz gauge: $\nabla \cdot A = \frac{1}{c^2} \frac{\partial \nu}{\partial t}$. To note there is also the Coulomb gauge $\nabla \cdot A = 0$, $\nabla \times A = B$. Using these gauges one can generate wave equations. Starting with the fact $\nabla \times F = 0$ there exists a scalar field such that $F = \nabla \psi$, in this case substitute $V$ for $\psi$.

Combining this with Faraday’s law $\nabla \times E = -\frac{\partial B}{\partial t}$ and $B = \nabla \times A$:

\[
\begin{align*}
\nabla \times E &= -\frac{\partial}{\partial t} (\nabla \times A) \\
&= \nabla \times \left( -\frac{\partial A}{\partial t} \right) \\
\nabla \times E + \nabla \times \left( -\frac{\partial A}{\partial t} \right) &= 0 \\
\nabla \times \left( E + \frac{\partial A}{\partial t} \right) &= 0
\end{align*}
\]

Let

\[
-\nabla V = E + \frac{\partial A}{\partial t}
\]

\[
E = -\nabla V - \frac{\partial A}{\partial t}
\]
Substitute this into Maxwell’s first equation

\[ \nabla \cdot E = \nabla \cdot \left( -\nabla V - \frac{\partial A}{\partial t} \right) \]

\[ = -\nabla^2 V - \frac{\partial}{\partial t} \nabla \cdot A = \frac{\rho}{\varepsilon_0} \]  

-2.54c

Using Gauge:

\[ \nabla \cdot A = \frac{1}{c^2} \frac{\partial V}{\partial t} \]

\[ -\nabla^2 V - \frac{\partial}{\partial t} \left( \frac{1}{c^2} \frac{\partial V}{\partial t} \right) = \frac{\rho}{\varepsilon_0} \]

\[ -\nabla^2 V + \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} = \frac{\rho}{\varepsilon_0} \]

\[ \text{or} \]

\[ \nabla^2 V - \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon_0} \]  

-2.54d

This states that in the absence of sources, Maxwell’s equations predict that the scalar potential and vector potential propagate as waves in space with a velocity \( c \); the speed of light, viz:-

\[ -\frac{\rho}{\varepsilon_0} = 0 \]

\[ \therefore \]

\[ \nabla^2 V - \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} = 0 \]

Thus Maxwell’s equation with no sources become (let \( \rho = J = 0 \))

\[ \nabla \cdot E = 0 \]  

-2.55a

\[ \nabla \times E = -\frac{\partial B}{\partial t} \]  

-2.55b

\[ \nabla \cdot B = 0 \]  

-2.55c

\[ \nabla \times B = \varepsilon_0 \mu_0 \frac{\partial E}{\partial t} \]  

-2.55d

with general solutions
\[ \nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = 0 \]
\[ \nabla^2 \mathbf{V} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = 0 \]

\textit{i.e.} wave equations.

The significance of \( \nabla \mathbf{E} \) or \( \nabla \mathbf{B} \) being zero, is that the divergence is zero. This means the flow through the volume can not accumulate; using Gauss theorem

\[ \int_{\partial V} \mathbf{E} \cdot \mathbf{n} \, ds = \int_V \nabla \cdot \mathbf{E} \, dV = 0 \text{ for example} \]

\( dV = dx dy dz \). The term often used is that \( \mathbf{E} \) or \( \mathbf{B} \) is incompressible.

If on the other hand \( \nabla \times \) lead to a zero result as in \( \nabla \times \mathbf{B} = 0 \) for a stationary \( \mathbf{E} \) field, \( \text{i.e.} \ \frac{\partial \mathbf{E}}{\partial t} = 0 \); it is said to be irrotational in that region.

Using Stokes’s theorem

\[ \oint \mathbf{B} \cdot d\mathbf{r} = 0 \quad (\text{for any closed curve}) \]

Physically it means the force field considered has no \textit{eddies} or vortices. The field never curls back on itself; there is no rotation of thing pushed by it. (Bressoud, 1991).

If one starts with Faraday’s law, Maxwell’s second equation and take the curl of both sides

\[ \nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \]
\[ \nabla^2 \mathbf{B} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0 \]

noting that

\[ c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \]

-2.55e
-2.56a
-2.56b

53
A very important and noted step in physics, that this expression gave the speed of light equal to that found experimentally from work by Ole Romer, or terrestrially *i.e* in air, by Jean-Bernard-Leon-Foucault.

In the case of statics the $E, B$ – fields and $V, A$ are related

$$E = -\nabla V \quad (since \quad \frac{\partial A}{\partial t} = 0)$$

$$B = \nabla \times A$$

- 2.56c

Relating back to chapter one; integral forms and differential forms of Maxwell’s equations, can be produced one from the other, using our tools, Gauss Divergence Theorem and Stokes Theorem. Further, Maxwell’s equations can be stated in simplistic or differential forms.

Taking two examples –

(1) Law of conservation of charge states that

$$\nabla \cdot J = \frac{\partial \rho}{\partial t}$$

- 2.57a

By taking the volume integral of both sides.

$$\int_V \nabla \cdot J \, d\tau = \int_V \frac{\partial \rho}{\partial t} \, d\tau = -\frac{\partial}{\partial t} \int_V \rho \, d\tau$$

- 2.57b

Gauss’ Divergence Theorem:

$$\int (\nabla \cdot F) \, d\tau = \oint F \cdot ds$$

where $s$ is a closed surface which bounds the volume $\tau$. 

54
\[ \int \nabla \cdot \mathbf{J} \, d\tau + \frac{\partial}{\partial t} \int \rho \, d\tau = 0 \quad - 2.57c \]

By applying the theorem to the first part and using the definition of the volume charge density to the second part

\[ \oint \mathbf{J} \cdot d\mathbf{s} = -\frac{\partial Q}{\partial t} \quad -2.57d \]

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{by Stokes Theorem} \]

\[ \int (\nabla \times \mathbf{E}) \cdot d\mathbf{s} = \int -\frac{\partial \mathbf{B}}{\partial t} \, d\mathbf{s} \quad -2.57e \]

\[ \therefore \oint \mathbf{E} \cdot d\mathbf{L} = -\frac{\partial}{\partial t} \int \mathbf{B} \cdot d\mathbf{s} = -\frac{\Phi}{\partial t} \quad \text{from 2.57e} \]

Maxwell's equations went further to show what was lacking in the earlier work from which they were formulated. For example, Ampere's circuit law (differential form, \( \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \)) does not allow or express every physical situation—especially the Law of Conservation of Change. Hence Maxwell's modification's.
Ampere’s Law and displacement current

\[ \oint \mathbf{B} \cdot d\mathbf{L} = \mu_0 \int \left( \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \cdot ds \]  

or

\[ \nabla \times \mathbf{B} = \mu_0 \left( \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \]  

Maxwell’s Four Equation

This predicts that a time varying \( E \) field creates a \( B \) field and solves the problem with conservation of charge, vis:-

\[ \nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 \nabla \mathbf{J} + \mu_0 \varepsilon_0 \nabla \cdot \frac{\partial \mathbf{E}}{\partial t} = 0 \]  

\[ \therefore \nabla \cdot \mathbf{J} + \varepsilon_0 \frac{\partial \rho}{\partial t} \nabla \cdot \mathbf{E} = 0 \]  

\[ \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \]  

\[ \text{from } \oint \mathbf{E} \cdot ds = \frac{\rho}{\varepsilon_0} \]  

or more precisely here

\[ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \]  

Maxwell’s First Equation

Hence charge is conserved. One can further generalise the above \( \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \) to vector fields in all cases, use \( \mathbf{B} = \nabla \times \mathbf{A} \):

\[ \nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J} \quad [\text{using } \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}] \]

then \( \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \)  

Hence \( \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} \)  

Using Coulomb gauge \( \nabla \cdot \mathbf{A} = 0 \)

\[ \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \]  

Component wise it looks like

\[ \nabla^2 \mathbf{A}_x = -\mu_0 \mathbf{J}_x \quad \text{etc} \]

Each being mathematically identical to

56
\[ \nabla^2 V = -\frac{\rho}{\varepsilon_0} \]  

-2.58g

This will be useful to this work, and is known as Poisson’s equation. A solution being, as:

\[ dV = \frac{dQ}{4\pi\varepsilon_0 R} = \frac{Pd\tau}{4\pi\varepsilon_0 R} \]

\[ V = \int \frac{Pd\tau}{4\pi\varepsilon_0 R} \]  

-2.58h

This also has component solutions

\[ A_x = \frac{\mu_0}{4\pi} \int \frac{J_x d\tau}{R} \text{ etc.} \]  

-2.58i

Example (2) (Bressoud, 1991), differential forms for the electromagnetic potential.

From Table 1, chapter one, Maxwell’s equations can be stated: for the first two

\[ d(B + E dt) = 0 \quad \text{B is a 2-form} \]  

\[ E \text{ is a 1-form} \]  

-2.59a

The last two

\[ d \left( E - \frac{1}{\varepsilon \mu} B dt \right) = \frac{1}{\varepsilon} J \quad \text{B is a 1-form} \]  

\[ E \text{ is a 2-form} \]  

-2.59b

Noting “\( \varepsilon, \mu \)” is a constant dependant upon medium. 2.11b tells about charge density and current, while 2.11a that the electromagnetic field is exact, the 1-form gives the electromagnetic potential

\[ \phi = A_1 dx + A_2 dy + A_3 dz + A_4 dt \]  

-2.59c

Knowledge of \( \phi \) and \( J \) and ‘\( \rho \)’ can be found by finding the partial differential equations for \( A_i \) in terms of \( \rho \) and \( J \). The assumption being the coefficients are twice continuously, second mixed partials then being taken in any order. \( \phi \) is defined by

\[ E = -\nabla \phi; \]
\[ \mathbf{B} + \mathbf{E} \, dt = -d\phi \]
\[ = \left( \frac{\partial A_2}{\partial z} - \frac{\partial A_3}{\partial y} \right) dydz + \left( \frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} \right) dx \]
\[ + \left( \frac{\partial A_1}{\partial y} - \frac{\partial A_2}{\partial x} \right) dx dy + \left( \frac{\partial A_2}{\partial t} - \frac{\partial A_1}{\partial x} \right) dx dt \]
\[ + \left( \frac{\partial A_2}{\partial t} - \frac{\partial A_4}{\partial y} \right) dy dt + \left( \frac{\partial A_1}{\partial t} - \frac{\partial A_4}{\partial z} \right) dz dt \]

\[-2.59d\]

\[ \frac{1}{\varepsilon} \mathbf{J} = d \left( \mathbf{E} - \frac{1}{\varepsilon \mu} \mathbf{B} \, dt \right) \]
\[ = d \left[ \left( \frac{\partial A_1}{\partial T} - \frac{\partial A_4}{\partial x} \right) dydz + \left( \frac{\partial A_2}{\partial t} - \frac{\partial A_4}{\partial y} \right) dzdx \right] \]
\[ = d \left[ \frac{1}{\varepsilon \mu} \left( \frac{\partial A_3}{\partial x} - \frac{\partial A_4}{\partial y} \right) dx dy dt - \frac{1}{\varepsilon \mu} \left( \frac{\partial A_1}{\partial x} - \frac{\partial A_2}{\partial y} \right) dz dt \right] \]

\[-2.59e\]

\[ \phi \] is not uniquely determined by \( J \). \( R^4 \) naturally corresponds back between in 1-form and 3-form. Therefore the electromagnetic potential can be represented as a space-time flow

\[ \phi = A_1 dx dy dz + \frac{1}{\varepsilon \mu} A_2 dy dz dt + \frac{1}{\varepsilon \mu} A_3 dz dx dt + \frac{1}{\varepsilon \mu} A_4 dx dy dt \]

if \( d\phi = 0 \) the flow is closed, i.e.
\[ 0 = \left[ -\frac{\partial A_4}{\partial t} + \frac{1}{\epsilon \mu} \left( \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) \right] dx dy dz dt \]

Thus the R.H.S of 2.11e, and using the d'Alembertian defined as

\[ \Box^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} - \epsilon \mu \frac{\partial^2 f}{\partial t^2} \]

\[ \frac{1}{\epsilon} J = -\left( \frac{\partial^2 A_4}{\partial x^2} + \frac{\partial^2 A_4}{\partial y^2} + \frac{\partial^2 A_4}{\partial z^2} - \epsilon \mu \frac{\partial^2 A_4}{\partial t^2} \right) dx dy dz \]

Firstly

\[ -\frac{1}{\epsilon \mu} \left( \frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2} - \epsilon \mu \frac{\partial^2 A_1}{\partial t^2} \right) dy dz dt \]

\[ -\frac{1}{\epsilon \mu} \left( \frac{\partial^2 A_2}{\partial x^2} + \frac{\partial^2 A_2}{\partial y^2} + \frac{\partial^2 A_2}{\partial z^2} - \epsilon \mu \frac{\partial^2 A_2}{\partial t^2} \right) dz dx dt \]

\[ -\frac{1}{\epsilon \mu} \left( \frac{\partial^2 A_3}{\partial x^2} + \frac{\partial^2 A_3}{\partial y^2} + \frac{\partial^2 A_3}{\partial z^2} - \epsilon \mu \frac{\partial^2 A_3}{\partial t^2} \right) dx dy dt \]

this becomes:

\[ \Box^2 \left( A_4 dx dy dz + \frac{1}{\epsilon \mu} A_1 dy dz dt + \frac{1}{\epsilon \mu} A_2 dz dx dt + \frac{1}{\epsilon \mu} A_3 dx dy dt \right) + \frac{1}{\epsilon} J = 0 \]  \hspace{1cm} -2.59g

A whole system of P.D.E is now an equation in the d'Alembertian of the 3-form \( \phi \).

\[ \epsilon \Box^2 \phi + J = 0 \]  \hspace{1cm} -2.59h

The d'Alembertian use in electromagnetism shows it be a wave phenomenon. This can be stated as a result of his solutions to partial differential equations of the form:

\[ \frac{\partial^2 h}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 h}{\partial t^2} = 0 \]

being related to the amplitude of fixed string vibration; \( h(x,t) \). Euler showed the solution to be the sum of periodic functions.

\[ \frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 A_1}{\partial t^2} = 0 \]  \hspace{1cm} -2.59i
Note $\frac{1}{c^2} = \frac{1}{\varepsilon \mu}$, the important step by Maxwell regarding the E.M waves travelling at the speed of light was an E.M wave.

It now remains to demonstrate to Maxwell’s equations can have, and do have a tensor nature. This will provide a fuller, more exact description of the wave; the energy to the plasma. It is the wave that imparts the momentum, that by it’s electromagnetic nature effects the electromagnetic components, the ions and electrons that make up a plasma. Hence the space devoted to Maxwell’s work. (Harper, 1978; Jackson, 1999; Landau and Lifshitz, 2000).

The tool needed is the electromagnetic field strength tensor.

$$F_{ik} = \frac{\partial A_i}{\partial x^m} - \frac{\partial A_m}{\partial x^i} \quad (m, n = 1, 2, 3, 4)$$

the latter: $F^{\alpha \beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$

-2.60

Same thing, different format. In actuality, it is a matrix built from the 4 vector, Minkowski space. The quantities are position $x^i(x, y, z,ict)$, current density $J_r = (J_x, J_y, J_z, icp)$, potential $A_\mu = (A_x, A_y, A_z, icp)$ as was stated in chapter 1, p17.

$$F_{11} = \frac{\partial A_1}{\partial x_1} - \frac{\partial A_1}{\partial x_1} = F_{22} = F_{33} = F_{44} = 0$$

$$F_{12} = \frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2} = \frac{\partial A_1}{\partial x} - \frac{\partial A_1}{\partial y} = B_z$$

$$F_{13} = \frac{\partial A_1}{\partial x^1} - \frac{\partial A_1}{\partial x^2} = \frac{\partial A_2}{\partial x} - \frac{\partial A_3}{\partial z} = -B_y$$

$$F_{14} = \frac{\partial A_1}{\partial x^1} - \frac{\partial A_1}{\partial x^4} = \frac{\partial (icp)}{\partial x} - \frac{\partial A_3}{\partial t} = \frac{i}{c} \left[ \frac{\partial \phi}{\partial x} + \frac{\partial A_z}{\partial t} \right] = -\frac{i}{c} E_x$$

etc. The matrix

$$F_{\mu \nu} = \begin{pmatrix}
0 & B_z & -B_y & -\gamma c E_x \\
-B_z & 0 & B_x & -\gamma c E_y \\
B_y & -B_x & 0 & -\gamma c E_z \\
\gamma c E_x & \gamma c E_y & \gamma c E_z & 0
\end{pmatrix}$$

$$\frac{\partial F_{\alpha \beta}}{\partial x^\mu} + \frac{\partial F_{\beta \mu}}{\partial x^\alpha} + \frac{\partial F_{\mu \alpha}}{\partial x^\beta} = 0 (\alpha, \beta, \mu = 1, 2, 3, 4)$$

This a generalised anti-symmetric tensor, sixty-four possible equations are indicated.
As well was stated in chapter one this can be reduced, to in this case to four. Values for \( \alpha, \beta \) and \( \mu \) are sets like

a) 1, 2, 3; b) 4, 2, 3; c) 4, 3, 1; and d) 4, 1, 2. So

\[
\frac{\partial F_{12}}{\partial x^3} + \frac{\partial F_{23}}{\partial x^1} + \frac{\partial F_{31}}{\partial x^2} = 0 \quad \text{or} \quad \frac{\partial B_z}{\partial z} + \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0
\]

i.e \( \nabla \cdot \mathbf{B} = 0 \)

b)

\[
\frac{\partial F_{42}}{\partial x^3} + \frac{\partial F_{23}}{\partial x^1} + \frac{\partial F_{34}}{\partial x^2} = 0 \quad \text{or} \quad \frac{\partial (\gamma \cdot E_y)}{\partial z} + \frac{\partial B}{\partial (ict)} + \frac{\partial (\gamma \cdot E_z)}{\partial y} = 0
\]

or \( \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_z}{\partial t} \)

c)

\[
\frac{\partial F_{43}}{\partial x^3} + \frac{\partial F_{31}}{\partial x^1} + \frac{\partial F_{14}}{\partial x^2} = 0 \quad \text{or} \quad \frac{\partial (\gamma \cdot E_x)}{\partial x} + \frac{\partial B_y}{\partial (ict)} + \frac{\partial (\gamma \cdot E_z)}{\partial z} = 0
\]

or \( \frac{\partial E_z}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t} \)

d)

\[
\frac{\partial (\gamma \cdot E_x)}{\partial y} + \frac{\partial B}{\partial (ict)} + \frac{\partial (\gamma \cdot E_z)}{\partial x} = 0 \quad \text{or} \quad \frac{\partial E_y}{\partial x} - \frac{\partial E_z}{\partial y} = -\frac{\partial B}{\partial t}
\]

b), c) and d) combine to produce

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}
\]

The other two equations are derived from \( \sum_{\mu=1}^{4} \frac{\partial F_{\mu \nu}}{\partial x^\nu} = \mu_0 J_\mu \) (Notably Landau and Lifshitz, 2000).
There are more “sophisticated” treatments; for example, from the principle of least action, this looks more at momentum.

\[
\delta S = \delta \int_a^b \left( -mc ds - \% A_i dx^i \right) = 0
\]

\[ds = \sqrt{dx_i dx^i}\]

\[
\delta S = - \int \left( mc \frac{dx_i d\delta x^i}{ds} + \% A_i d\delta x^i \right) = 0
\]

Deletion boundaries and using integration by parts, changing \( dx^i/\text{ds} = u_i \) (four-velocity)

\[
\int \left( mc u_i \delta x^i dA_i - \% \delta A_i dx^i \right) - \left( mc u_i + \% A_i \right) \delta x^i = 0
\]

(2.61c)

\[
\delta A_i = \frac{\partial A_i}{\partial x^k} dx^k, \quad dA_i = \frac{\partial A_i}{\partial x^k} dx^k \quad \text{second term is zero}
\]

\[
\therefore \int \left( mc u_i \delta x^i + \% \frac{\partial A_i}{\partial x^k} \delta x^k dx^k - \% \frac{\partial A_i}{\partial x^k} \delta x^k \delta x^k \right) = 0
\]

Changing \( i \) for \( k \) in the third term, indices, which are summed over. Using \( du_i = (du_i/\text{ds}) \) (part TWO) and in part three \( dx^i = u^i \text{ds} \)

\[
\int \left[ mc \frac{du_i}{ds} - \% \left( \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k} \right) u^k \right] \delta x^i \text{ds} = 0
\]

The integrand is zero: \( mc \frac{du_i}{ds} = \% \left( \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k} \right) u^k \), this can , e.g. 2.12, electromagnetic field tensor. Hence

\[
mc \frac{du_i}{ds} = \% F^{ik} u_k
\]

Note: \( A_i = (\phi, -A) \)

\[
\begin{pmatrix}
0 & E_x & E_y & E_z \\
-E_x & 0 & -H_z & H_y \\
-E_y & H_z & 0 & -H_y \\
-E_z & -H_y & H_x & 0
\end{pmatrix}
\]

\[
F_{jk} = \begin{pmatrix}
0 & -E_z & -E_y & -E_z \\
E_x & 0 & -H_z & H_y \\
E_y & H_z & 0 & -H_x \\
E_z & -H_y & H_x & 0
\end{pmatrix}
\]

or \( F_{jk} = (E, H), F^{ik} = (-E, H) \)
The components of electric and magnetic field strengths, components of the same electromagnetic field four-tensor. This in three dimensions is:

\[ \frac{dp}{dt} = eE + \gamma \times x \times H \quad \text{(R.H.S is the Lorentz force)} \]

of the time component is employed, the work equation results.

These same tensor patterns can be developed for momentum (M.V. Laue, 1950) in the Minkowski space-time. Always for this works focus the emphasis is how do these equations describe the importing of energy, momentum a change, a medium, (especially a plasma) when electromagnetic radiation penetrates it.

2.5 FIELD ENERGY

It is known that the radiation of light by a medium results in the medium loosing energy. Energy is supplied to the laser; but this energy can be absorbed and re-emitted by the receiving medium. In affect, it can alter and react with the incoming energy. Any law of conservation of electromagnetic field energy must take this into account.

The law of conservation of charge

\[ \nabla \cdot J = -\frac{\partial \rho}{\partial t} \]

leads to ideas of a law of conservation of energy of the electromagnetic field, by analogy.

\[ \nabla \cdot S = -\frac{\partial u}{\partial t} \]

-2.63a

Let \( S \) be energy flux density (i.e. energy per unit, time unit area)

Let \( U \) be the energy density at the same point.

Using the integral form from Gauss’ divergence

\[ \int (\nabla \cdot S) d\tau = \oint S \cdot ds = \int \frac{\partial u}{\partial t} d\tau \]

-2.63b

\( ds \) being an infinite small unit of surface area.

So

\[ \oint S \cdot ds = -\frac{\partial}{\partial t} \int u d\tau \]

-2.63c
First term being the flow. Second term being total of field energy out of the volume energy in volume \( \tau \).

This expresses the charge in field energy resulting from an outward flow of energy from the volume, but it does not take into account the fact that electromagnetic radiation can be absorbed by and emitted by matter within the volume.

The expression needs to be modified.

\[
- \frac{\partial}{\partial t} \int_r \mu d \tau = \oint_S \vec{S} \cdot ds + \text{Work done by field on matter in } S.
\]

Work done by the field on a particle inside \( s \) is \( \vec{F} \cdot \vec{r} \)

So rate of doing work is

\[
\frac{d}{dt} \vec{F} \cdot \vec{r} = \vec{F} \cdot \vec{v} \quad \vec{r} \text{ is the vector in the direction of the force length equal to the distance moved.}
\]

If the particle has charge \( Q \) (ions positive, electrons negative) the force experienced is given by the Lorentz force law

\[
\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B})
\]

So the rate of doing work on the charge is

\[
Q(\vec{E} + \vec{v} \times \vec{B}) \cdot \vec{v} = Q\vec{E} \cdot \vec{v} + Q(\vec{v} \times \vec{B}) \cdot \vec{v} \quad \text{(from vector relationship)}
\]

\[
= Q\vec{E} \cdot \vec{v} + (\vec{B} \times \vec{v}) \cdot \vec{v} \quad \text{(Using } a \cdot (b \times c) = (a \times b) \cdot c)\]

\[
= Q\vec{E} \cdot \vec{v} - \vec{Q} \cdot (\vec{v} \times \vec{v}) \quad \text{(vector product } i \times i = 0)\]

\[
= Q\vec{E} \cdot \vec{v}
\]

If there are \( N \) particles per unit volume than the rate of doing work is \( N \cdot Q \cdot E \cdot \vec{v} \) per unit volume. Now \( N \cdot Q \cdot \vec{v} = \vec{J} \), so the rate of doing work is \( E \cdot \vec{J} \) per unit volume.

\[
i.e. \quad \frac{dW}{dt} = E \cdot \vec{J} \text{ per unit volume}
\]
\[ \mathbf{E} \cdot \mathbf{J} = \text{loss of energy by the field per unit volume per unit time,} \]

and \[ \int \mathbf{E} \cdot \mathbf{J} \, d\tau = \text{volume per unit time, work done by field on matter in } \tau. \]

Thus
\[ -\frac{\partial}{\partial t} \int u \, d\tau = \frac{d}{2} s \cdot ds + \int \mathbf{E} \cdot \mathbf{J} \, d\tau \]

becomes the conservation of electromagnetic energy. In differential form
\[ -\frac{\partial u}{\partial t} = \nabla \cdot \mathbf{S} + \mathbf{E} \cdot \mathbf{J} \]

This needs to be expressed in terms of \( E \) and \( B \) fields. Using Maxwell's laws to eliminate \( J \).

Since Maxwell's version of Ampere's law is
\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \]

\[ \mu_0 \mathbf{J} = \nabla \times \mathbf{B} - \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \]

\[ \mathbf{J} = \frac{1}{\mu_0} \nabla \times \mathbf{B} - \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \]

and since \( \mu_0 \varepsilon_0 = \frac{1}{c^2} \Rightarrow \frac{1}{\mu_0} = \varepsilon_0 c^2 \)

\[ J = \varepsilon_0 c^2 \nabla \times \mathbf{B} - \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \]

So
\[ -\frac{\partial u}{\partial t} = \nabla \cdot \mathbf{S} + \mathbf{E} \left( \varepsilon_0 c^2 \nabla \times \mathbf{B} - \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \]

\[ = \nabla \cdot \mathbf{S} + \varepsilon_0 c^2 \mathbf{E} \cdot (\nabla \times \mathbf{B}) - \varepsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} \]

Now
\[ \mathbf{E} \cdot (\nabla \times \mathbf{B}) = \nabla \cdot (\mathbf{B} \times \mathbf{E}) + \mathbf{B} \cdot (\nabla \times \mathbf{E}) \]

And
\[ \varepsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{\partial}{\partial t} (\frac{1}{2} \varepsilon_0 \mathbf{E} \cdot \mathbf{E}) \]
\[
\frac{\partial}{\partial t} \frac{1}{2} \varepsilon_0 E \cdot E = \varepsilon_0 \left( E \cdot \frac{\partial E}{\partial t} + E - \frac{\partial E}{\partial t} \right) \\
= \varepsilon_0 \frac{\partial}{\partial t} \left( 2E \frac{\partial E}{\partial t} \right) \\
= \varepsilon_0 E \frac{\partial E}{\partial t}
\]

So \[-\frac{\partial u}{\partial t} = \nabla \cdot S + \varepsilon_0 c^2 \nabla \cdot (B \times E) + \varepsilon_0 c^2 B \cdot (\nabla \times E) - \frac{\partial}{\partial t} \left( \frac{1}{2} \varepsilon_0 E \cdot E \right)\]

And since \(\nabla \cdot E = -\frac{\partial B}{\partial t}\) Maxwell’s Second Equation

\[-\frac{\partial u}{\partial t} = \nabla \cdot S + \varepsilon_0 c^2 \nabla \cdot (B \times E) + \varepsilon_0 c^2 B \cdot \left( -\frac{\partial B}{\partial t} \right) - \frac{\partial}{\partial t} \left( \frac{1}{2} \varepsilon_0 E \cdot E \right)\]

or

\[-\nabla \cdot S - \frac{\partial u}{\partial t} = \varepsilon_0 c^2 \nabla \cdot (B \times E) - \varepsilon_0 c^2 \frac{\partial}{\partial t} \left( B \cdot B \right) - \frac{\partial}{\partial t} \left( \frac{1}{2} \varepsilon_0 E \cdot E \right)\]

\[(1-1)\]

\[\nabla \cdot S = \frac{\partial u}{\partial t} = \nabla \cdot \left[ \varepsilon_0 c^2 E \times B \right] + \frac{\partial}{\partial t} \left[ \frac{\varepsilon_0 c^2}{2} B \cdot B + \frac{\varepsilon_0}{2} E \cdot E \right]\]

\[S = \varepsilon_0 c^2 E \times B\]

Energy flow vector field of an electromagnetic field.

\[U = \frac{\varepsilon_0}{2} E \cdot E + \frac{\varepsilon_0 c^2}{2} B \cdot B\]

Energy density is due to “electric” energy density and “magnetic” energy density.

\(S\) is the Poynting vector after its discoverer (in 1884), \(S\) is the energy density flowing per unit time so \(Sds\) is the energy per unit time flowing across an element of area \(S\).

The units of which are:

\[\left[ S \right] = \frac{kgm}{s^2 C \ ms} = \frac{joule}{m^2 s} = \frac{watt}{m^2} \] (power per area)
2.15 i can also be expressed as

\[
\mathbf{E} \cdot \nabla \times \mathbf{H} - \mathbf{H} \cdot \nabla \times \mathbf{E} = - \nabla \cdot \mathbf{E} \times \mathbf{H} = - \nabla \cdot \mathbf{S}
\]

where \( \mathbf{H} = \frac{1}{\mu} \mathbf{B} \)  

\[
\frac{\partial}{\partial t} \varepsilon_0 \mathbf{E} \cdot \mathbf{E}/2 + \frac{\partial}{\partial t} \mu_0 \mathbf{H} \cdot \mathbf{H}/2 + \sigma E^2 = - \nabla \cdot \mathbf{S}
\]

\[\mu: \text{being permeability of magnetic field.}\]

\[\mu_0: \text{absolute permeability } \mu = B/H \text{ in } H \cdot m^{-1}\]

\[\varepsilon: \text{permittivity}\]

\[\varepsilon_0: \text{absolute permittivity } F \cdot m^{-1} = H \cdot s \cdot V^{-1} \cdot m^{-1}\]

\[\varepsilon = D/E\text{ in an isotropic medium (Ogrim and Vaughan, 1977).}\]

The exchange of electromagnetic field energy into mechanical energy of motion, i.e. laser beam energy to particles of the plasma which are actually set in motion, is seen from the Lorentz force. There can not be energy charged in to heat (as per Ohm’s law) or losses or gains by electromagnetic radiation. Thus the energy law for lasers (Hora, 2000) in cgs units:-

\[
\frac{\partial}{\partial t} \left[ \varepsilon \mathbf{E}^2 + \mu \mathbf{H}^2 \right] \frac{1}{8\pi} + \sigma \mathbf{E}^2 \frac{1}{4\pi} + \nabla \cdot \mathbf{S} = 0
\]

The energy transferred is given by

\[
\frac{1}{8\pi} \left( \mathbf{E}^2 + \mathbf{H} \right)
\]

Gradients f energy densities give force densities.

\[
f_i = \frac{1}{8\pi} \nabla \left( \mathbf{E}^2 + \mathbf{H}^2 \right) - \frac{1}{4\pi} \mathbf{E} \nabla \cdot \mathbf{E}
\]

Using the energy tensor as per chapter one

\[
\frac{1}{4\pi} \nabla \cdot \left[ \mathbf{E} \cdot \mathbf{E} + \mathbf{H} \mathbf{H} - \frac{1}{2} \left( \mathbf{E}^2 + \mathbf{H}^2 \right) \right] \approx \frac{1}{8\pi} \nabla \left( \mathbf{E}^2 + \mathbf{H}^2 \right)
\]
which can be approximated as (when spatially derived)

\[ \nabla \cdot (\mathbf{EE} + \mathbf{HH}) = \nabla (\mathbf{E} \cdot \mathbf{E} + \mathbf{H} \cdot \mathbf{H}) \]

Taking Poynting term to be zero

\[ f_1 = \frac{1}{4\pi} \nabla \cdot \left[ EE + HH - \frac{1}{2} \left( E^2 + H^2 \right) \right] - \frac{1}{4\pi} \frac{1}{\epsilon_0} \frac{\partial}{\partial t} \mathbf{E} \times \mathbf{H} \]

\[ f_1 = -\frac{1}{8\pi} \nabla \cdot \left( \mathbf{E}^2 + \mathbf{H}^2 \right) - \frac{1}{4\pi} \mathbf{E} \nabla \cdot \mathbf{E} - \frac{1}{4\pi} \left( \frac{\partial}{\partial t} \mathbf{E} \right) \times \mathbf{H} - \frac{1}{4\pi} \mathbf{E} \times \left( \frac{\partial}{\partial t} \mathbf{H} \right) + \frac{1}{4\pi} \left( \mathbf{E} \cdot \nabla \mathbf{E} + \mathbf{E} \nabla \cdot \mathbf{E} \right) + \frac{1}{4\pi} \mathbf{H} \cdot \nabla \mathbf{H} \]

where \( \nabla \cdot \mathbf{EE} = \mathbf{E} \cdot \nabla \mathbf{E} + \mathbf{E} \nabla \cdot \mathbf{E} \)

\( \nabla \cdot \mathbf{HH} = \mathbf{H} \cdot \nabla \mathbf{H} + \mathbf{H} \nabla \cdot \mathbf{H} \)

and \( \nabla \cdot \mathbf{H} = 0 \) as its vacuum situation.

As \( \mathbf{E} \nabla \cdot \mathbf{E} / 4\pi \) terms cancel; using the relationship

\[ \left( \nabla \times \mathbf{H} \right) \times \mathbf{H} = \mathbf{H} \cdot \nabla \mathbf{H} - \frac{1}{2} \nabla \mathbf{H}^2 \]

the differentiation being only on the first vector in the brackets.

\[ f_1 = \frac{1}{4\pi} \left( \nabla \times \mathbf{H} \right) \times \mathbf{H} + \frac{1}{4\pi} \left( \nabla \times \mathbf{H} \right) \times \mathbf{E} - \frac{1}{4\pi} \left( \frac{\partial}{\partial t} \mathbf{E} \right) \times \mathbf{H} - \frac{1}{4\pi} \mathbf{E} \times \frac{\partial}{\partial t} \mathbf{H} \]

\[ = \frac{1}{4\pi} \left[ \nabla \times \mathbf{H} - \frac{1}{\epsilon_0} \frac{\partial}{\partial t} \mathbf{E} \right] \times \mathbf{H} + \frac{1}{4\pi} \left[ \nabla \times \mathbf{E} + \frac{1}{\epsilon_0} \frac{\partial}{\partial t} \mathbf{H} \right] \times \mathbf{E} \]

\[ 4\pi \frac{j}{c} \]

hence

\[ f_1 = \frac{1}{\epsilon_0} \mathbf{s} \times \mathbf{H} \]

From

\[ \nabla \cdot \mathbf{D} = 4\pi \rho \quad \text{where} \quad \mathbf{D} = \varepsilon \mathbf{E}, \]

\[ 4\pi \rho = \nabla \cdot \mathbf{E} \]

Then the first adiabatic transfer of electromagnetic field energy to mechanical energy becomes:

\[ f = \frac{1}{4\pi} \nabla \left( E^2 + H^2 \right) = \frac{1}{\epsilon_0} j \times H + \rho \mathbf{E} \]
This being the mechanical movement force, the added term (2.17i) the Coulomb law, for the electrostatic forces. The force normally associated with the $\nabla \mathbf{E}^2$ (or $\nabla \mathbf{H}^2$) is the pondermotive force. If electric or magnetic fields bring about a force which causes physical movement. (Osman et al, 1999), The term comes from ponderable – weight; heavy matter, as opposed to electromotive (Webster’s Dictionary of the English Language – unbridged, Encyclopedic Edition). Electrostriction (or magnetostriction) as per

$$f_k = \frac{1}{4\pi} \left( \frac{n^2 - 1}{2} \right) \nabla E^2 - \frac{1}{4\pi} \left( n^2 - 1 \right) \mathbf{E} \times (\nabla \times \mathbf{E})$$

($k$ for formulator Lord Kelvin, 1845)

If a dielectric material (such as a plasma) is considered, there is a requirement upon the Lorentz force to show the attraction of such a material is to have ‘ν’ and ‘B’ terms. (see fig 2.2, after Hora, 2000).

Dielectric material, $\varepsilon > 1$

The pondermotive force here is equivalent to the electrostrive

$$f_k = \frac{1}{8\pi} (\varepsilon - 1) \nabla \mathbf{E}$$

There is no Coulomb force.

The situation of force imparting brakes down in an inhomogeneous plasma. The behaviour and resulting behaviours of a laser beam put into a plasma tells us about the plasma, if that side of the interaction is focused on. The pondermotive force, as was noted, predates lasers, 1845, by Kelvin. It is produced in dielectric medium, e.g. a plasma; this allows the subscribing to the plasma the property of refractive index $n(n^2 = \varepsilon)$(Hora, 2000, 1991). The force results in such medium in polarisation $P$

$$f_k = p \cdot \nabla \mathbf{E}$$

$\nabla \mathbf{E}$ being a tensor.

which becomes, using

$$p = (n^2 - 1) \mathbf{E}$$
It will occur at times, that a tensor results from a product (dyadic or indefinite) of two nuclear vectors. This is the “indefinite product” (Chorlton, 1977), leads to the tensor \( b_i c_j \). The scalar product of them being \( b \cdot c \). Skew-symmetry of \( b_i c_j \) is \( \frac{1}{2} (b_i c_j - b_j c_i) \); this is associated with the axial vector 3-D Euclidean vector \( w \), satisfying

\[
W_x = \omega \times x \text{ being } 2^n \text{ order Cartesian tensor.} \quad (\text{Borowski and Borwein, 1989})
\]

\[
a_k = \varepsilon_{kij} b_i c_j = \frac{1}{2} \varepsilon_{kij} (b_i c_j - b_j c_i)
\]

Pre-scalar multiplication of the dyadic \( b \cdot c \) by a vector \( a \) gives vector \(( a \cdot b ) \cdot c \) and post-multiplication gives, a different vector \( b(c \cdot a) \).

**The dyadic \( b \cdot c \) in matrix form**

\[
\begin{bmatrix}
  b_1 c_1 & b_1 c_2 & b_1 c_3 \\
  b_2 c_1 & b_2 c_2 & b_2 c_3 \\
  b_3 c_1 & b_3 c_2 & b_3 c_3
\end{bmatrix}
\]

\( b_i^l = l_i b_i, \ b_i^l = l_j c_j \) being the laws of orthogonal transformation. (Akivis and Goldberg, 1977, Chorlton, 1977), where \( l_i l_j = \delta_{rs} = l_r l_s \)

therefore

\[
b_i^l b_i^l (l_i l_j) (b_i c_j)
\]

Thus certifying \( b : c \) to be a second-order tensor.

\[
f_k = \left( \frac{n^2 - 1}{2} \right) \nabla E^2 - \left( n^2 - 1 \right) E \times (\nabla \times E)
\]

-2.67

The pondermotive force (electrostrictive) being \( E \cdot E \) while \( \nabla \times E = 0 \) in the static case.
2.6 REFRACTIVE INDEX, PLASMA FREQUENCY

In equation 2.67 the \( n \) is the refractive index; usually light taken as the ratio of the velocity of, or \( E.M \) radiation in incident medium to the velocity of \( E.M \) radiation in the refracted medium. This gives the first parameter or properties of a plasma that needs to be considered. Dealing with \( n' \) will also involve the frequency of the incident \( E.M \) source and any frequencies of oscillating charges in the plasma. One would expect such as there are electric fields incident on the electric fields associated with electrons (negative) moving around the ions, which have positive electric fields associated with them.

The field in region of space associated with a plasma will be determined by electrostatics – electric field to change distribution, and by the motions of the charges in the field. Due to the large mass difference between electrons and ions; the ion movement is negligible compared to that of the electron. The electron density \( n_e \) is uniform until disturbed by an \( E.M \) field. (1) The disturbance causes a bunching up of electrons, this in turn results electrostatic repulsion, pushing electrons back into place. As they move back they gather kinetic energy, which results in their moving past their equilibrium position. (2) Then back again, \text{i.e.} they oscillate to and fro.

<table>
<thead>
<tr>
<th>Fig 2.3a</th>
<th>Fig 2.3b</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="equilibrium_position" alt="Diagram" /></td>
<td><img src="equilibrium" alt="Diagram" /></td>
</tr>
</tbody>
</table>

This one on one situation needs to multiple by the number of electrons in the space cornered.
Consider the shift in density on the $x$ plane only, the density becomes

\[ n = \frac{n_0}{1 + (\Delta s / \Delta x)} \]  

-2.68a

Magnifying the situation in Fig 2.3, the electrons around the equilibrium between two firm lines, have moved to the space between the dashed lines. The number between the firm lines is $n_0 \Delta x$, this number is moved and then located in the space $\Delta x + \Delta s$, hence equation 2.68a.

Using the binomial theorem for negative indices, which needs to be rounded as it does not terminate. This can be used as the density is a small value.

\[ n = n_o \left( 1 - \frac{\Delta s}{\Delta x} \right) \]  

-2.68b

As the ions have a larger inertia, the argument can concentrate on the electron carrying charge $-q$, the average charge density being

\[ \rho = -(n - n_o)q \]  

\[ \rho = n_o q \frac{ds}{dx} \]  

-2.69

From

\[ \nabla \cdot E = \frac{\rho}{\varepsilon_0} \]  

Maxwell’s Equation for Gauss’ Law (eq 2.49a)

Pursuing the $x$-dimensional argument and assuming no other fields bar the one set by the electron displacement.

\[ \frac{\partial E_x}{\partial x} = \frac{n_o q}{\varepsilon_0} \frac{\partial s}{\partial x} \]  

-2.70a

Integrating both sides

\[ E_x = \frac{n_o q}{\varepsilon_0} s + k \]  

-2.70b

As $E_x = 0$ when $s = 0$, $k$ is zero. The force in the displaced position on the electron is
\[ F_x = -\frac{n_0 q^2}{\varepsilon_0} s \]  

-2.70c

this being a like the force on a spring, a restoring force proportional to displacement. Hence, as with a spring, a harmonic oscillation ensues. Equation of motion is

\[ m_e \frac{d^2 s}{dt^2} = -\frac{n_0 q^2}{\varepsilon_0} s \]  

-2.70d

As varies harmonically

\[ e^{i\omega t} \]  

-2.71

The frequency of this oscillation \( \omega^p \), comes from 2.70d

\[ \omega_p^2 = \frac{n_0 q^2}{\varepsilon_0 m_e} \]  

-2.72a

This then is termed the plasma frequency, it is a characteristic parameter of a plasma and relates to the phenomena so described. Usually expressed as

\[ \omega_p^2 = \frac{4\pi e^2 n_e}{m_e}; \quad \omega_p = 5.65 \times 10^4 \sqrt{n_e} \]  

-2.72b

\( n_e \) is in cm\(^{-3}\) (Hora, 1991)

Or

\[ \omega_p = \left( \frac{n e^2}{\varepsilon_0 m_e} \right)^{\frac{1}{2}} = 56.4 n_e^{\frac{1}{2}} \sec^{-1} \text{(S1 units)} \]

As Tonks and Langmuir found in 1929 (Brennan, 1964).

Plasma frequency as a viable characteristic describing a meaningful parameter, is corroborated from other areas of physics such as the study of ionosphere, a plasma region. This area will be revisited when next dealing with refractive index for a plasma, the work of Langmuir and total reflection of radio waves off the ionosphere. This leads to the question again of what happens to a plasma, when E.M. radiation passes into it, or is absorbed by it. The refractive index in a dispersive medium, means it is frequency dependent. (See Fig 2.4) This being a microscopic effect, is not discussed by Maxwell’s equations. If the medium is a homogenous, isotropic dielectric, the constants change in Maxwell’s equation from \( \varepsilon_0 \) to \( \varepsilon \), \( \mu_0 \) to \( \mu \), thus going the phase speed as (Hecht, 1998)
\[ v = \sqrt{\frac{\varepsilon \mu}{\varepsilon_0 \mu_0}} \]  

-2.73

The refractive index comparing wave speed in vacuum to a medium is the absolute index of refraction

\[ n = \frac{c}{v} \sqrt{\frac{\varepsilon_0 \mu_0}{\varepsilon \mu}} \]  

-2.74

The frequency of the incident E.M radiation, the frequency of the plasma and the refractive index allows links to the group velocity of the E.M radiation in the plasma, thus

\[ \nu_g = \frac{d \omega}{dk} = \frac{c}{d(n \omega)d \omega} \]  

-2.75

\( k \) being the wave vector. The total E.M energy density is given by (Landau and Lifshitz, 1993)

\[ \mathcal{U} = \frac{1}{16 \pi \mu \omega} \frac{d}{d \omega} (\omega^2 \varepsilon \mu) \mathbf{E} \cdot \mathbf{E}^* = \frac{c}{8 \pi} \sqrt{\frac{\varepsilon}{\mu}} \frac{dk}{d \omega} \mathbf{E} \cdot \mathbf{E}^* \]  

-2.75

Thus

\[ \nu_g = \frac{\mathbf{S}}{\mathcal{U}} \text{ where } \mathbf{S} = \frac{c}{8 \pi} \sqrt{\frac{\varepsilon}{\mu}} \mathbf{E} \cdot \mathbf{E} \text{ (Mean Poynting Vector)} \]  

-2.76

Being the velocity of energy transfer.

One requirement of the plasma is that it is an absorbing medium. The wave vectors \( k' \) for \( \mathbf{E} \) and \( k' \) for \( \mathbf{H} \) are parallel allowing discussion of a plane wave. Indeed \( k = k l \), using the unit length in direction of the wave vector, and

\[ k^2 = k'^2 - k'^2 + 2i k \cdot k' = \varepsilon \mu \omega^2 / c^2 \]  

-2.77a

\( k \) becomes

\[ \sqrt{(\varepsilon \mu)} \omega / c \]  

-2.77b

The quantity \( \sqrt{\varepsilon \mu} \) is a complex quantity usually written as a typical complex number \( n + ik \).
Thus
\[ k\sqrt{\varepsilon\mu} \omega/c = (n + ik) \omega/c \]  

Now \( n \) is the refractive index, and \( k \) is the absorption coefficient; another parameter that can be used to describe the plasma. The absorption coefficient is referring to energy dissipation, dissipation as actual absorption when \( \varepsilon \) or \( \mu \) can be taken as one. This modifies the Maxwellian equations concerned to

\[ \nabla \cdot \mathbf{E} = -\frac{1}{c} \frac{\partial}{\partial t} \mathbf{H} \]  
\[ \nabla \times \mathbf{H} = \frac{4\pi}{C} \mathbf{J} + \frac{1}{c} \frac{\partial}{\partial t} \mathbf{E} \]

Following the Lorentz "electric glue" model offered for situations like a metal, the free electric can be caused to follow as if constituting an electric current. This implies an Ohm's law situation for a plasma. Such a concept will be revisited in chapter three.

The current density resulting is described by (Hora, 2000, 1991)
\[ \frac{d\mathbf{J}}{dt} + \nu \mathbf{J} = \frac{\omega_p^2}{4\pi} \mathbf{E} \]

The field quantities become
\[ \mathbf{E} = \mathbf{E}_r e^{i\omega t} \]
\[ \mathbf{H} = \mathbf{H}_r e^{i\omega t} \]
\[ \mathbf{J} = \mathbf{J}_r e^{i\omega t} \]

\( r \) being dependant on spatial coordinates \( x, y, z \); \( \omega \) being radian frequency and referring to monochromatic light. Integrating equation 2.80
\[ \mathbf{J} = \frac{\omega_p^2}{4\pi \nu (1 - i\nu/\omega)} \mathbf{E} \]

2.82 can be substituted into the Maxwell's equation above:-

\[ \nabla \times \mathbf{E}_r = -\frac{i\omega}{c} \mathbf{H}_r \]  
\[ \nabla \times \mathbf{H}_r = -\frac{i\omega_p^2}{c\omega (1 - i\nu/\omega)} \mathbf{E}_r + \frac{1}{c} \omega \mathbf{E}_r \]
From past sections it has been shown how this material lends itself to the development of wave equations from Maxwell’s. Here the $\nabla \times$ is used on 2.36b and making use of $\nabla \times \mathbf{E}$ from 2.36a:

$$
\nabla^2 \mathbf{H} + \frac{\omega^2 n^2}{c^2} \mathbf{H} - i \frac{\omega}{c} \mathbf{E} \times \nabla \left[ 1 - \frac{\omega_p^2}{\omega(1 - i \nu/\omega)} \right] = 0
$$

-2.83c

The last part before the equal sign being $n^2$, the time independent complex refractive index. Using $\nabla \cdot \mathbf{H} = 0$ and referring to 2.34, the following wave equation with phase velocity

$$
c_\phi = c \sqrt{\overline{\epsilon}(n)}
$$

-2.84

$n$ again being the complex, time independent, refractive index, gives:

$$
\nabla^2 \mathbf{H} - \frac{n^2}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{H} + 2(\nabla \mathbf{H}) \cdot \nabla \ln n - 2(\nabla \ln n) \cdot \nabla \mathbf{H} = 0
$$

-2.85

The complex dielectric constant $\epsilon$ relates to this refractive index via

$$
\epsilon = n^2 = 1 - \frac{\omega^2}{\omega^2 (1 - i \nu/\omega)}
$$

-2.86

Homogenous plasma’s have the first order term of $\mathbf{H}$ in 2.39 as zero, while $\nabla n^2 = 0$. In homogenous plasma’s, the refractive index is a function of $x$, $y$, $z$, as $n_e$ is spatially dependant. Temperature is also a governing factor here, as will be shown in chapter three when dealing with microscopic views of plasma and kinetic theory. Equation 2.83 becomes:

$$
\nabla^2 \mathbf{H} + \frac{\omega^2 n^2}{c^2} \mathbf{H} + 2(\nabla \mathbf{H}) \cdot \nabla nn - 2(\nabla / nn) \cdot \nabla \mathbf{H} = 0
$$

-2.87

From parallel arguments as above, and assuming time-dependence as reasonable approximation, a wave equation can be obtained. It refers to the refractive index and the $\mathbf{E}$ field (Hora, 1991).

$$
\nabla^2 \mathbf{E} + 2(\nabla \mathbf{E}) \cdot \nabla nn - \frac{1}{\epsilon} \left[ n^2 + 2 \left( \frac{c}{\omega} \right)^2 \nabla^2 nn \right] \frac{\partial^2}{\partial t^2} \mathbf{E} = 0
$$

-2.88
The length of time an \( E.M \) radiation beam is allowed to interact with a plasma is important, as that will affect energy absorption and, response by the plasma. Various time scales can operate within a plasma, due to the \( E.M \) field can act simultaneously with the fields of particles, and their many particle-particle interactions (Gross, 1984).

For example, the homogenous plasma, which is collisionless, has the refractive index of a transparent media. If particle-particle interactions resolve as a collision, the electromagnetic waves are damped hence the absorption constant rises. Temperature will affect refractive index and corresponding absorption.

The vacuum properties, such as magnetic permeability, dielectric constant; arising from the vacuum assumed to be between the ions and electrons, would bring into question the refractive index. The same applies to a high frequency dielectric constant. As stated earlier, the result equation 2.40, derives from and explains Langmuir's total reflection of radio waves at the ionosphere, a well used phenomena. At this point parameter, such as those belonging to waves (an important plasma characteristic) (Gross, 1984); \textit{i.e.} frequency, refractive index and in turn absorption are now identified. As well, the role of temperature and hence via kinetic theory, pressure, density and volume, will need to be viewed as keys to plasma behaviour. Plasma are used to see the factors affecting it to fuse – nuclear fusion specifically initiated by an intense laser beam, The energy so delivered can have various effects.

The absorption process is vital to harnessing the required energy. Electric fields of the incident beam will cause electron oscillation as has been discussed. This energy will go to thermal energy through electron, ion collisions. This is not the only process for the energy in this case material confinement fusion. From the above material, it can be seen that collisional processes decreases in importance with the rise in temperature. (Gross, 1984; Hora, 1991, 2000).
CHAPTER 3

NATURE AND PROPERTIES OF PLASMA.

3.1 DEBYE LENGTH AND PLASMA NEUTRALITY

One of the first questions to be asked about the nature of the plasma is how far apart are the particles? By particles, what is meant, the ions or ions included with surrounding electrons? In the case of deuterium, one electron for every ion, one proton and accompanying neutron. Indeed this idea of electron and ions leads to an addition to the plasma definition. Looking at it as a dielectric medium whose properties are determined by free charges, with no dipoles involved. (Hora, 1999) This describes a fully ionised plasma, a condition that can not always be considered to be the case. Examining some plasma’s it can be found that the electron density varies. (See Table 3.1) (After Ginzburg,1970).

TABLE 3.1

<table>
<thead>
<tr>
<th>Plasma Situation</th>
<th>Electron Density (N)</th>
<th>Temperature $\approx ^{0}\text{K}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intergalactic Space</td>
<td>$10^5$ per cm$^3$</td>
<td>$10^5 - 10^6$</td>
</tr>
<tr>
<td>Interstellar medium</td>
<td>$10^2$ and 10 per cm$^3$</td>
<td>100 weak ionised $10^4$ strongly ion.</td>
</tr>
<tr>
<td>Solar corona</td>
<td>$10^4$ and $3\times10^8$ per cm$^3$</td>
<td>$10^6$</td>
</tr>
<tr>
<td>Interplanetary Space</td>
<td>1 and $10^4$ per cm$^3$. Latter value occurring most when powerful corpuscular emissions occur from the sun.</td>
<td></td>
</tr>
<tr>
<td>Ionosphere</td>
<td>$10^3$ and $3\times10^6$ per cm$^3$</td>
<td>300 to 3,000</td>
</tr>
<tr>
<td>Thermonuclear Reaction</td>
<td>$\sim10^{12}$ per cm$^3$ (Estimated by some for controlled fusion.)</td>
<td>$10^8$ to $10^9$ Current apparatus it is $10^6$ to $10^7$</td>
</tr>
<tr>
<td>Various gas discharge apparatus.</td>
<td>$10^{12}$ per cm$^3$</td>
<td></td>
</tr>
</tbody>
</table>
If there is a density for the electrons, then in a neutral plasma that should be approximately the density of the positive ions, there could also be neutral particles which introduces the idea of the degree of ionisation. For example, in Earth’s lower D layer of the ionosphere, the neutral particle density is around $10^{15}$. The ratio of electron density to neutral particle density is $10^{-11}$ to $10^{-12}$. In the Sun’s corona is almost zero. If the plasma is not fully ionised, then $n_e \neq n_i$ electron number will not equal ion numbers. This being the case a potential difference will exist. The average $\overline{V}$ of which, taken at any point in the plasma, can be represented by Poisson’s equation. (Macker, 1964; Hora, 1991).

$$\nabla^2 \overline{V} = -(n_i - n_e)e/\varepsilon_0$$  \hspace{1cm} (3.1)

On the other hand if the plasma is fully ionised there could be $3N$ equations for $N$ particles, thus:

$$m_n \frac{d^2 x_{1}}{dt^2} = f_{1n} \left(x, \ldots, z_N, \frac{\partial x_1}{\partial t}, \ldots, \frac{\partial z_n}{\partial t}\right); n = 1, \ldots, N$$  \hspace{1cm} (3.2a)

$$m_n \frac{d^2 x_{2}}{dt^2} = f_{2n} \left(x, \ldots, z_N, \frac{\partial x_1}{\partial t}, \ldots, \frac{\partial z_n}{\partial t}\right); n = 1, \ldots, N$$  \hspace{1cm} (3.2b)

$$m_n \frac{d^2 x_{3}}{dt^2} = f_{3n} \left(x, \ldots, z_N, \frac{\partial x_1}{\partial t}, \ldots, \frac{\partial z_n}{\partial t}\right); n = 1, \ldots, N$$  \hspace{1cm} (3.2c)

$x_1, x_2, x_3$ being the particle coordinates, $t$ being time, $f_n$ etc being the forces. These forces are determined by the position coordinates and distance within which a slight departure from neutrality, say within a spherical region radius $r_0$ then

$$(n_i - n_e)e = \rho \text{ or } 0 \text{ for } 0 < r < r_0, r > r_0$$  \hspace{1cm} (3.3a)

Using $E$ as the electric field intensity $\left(-\frac{\partial V}{\partial r}\right)$, Gauss’ theorem gives

$$4\pi r^2 E = 4\pi \rho /3\varepsilon_0 \text{ for } 0 < r < r_0$$

or

$$4\pi \rho r^3 /3\varepsilon_0 \text{ for } r > r_0$$  \hspace{1cm} (3.3b)
Converting these to potential’s by integration

\[ v = -\frac{\rho r^2}{\sigma \varepsilon_0} + K_1 \quad 0 < r < r_0 \]
\[ = \frac{\rho_0^2}{3\varepsilon_0} r + K_2 \quad r > r_0 \]

Taking \( V \rightarrow 0 \) when \( r \rightarrow \infty, K_2 = 0; V \) being continuous at \( r = r_0 \)

\[ -\frac{\rho_0^2}{\sigma \varepsilon_0} + K_1 = \frac{\rho_0^2}{3\varepsilon_0} \]
\[ K_1 = \frac{\rho R_0^2}{2\varepsilon_0} \]

thus

\[ v = \frac{\rho_0^2}{\sigma \varepsilon_0} \left[ 1 - \left( \frac{r^2}{3r_0^2} \right) \right] / 2\varepsilon_0 \quad 0 < r < r_0 \]
\[ = \frac{\rho_0^2}{3\varepsilon_0} r \quad r > r_0 \]

-3.4

See Figure 3.1

The interesting point is that the space-charge fields can actually maintain a condition close to neutrality – Sommerville offers the numerical example of \( 10^{20}/m^3 \) of equal electron to ion particles, and goes on to show using the above equations that a small electron movement ends up requiring a field of \( 10^3 V/m \), too large to maintain, so conditions close to neutrality are maintained. He establishes departure being no more than a few parts in a million over volumes such as \( 10^{-9} m^3 \).

By examining the proposition of what happens when a test charge is placed in it, the particle, being an ion, small so that no plasma particles approaching within the boundaries of inter-particle distance, suffer a potential energy charge in comparison to
the $kT$ thermal energy (Sommerville, 1964). The ion would attract electrons, increasing their density in the region, while positive ions would be repelled. The beginning of an inhomogeneous situation, one that could be induced by use of ion/particle beams, instead of lasers, to induce fusion. Using Boltzmann statistics.

$$n_e(r) = n \exp(-eV/kT_e)$$  
$$n_i(r) = n \exp(-eV/kT_i)$$

Here $r$ represents the radial distance from the considered charge. Equation 3.1, $k / y$, Boltzmann’s constant, then becomes

$$\nabla^2 V = n\left\{\exp(eV/kT_e) - \exp(-eV/kT_i)\right\}e/e_0$$

From symmetrical of a sphere:

$$\nabla^2 V = n\left\{\exp(eV/kT_e) - \exp(eV/kT_i)\right\}e/e_0$$

The condition $Qe/4\pi\varepsilon_0 n^\frac{1}{2} \ll kT$, $Q$ being the charge in question, being required to hold, then :-

$$\nabla^2 V = n\left\{\exp(eV/kT_e) + eV/kT_i\right\}e/e_0$$

$$= \frac{ne^2V}{k\varepsilon_0 \left[ \frac{1}{T_e} + \frac{1}{T_i} \right]}$$

Clearly this tends to zero as $r \to \infty$

$$V = \left(\frac{A}{r}\right)\exp\left\{-r\left[\frac{k\varepsilon_0}{ne^2} \left(\frac{1}{T_e} + \frac{1}{T_i}\right)\right]\right\}$$

$$= \left(\frac{A}{r}\right)\exp\left(-r/d\right)$$

$$d = \left\{\frac{k\varepsilon_0}{ne^2} \left(\frac{1}{T_e} + \frac{1}{T_i}\right)\right\}$$

$A$ = constant

In the immediate volume around $Q$, $A$ is $Q/4\pi\varepsilon_0$, making 3.9b

$$V = \frac{Q}{4\pi\varepsilon_0 r} \exp\left(-r/d\right)$$
The Coulomb potential is thus reduced here by virtue of the movement of charges making the plasma. When \( r \gg d \) the potential becomes zero. A shielding effect ensues, as screened potential from the plasma screening effect. See Fig 3.2 (Sommerville, 1964).

A few special situations arise from this, but the one of interest here is, if the charge is moving with a velocity equal to or greater than the ions, but less than the electron velocity. In this case, the screening will be due to the electrons only as the ions can not keep pace. The ‘d’ of 3.10 will be equal to \( (kT_e \varepsilon_0 / ne^2)^{1/2} \). This is the Debye Length, which due to above derivation, can be seen to earn the name Debye shielding distance.

\[
\lambda_D = \left( kT_e \varepsilon_0 / ne^2 \right)^{1/2} = 69 \sqrt{T_e / n} \quad \text{(S.I units)}
\]  

-3.11

Should the charge move faster than the electrons or ions, nothing screens it. The Debye length can be seen also the size of a sphere, so the number of charged particles within will be

\[
N_0 = \frac{\varepsilon_0 n \lambda_D^3}{2 \pi} = 4\pi \left( \varepsilon_0 kT_e / e^2 \right)^{3/2} / e^2 n^{3/2}
\]

-3.12

It can seen to be an important parameter for the distance screening of charges in plasma, the interchange distance for effective interaction. It gives the greatest distance for which noticeable deviations from neutrality will occur. Thus a sphere where \( r = \lambda_D \) the potential would be from the sphere case from before (equation 3.4):

\[
\frac{n e kT_e \varepsilon_0}{\varepsilon_0 ne^2} = \frac{\varepsilon_0 kT_e}{e}
\]

-3.13
Charge $e$, at the centre has potential $\frac{1}{2} kT_e$, the order of the thermal energy. This dimension is also relevant for the space-charge sheaths that develop where plasmas come in contact with the walls of their confinement vessel.

The Debye length also relates to the plasma frequency (Chapter 2), due to inertia causing the overshoot of the oscillating charges. The perturbation of the electrons from neutrality and subsequent oscillation, relative to the ions is situation specifically determining plasma frequency.

$$\frac{\omega_p}{2\pi} = \left( n_0 e^2 / 4\pi^2 \varepsilon_0 m_e \right)^{1/2} \approx 9\sqrt{n_0} \text{ per sec} \quad -3.14a$$

from 2.24: \( \omega_p = \left( n_0 e^2 / \varepsilon_0 m_e \right)^{1/2} \quad -3.14b \)

If the thermal motion of the plasma is allowed for, many oscillation frequencies can be viewed. Each of these relate to the wavelength of "electrostatic" waves (Sommerville, 1964)

$$\omega^2 = \omega_p^2 \left( 1 + 12\pi^2 \lambda_D^2 / \lambda^2 \right) \quad -3.15$$

$\lambda$ being their wavelength. The link being $\lambda \to \infty$.

Deleting thermal motion the plasma frequency can be seen as a natural resonant frequency for particular plasmas.

$$\sqrt{\omega_p^3}$$, gives the shortest time a plasma has to shield itself against loss of neutrality by externally applied fields. (Sommerville, 1964)

$$\omega_p \lambda_D = \left( kT_e / m_e \right)^{1/2} \sim \nu_e \quad -3.16$$

This being the means electron speed, so electrons with the average thermal energy will cross the Debye length in approximately, $\sqrt{\omega_p^3}$; thus re-establishing screening. The refractive index's relation to the frequency was discussed in chapter 2, and now can be seen to relate to Debye length.

It should be noted that ions in a plasma have a similar aspect

$$\omega_{pi} = \left( n e^2 / \varepsilon_0 m_i \right)^{1/2} \quad -3.17$$
One small, but important point, that will be dealt with later; these velocities can be relativistic if $kT = mc^2$. This may occur at $T \sim 10^{10}$ K. By adding thermal effects the mechanism to allow wave packets to propagate is admitted. This is done by including the plasma pressure term (Brennan, 1964) in the equation of motion. Thus the dispersion term, from the frequency term becomes:

$$\omega^2 = \omega_p^2 + \frac{3kT_e k_z}{m_e}$$  \hspace{1cm} (3.18)

$T_e$ being electron temperature. And $k = \frac{2\pi}{\lambda}$, the wave number.

A similar case for ions oscillating can be drawn up in this collisionless plasma situation. Bearing mind the electrons can re-establish a Boltzmann distribution. The overall equation, allowing for thermal energy is (Brennan, 1964)

$$\omega^2 = \frac{\omega_{pi}^2}{1 + (k\lambda_D)^2} + \frac{3kT_i^2}{m_i} k^2$$  \hspace{1cm} (3.19)

$$\left(\frac{ne^2}{\varepsilon_0 m_i}\right)^{\frac{1}{2}}$$ being the ion plasma frequency, in this case a low one. This can be resolved to two forms, for long wavelengths ($k\lambda_D << 1$)

$$\omega^2 = \frac{k^2}{m_i} (kT_e + 3kT_i)$$  \hspace{1cm} (3.20a)

This being in the form of an acoustic wave that is moving with the speed of sound. After all it is a longitudinal wave, travelling through the ions and electrons. Electrons having a ratio of specific heats of one, ions three. Another form is for $T_e >> T_i$:

$$\omega^2 = \frac{\omega_{pi}^2}{1 + (k\lambda_D)^2}$$

This is where the ion frequency oscillation approaches the ion plasma frequency.
It should be noted here, all the above discussion ignores Landau damping, which is more prominent at short wavelengths. Briefly Landau damping where collisions play no part. It is a trapping in the potential well of the wave of particles with thermal velocity near that of their wave velocity. Energy passes from the wave velocity; then those, whose velocity exceeds wave velocity. This is sometimes referred to as weak collision-less damping (Ginzburg, 1970, Brennan, 1964). This actually forms another restriction on the wave field energy (briefly discussed in chapter two):

\[
\frac{eE_0}{k} \sim eE_0 \lambda \ll \left( \frac{12\pi \nu^4 N_x T}{k m^2 \nu_{ph}^3} \right)^{3/2} \ll \left( \frac{12\pi \nu^4 N}{k (\lambda T)^{3/2}} \right)
\]

omitting the factor \( n \frac{mv_p^2}{e^2 N^2} \). The damping is given by

\[
\gamma = \frac{\gamma_0}{1 + A \nu_{ph}/\nu}
\]

\[
= \frac{\gamma_0}{1 + A k (e\phi_0)^{3/2} / \sqrt{m v_{eff} \lambda T}}
\]

\( A = \frac{\sqrt{2\pi}}{3(7\pi + 6)} \approx 0.053 \) the limit \( \nu_{ph} \gg \nu \) being

\[
\gamma = \frac{3(7\pi + 6)}{4\sqrt{\pi}} \nu_{eff} e^{\frac{\nu_{ph}}{\nu}} \left( \frac{e\phi_0}{\nu} \right)^{3/2} e^{-\frac{e\phi_0}{\lambda T}}\left( \frac{e\phi_0}{\lambda T} \right)^{-\gamma_0}
\]

\[
E = -\frac{d\Phi}{dz}
\]

\[
\Phi = \phi_0 \cos(\alpha x - k z) = (E_0 / k) \cos(\alpha x - k z)
\]

and \( \nu_{ph} = V(eE_0, k/m) \). For more detail see chapter eight of Ginzburg’s “Propagation of Electromagnetic Waves in Plasmas”. As stated earlier, the main work on plasma frequency comes from Langmuir. He was able to calculate electron densities for the ionosphere. Further, plasmas came to be viewed as dielectric, as has been discussed and the constant so associated is also linked to the plasma frequency. For example (Hora, 1991); a neodymium glass laser at 1.78 × 10^{15} Hz, gives from equation 2.72b; an
electron density of $10^{18} \text{m}^{-3}$. Thus different cut off densities can be calculated for different frequency lasers, as can the Debye length, thus:

$$\lambda_D = \frac{v_e}{\omega_p} = \left[ \frac{kT}{4n_e^2 e^2} \right]^{\frac{1}{2}}$$

where $k$= Boltzmann constant of $1.38 \times 10^{-16} \text{cm}^2 \text{ sec}^{-2}$

And calculating with $kT/2$ for every degree of freedom.

$$\lambda_D = 6.9 \left[ \frac{T}{n_e} \right]^{\frac{1}{2}} \lambda_D \text{ in cm, } T \text{ in } ^0\text{Kelvin, } n_e \text{ per cm}^3.$$

or $743 \left[ \frac{T}{n_e} \right]^{\frac{1}{2}}$ Where $T$ is in $eV$.

Note: $^0K = 1.16 \times 10^4 eV$.

This is again the limit length for space charge neutrality. Over this length, $\lambda > \lambda_D$, space charge neutrality can be taken as a macroscopic phenomena as well as microscopic. In terms a plasma/laser situation the thermostatics translate to (fig 3.3) a plasma expanding into a vacuum. The electrons escape faster, so a concentration of ions builds up at the plasma surface. This positive charge barrier once built prevents any further electrons escaping. (Hora, 1991, 2000).
- Fig 3.3
3.2 DEBYE LENGTH AND COLLISION FREQUENCY

The collisions are those between electrons and ions (being a positively charged nucleus, Z protons). Electrons are attracted to ions, with a lateral separation of \( r_0 \) between them, the impact parameter (see fig 3.4). This is a Coulomb force situation

\[
F_e = -\frac{2e^2r}{r^3}
\]

Time allowed for interaction : \( t = \frac{r_0}{v} \), now over this time the electron’s momentum will change, giving the angle \( \phi_0 \)

\[
\Delta P_e = |Fi| = \frac{Ze^2}{r_0v}
\]

Taking only 90° collisions

\[
\Delta P_e \approx P_e \approx \frac{Ze^2}{r_0v} \quad \text{or} \quad r_0 = \frac{Ze^2}{P_e}
\]

90° is used as it links better too experimental results. The Coulomb cross section:

\[
A_e = \frac{m^2_0}{\pi r_0^2} = \frac{Z^2ne^4}{P_e^2} \quad \text{where} \quad P_e = mv
\]

Taking an ion density of \( n_i \), the collision frequency becomes

\[
\nu_{ei} = n_i A_e v = \frac{Zn_e ne^4}{m^2 v^4}
\]
In terms of temperature for the electrons

\[ \nu_{ei} = \frac{Zn_e e^4 3^{-\frac{3}{2}}}{m^{\frac{1}{2}} (kT)^{\frac{3}{2}}} \]  

-3.26a

This also can give the Ohmic conductivity of a plasma, after for nearly three chapters referring to charges moving and that is an electric current.

\[ \sigma = \frac{3(kT)^{\frac{3}{2}}}{2Z \sqrt{m/3\pi e^2}} \]  

-3.26b

Spitzer and Hårm in 1953, and later Spitzer 1962, allowed for collisions between electrons. Allowing for this and small angles, plus quantum factors at high temperatures. Spitzer (1962) gives the collision frequency as

\[ \nu_e = \frac{n_e Z e^4 \gamma_e / n \wedge}{(kT)^{\frac{3}{2}} \left( m^{\frac{1}{2}} 2^{\frac{3}{2}} \gamma(z) \right)} \]  

-3.27a

\[ \gamma_e \] being the e-e collision correction, between 0.5 and 1 for Z=1 and high Z values respectively. In \( \wedge \) is the Coulomb logarithm, and is the ratio of Debye length to impact parameter for \( 90^\circ \):

\[ \wedge = \frac{A_D}{r_0(90^\circ)} = 3 \frac{k^3 T^3}{2 Ze^4} \left( \frac{m}{m_e} \right)^{\frac{1}{2}} \]

Numerically

\[ \nu_{ei} = 8.513 \times 10^{-7} \frac{Zn_e}{\gamma_e(z) T_{ei}^{\frac{3}{2}}} \ln \left( 1.55 \times 10^{10} \frac{T_{ei}}{Zn_e^{\frac{3}{2}}} \right) \]
The quantum correction occurs when the impact parameter can not be considered a point in the classical sense. Here the de Broglie wavelength is larger than the impact parameter, leading to quantum mechanical diffraction of electrons by ions. The parameter now becomes

\[
r_0 = \frac{r_{Bohr}}{2Z} \frac{1}{\left(1 + 4T/T^*\right)^{\frac{1}{4}}} - 1
\]

\[
\left\{
\begin{array}{l}
\frac{r_{Bohr}T^*}{4ZT} = \frac{Ze^2}{3kT}\\
\frac{r_{Bohr}}{4Z} \sqrt{\frac{T^*}{T}} = \frac{h}{2\sqrt{3kT/m}}
\end{array}
\right. 
\text{for } T^* > T^* \quad -3.27b
\]

\(T^*\) being the plasma temperature for the transition

\[
T^* = \frac{4Z^2mc^2\alpha^2}{3k} = 4.176 \times 10^4 Z^2 \quad (\text{Kelvin})
\]

\(\alpha = 2\pi e^2/hc\) \(h\) being Planck's constant

\[
r_{Bohr} = \frac{\hbar}{(4\pi^2 me^2)}
\]

Collision frequency is then

\[
v_{ei} = \frac{\rho_{Bohr}n_e}{4Z^2} \sqrt{\frac{3kT}{m}} \left[1 + 4T/T^*\right]^{\frac{3}{4}} - 1
\]

\[
\left\{
\begin{array}{l}
\frac{Z\pi e^4}{3^{\frac{3}{2}} m^{\frac{1}{2}} (kT)^{\frac{3}{2}}} = v_{ni} \quad T << T^* \quad -3.27c\\
\frac{\pi \hbar^2}{3^{\frac{3}{2}} Z n_e^2 (kT)^{\frac{3}{2}}} = v_{ei} \frac{T}{T^*} \quad T >> T^*
\end{array}
\right.
\]

\(T^* = 35.6Z^2 eV\) and scatting at 90° gives the result 3.26a. The suitably adjusted electrical conductivity becomes

\[
\sigma = \frac{en_s}{2mv_{ei}} = \frac{eZ^3}{\sqrt{3mkT_{Bohr}}} \left[1 + 4T/T^*\right]^{\frac{3}{4}} - 1
\]

\[
\left\{
\begin{array}{l}
\frac{3^{\frac{3}{2}} (kT)^{\frac{3}{2}}}{2Z m^{\frac{1}{2}} e^2} = \sigma_{ei} \quad T << T^* \quad -3.27d\\
T^* \sigma_{ei}/T \quad T >> T^*
\end{array}
\right.
\]

Confirmation of these comes from measurements with stellartators and deuterium at a temperature 800eV.
3.3 SCHLÜTER’S TWO FLUID EQUATIONS, NONLINEAR FORCE EQUATIONS.

In discussing the effects of electromagnetic radiation on a plasma it will be noticed, that a dichotomy exists. The energy absorbing, or reflecting, depends on whether the ions or electrons are being discussed. Clearly if fusion is the direction of the discussion, than the ions, the nuclei are going to the highlighted more. The input energy to compress the plasma will be spread over both component, hence both need to be fully treated before focusing on the actual nuclear processes. The model of envisioning a plasma as two fluids (and using Boltzmann’s kinetic theory, as been used throughout here), is the work of Van Arnulf Schlüter.

In his 1950 paper. “Dynamik des Plasmas I Grundgleichungen, Plasma in gekreuzten Feldern”, he explains that this dichotomy allows the plasma to be seen as an entity. “Das Plasma zieht sich dann zu einem Faden zusammen”. Essentially, plasma has the ability to draw itself together to one single thread (or unit).

Taking the Euler equation, and setting it for electrons (Hora,1991,2000):

\[
\frac{dn_t}{dt} = -n_t eE - n_t \frac{e}{c} \mathbf{v}_t \times \mathbf{H} - \nabla \frac{1}{2} n_t kT_t \\
+ mn_e \nu_{ei} (\mathbf{v}_i - \mathbf{v}_e) + \mathbf{K}_e
\]

-3.28a

While for ions:

\[
\frac{dn_i}{dt} = Zn_i eE + n_i \frac{Ze}{c} \mathbf{v}_i \times \mathbf{H} - \nabla \frac{1}{2} n_i kT_i \\
- mn_e \nu_{ei} (\mathbf{v}_i - \mathbf{v}_e) + \mathbf{K}_i
\]

-3.28b

\(v_i\) being ion velocity, \(v_e\) electron velocity, \(m_i\) ion and electron mass respectively. \(E\) is the electric field, while \(j_e = n_e v_e\) is the current density involved in the Lorentz force. \(T_e\) and \(T_i\) terms are thermo-kinetic pressure terms, relating to the temperature of electrons and ions, as has been used before. The \(V_{ei}\), electron-ion collision frequency which will help determine the viscosity. This term is

\[
\nu_{ei} = n_i A_e \nu = \frac{Zn_e m_e e^4}{m_i^2 v_e^3} \quad A_e : \text{Coulomb Cross Section}
\]
$K_e$ and $K_i$ and force densities due to gravitation, centrifugal, Coriolis forces and alike. This is considered for a fully ionised plasma, to avoid a neutral particle or third fluid equation. These equations can be derived from the kinetic theory, which forms the bridge between microscopic and macroscopic plasma properties and has been stated. There is a further derivation offered as Appendix C in Hora, 1991. The points of interest here are that by adding the two Euler equations and using

$$
\mathbf{v} = \frac{m_i \mathbf{v} + Zm \mathbf{v}_e}{m_i + Zm} \quad \text{as a net velocity and for the current density.} \quad -3.28d
$$

$$
\mathbf{j} = e(n_i \mathbf{v}_i - n_e \mathbf{v}_e) \quad -3.28e
$$

(These being more in line with Appendix, than the body of Hora, 1991 or 2000).

Also using the neutrality (space charge) explored earlier, the equation of motion for a plasma becomes

$$
f = m_i n_i \frac{d\mathbf{v}}{dt} = -\nabla p + \frac{1}{c} \mathbf{j} \times \mathbf{H} + \frac{1}{4\pi} \left( \frac{\omega_p}{\omega} \right)^2 \mathbf{E} \cdot \nabla \mathbf{E} \quad -3.29
$$

\(\nabla p\) being the pressure gradient.

Using Schlüter terms:

$$
\frac{1}{4\pi} \left( \frac{\omega_p}{\omega} \right)^2 \mathbf{E} \cdot \nabla \mathbf{E} = \mathbf{j} \cdot \nabla \frac{1}{n_e} \frac{m}{e^2} \quad -3.30
$$

This applies only too high-frequency cases. Again the movement of charge implies a current, hence Ohm's law. This can be obtained by subtracting the two Euler equations. Schlüter referred to this equation as the “diffusion equation”.

$$
\frac{m}{e^2 n_e} \left( \frac{d\mathbf{j}}{dt} + \mathbf{v} \mathbf{j} \right) = \mathbf{E} + \gamma_e \mathbf{v} \times \mathbf{H} + \frac{1}{en_c} \mathbf{j} \times \mathbf{H} + \frac{c}{en_e} \frac{\nabla p}{1 + 1/z} \quad -3.31
$$

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First term is Ohm’s law (see equation 2.33), the second; Hallsterm, while the third is the Lorentz term. The last term is the diffusion term dealing with ambipolar generation of electric field due to the gradient of pressure p. For a laser inducing relativistic intensities, the Ohm’s law term can be neglected. This then links back to the work on equation 2.31 onward, dealing with Maxwell’s equations and refractive index. If the above refers to only high-frequency, what about low-frequency, or that which is related to the electrostatic field:

\[
\mathbf{E} \nabla \cdot \mathbf{E} = \frac{1}{4\pi} \mathbf{E}_e (Zn_i - n_e) \tag{3.32a}
\]

Note: there is no refractive index.

In a rather extensive paper by M.M Novak “Interaction of Photons with Electrons in Dielectric Media” (Fortschritte der Physik – (Progress of Physics) vol 28, no. 6, 1980, pp285-353), the issue comes down to the use of the Minkowski theory field energy-momentum tensor for high frequency and the Abraham tensor and theory for low frequency case:

\[
S^\mu_\nu\ = \frac{1}{8\pi} \left[ (E_i D_k + E_k D_i) + (H_i B_k + H_k B_i) - \delta^{ik} (E \cdot D + H \cdot B) \right] \tag{3.33b}
\]

with force density \( f_n = \rho E - \frac{1}{8\pi} E_i E_k \nabla \delta_{ik} + \frac{1}{8\pi} \partial_k \left( E D_k - E_k D \right) \)

The Minkowski

\[
S^\mu_\nu_M = \frac{1}{4\pi} \left[ E_i D_k - \frac{\mu}{2} \delta^{ik} E \cdot D \right] \text{or}
\]

\[
= \frac{1}{4\pi} \left[ E_i D_k + H_i B_k - \frac{\mu}{2} (E \cdot D + H \cdot B) \delta^{ik} \right] \tag{3.32b}
\]

\[D_i = \varepsilon_{ik} E_k \]

\[B_i = \mu_{ik} H_k \]

With force density

\[f_n = \rho E - \frac{1}{8\pi} E_i E_k \nabla \delta_{ik}\]

Essentially the problem reduces to the ratio of momentum to energy. Minkowski stated \( n/c \), while Abraham: \( 1/nc \). The solution offered by Sir Rudolf Peierls (Peierls, 1976), after first correcting his earlier work (Burt and Peierls, 1973), is for the total momentum density.
\[ g = \left[ \frac{1 + n^2}{2} - \frac{\sigma}{2} \left( n^2 - 1 \right)^2 \right] g_{\text{Abraham}} \]

\[ = \frac{1}{\sqrt{\epsilon}} \left[ \sqrt{\epsilon} + n - \frac{\sigma (n^2 - 1)^2}{n} \right] g_{\text{vac}} \]

which is "the correct answer is then just the arithmetic mean of the Abraham and Minkowski results" (Peierls, 1976; p480). The large body of work comes down to the concept that a photon in a plasma has its momentum increased (Klima and Petzilka, 1972). It is a relativistic phenomena for the electromagnetic energy in a dielectric medium. In summary; (Hora, 2000; Novak, 1980)

\[ p_A = \frac{\hbar \omega}{nc} \quad \text{for Abraham} \]

\[ p_M = \frac{\hbar \omega}{c} n \quad \text{for Minkowski} \]

result

\[ p_{\Phi, p} = \frac{p_A + p_M}{2} \]

This will have bearing on the non-linear force. The equations of motion, etc., have been verified by the study of oblique incidence of the radiation. It can be said then that oblique incidence is a check on whether the non-linear force, described by:

\[ f_{nl} = \frac{1}{c} \mathbf{j} \times \mathbf{H} + \frac{1}{4\pi} \mathbf{E} \nabla \cdot \mathbf{E} + \frac{1}{4\pi} \nabla \cdot \left( n^2 - 1 \right) \mathbf{EE} \]

is complete. (Hora, 1991)

This lead to a more general case for the non-linear force than offered by perpendicular incidence. Equation 3.29, contains the Schlüter term:

\[ \frac{1}{4\pi} \left( \frac{\omega_p}{\omega} \right)^2 \mathbf{E} \cdot \nabla \mathbf{E} = \frac{1}{\sqrt{n_e}} \frac{m}{e^2} \mathbf{j} \]

To take into account dispersion, collisions that will give terms of dissipation and absorption, modification was required. (Hora, 1969). This modification lead to having to deal only with perpendicular (to plasma surface) acceleration
\[ f = -\nabla p + \frac{1}{c} j \times H + \frac{1}{4\pi} E \nabla \cdot E - \frac{\omega_p^2}{4\pi} \left( 1 + i \frac{\nu}{\omega} \right) E \nabla \cdot E - \frac{\omega_p^2}{4\pi} \frac{1}{\omega^2 + v^2} \left( 1 + i \frac{\nu}{\omega} \right) E \nabla \cdot E \]

\[ - \frac{1}{4\pi} \frac{\omega_p^2}{\omega^2 + v^2} \left( 1 + i \frac{\nu}{\omega} \right) E \cdot \nabla E - \frac{1}{4\pi} \nabla \cdot \nabla E \cdot \nabla - \frac{\omega_p^2}{\omega^2 + v^2} \left( 1 + i \frac{\nu}{\omega} \right) \]

-3.34

There is the addition of two non-linear terms.

3.4 THE STRESS TENSOR AND THE NON-TRANSIENT NONLINEAR FORCE.

The equation of motion, equation 3.34, has a direct connection with the Maxwellian stress tensor; briefly mentioned in chapter one. Such a tensor creates a way of solving problems of dispersion and dissipation. Note, if dealing with relativistic plasmas, Dragila (J.Physics D11, 1978, 683) added the term \((\partial E/\partial t) \times B/(4\pi)\). Starting with Schlüters two fluid equations, 3.28a and 3.28b and using 3.28d and 3.28e to 3.32. First Schlüters two equations are added giving (Hora, 2000)

\[ R.H.S. = - + p (Zn_i - n_i) E + (Zn_i e v_i - n_e e v_e) \times \frac{\nu}{c} + K_i + K_e \]

-3.36a

\[ L.H.S. = m_i n_i \frac{\partial}{\partial t} v_i + m_e n_i \frac{\partial}{\partial t} v_e + m_i n_i v_i + m_e n_e v_e \cdot \nabla v_e \]

\[ + m_i n_i \frac{Zm}{m_i} v_e \cdot \nabla v_i - m_i n_i \frac{Zm}{m_i} v_e \cdot \nabla v_i + m_i n_i Zm \frac{v_e}{v_e} \]

\[ - m_i n_i \frac{Zm}{m_i} v_e \cdot \nabla v_i - m_i n_i \frac{Zm}{m_i} v_i \cdot \nabla (v_i - v_i) \]

\[ + m_i n_i \frac{Zm}{m_i} v_i \cdot \nabla v_i \]

\[ - m_i n_i \frac{Zm}{m_i} v_i \cdot \nabla v_i + m_i n_i \left( \frac{Zm}{m_i} \right)^2 v_e \cdot \nabla v_e \]

-3.36b

By use of 3.28d and the electric current, 3.28e

\[ v_e - v_i \approx \frac{n_e v_e - Zn_i v_i e}{n_e e} = \frac{j}{en_e} \]

3.36b becomes

\[ m_i n_i \left[ \frac{\partial}{\partial t} v + v \cdot \nabla v \right] + m_i n_i \frac{Zm}{m_i} (v_e - v_i) \cdot \nabla (v_e - v_i) \]

-3.36c

and

\[ m Z n_i (v_e - v_i) \cdot \nabla (v_e - v_i) = \frac{m_i}{e} \nabla \cdot \frac{j}{en_e} = \frac{1}{4\pi} \left( \frac{\omega_p^2}{\omega} \right)^2 E \cdot \nabla E \]

-3.36d

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Making use of the Schlüter term for current densities in terms of oscillating fields, equation 3.30.

Noting \( n^2 - 1 = -\left(\frac{\omega_p}{\omega}\right)^2 \), the force density is found to be:

\[
f = m_i n_i \left[ \frac{\partial}{\partial t} v + v \cdot \nabla v \right] = -\nabla p + E \varepsilon (Z n_i - n_e) \\
+ j \times \frac{H}{c} + K_i + K_e - \frac{1}{4\pi} \left(\frac{\omega_p}{\omega}\right)^2 E \cdot \nabla E
\]

Schlüter assumed \( n_e \approx Z n_i \), again reasoned from plasma quasi-neutrality allowing for microscopic movement within one Debye length. As mentioned oblique incidence (especially with p-polarised laser light) as shown additional factors must be taken into account (Hora, 1969). From the Wentzel, Kramers, Brillouin, (sometimes, and Jordan) solution for the above, has shown a longitudinal electric field resulting. A high-frequency one, which must be allowed for, especially to account for conservation of momentum. It’s now necessary to include the dielectrically modified description. In a similar was as 2.65h and 2.65i:

\[
4\pi \varepsilon (Z n_i - n_e) = \nabla \cdot n^2 E = \nabla \cdot D
\]

The second and last term of 3.37 becomes

\[
\frac{1}{4\pi} \mathbf{E} \nabla \cdot \mathbf{E} - \frac{1}{4\pi} \left(1 - n^2\right) \mathbf{E} \nabla \cdot \mathbf{E} - \frac{1}{4\pi} \mathbf{E} \cdot \nabla \left(1 - n^2\right)
\]

To arrive at an equation for the non-transient non-linear force, 3.39 has the last two terms differentiated and the thermo-kinetic force reduced.

\[
f_{nl} = \varepsilon c j \times \mathbf{H} + \frac{1}{4\pi} \mathbf{E} \nabla \cdot \mathbf{E} + \frac{1}{4\pi} \nabla \cdot \left(n^2 - 1\right) \mathbf{E} \mathbf{E}
\]

Miller et al. 1979; Kentwell et al. 1980, worked this system for a collision plasma. They showed there was a force density along the plasma surface. Of importance here is the equivalence 3.40 to the Maxwell stress tensor. This is normally derived from elastomechanics, but can now be derived from the above hydromechanics. This approach was exercised in chapter two following the Landau and Lifshitz approach. Note here the Poynting vector \( \mathbf{S} \) will be given as
\[ \frac{\partial}{\partial t} \left( \frac{\mathbf{E} \times \mathbf{H}}{4\pi} \right) = \frac{\partial}{\partial t} \frac{1}{\epsilon_0} \mathbf{S} \]

-3.41

The procedure is to divide the non-linear force in A and B parts.

\[ f_{nl} = A + B \]

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
</table>
| \[ \begin{align*}
j &= \frac{\epsilon_0}{4\pi} (\nabla \times \mathbf{H}) - \frac{1}{4\pi} \frac{\partial}{\partial t} \mathbf{E} \\
&= \frac{1}{4\pi} (\nabla \times \mathbf{H}) \times \mathbf{H} - \frac{1}{4\pi} \nabla E \times \mathbf{H} \\
&= \frac{1}{4\pi} \mathbf{H} \times (\nabla \times \mathbf{H}) - \frac{1}{4\pi} \nabla E \times \mathbf{H}
\end{align*} \] | \[ \begin{align*}
&= \frac{1}{4\pi} \mathbf{E} \nabla \cdot \mathbf{E} + \frac{1}{4\pi} \nabla \cdot \left( \mathbf{n}^2 - 1 \right) \mathbf{E} \\
&= \frac{1}{4\pi} \mathbf{E} \nabla \cdot \mathbf{E} + \frac{1}{4\pi} \nabla \cdot \left( \mathbf{n}^2 \mathbf{E} + \frac{1}{4\pi} \mathbf{E} \end{align*} \] |

A1

B1

B2

B3

B4

Note: Space charge density \( \rho_e \) cannot be neglected. E.M. waves cause oscillation of frequency \( \omega \). This is the Langmuir wave (electrostatic in nature). Time averaged of which is zero, considering space charge of the plasma to be neutral. Adding zero, by addition of the last two terms, and combing the first terms of B3.

[\[ \frac{1}{4\pi} \left[ \nabla \cdot \left( \mathbf{n}^2 \mathbf{E} - \mathbf{E} \nabla \mathbf{E} + \frac{1}{2} \nabla \mathbf{E}^2 \right) \right] \]

B5

Combining second and third term

\[ \frac{1}{4\pi} \mathbf{E} \times (\nabla \times \mathbf{E}) = \frac{1}{8\pi} \mathbf{E} \nabla \mathbf{E} - \frac{1}{4\pi} \mathbf{E} \nabla \mathbf{E} \]

B6

Using Maxwell’s \( \nabla \times \mathbf{E} = \frac{1}{\epsilon_0} \frac{\partial}{\partial t} \mathbf{H} \)

\[ \Rightarrow \]

\[ \begin{align*}
&= \frac{1}{4\pi} \left[ \nabla \cdot \left( \mathbf{n}^2 \mathbf{E} - \frac{1}{2} \nabla \mathbf{E}^2 \right) \right] \\
&- \frac{1}{4\pi} \mathbf{E} \times \frac{\partial}{\partial t} \mathbf{H}
\end{align*} \]

B7

97
Returning to $f_{nl} = A + B$

$$f_{nl} = \frac{1}{4\pi} \nabla \left[ EE + H \cdot \nabla H - \frac{1}{2} (E^2 + H^2) \right] + \left( n^2 - 1 \right) EE$$

$$- \frac{1}{4\pi C} \frac{\partial}{\partial t} E \times H$$

where

i) $\mathbf{1} = i_x i_x + i_y i_y + i_z i_z$ being unity tensor $\mathbf{1}$

ii) plasma has a permeability $\mu = 1$.

iii) Using $H \cdot \nabla H = \nabla \cdot HH - HH \cdot H$

The last term of which is zero, because $\nabla \cdot B = 0$. Hence the tensor form of the non-linear force

$$f_{nl} = \frac{1}{4\pi} \nabla \left[ EE + HH - \frac{1}{2} (E^2 + H^2) \right] + \left( n^2 - 1 \right) EE$$

$$- \frac{1}{4\pi C} \frac{\partial}{\partial t} E \times H$$

Now the Maxwell stress tensor from chapter one can be employed

$$T = \frac{1}{4\pi} \left[ EE + HH - \frac{1}{2} (E^2 + H^2) \right]$$

where this is the situation described in equation 1.49 to equation 1.51 in chapter one. Equation 1.51 being the non-linear force involving the Maxwell stress tensor

$$f_{nl} = \nabla \cdot \left[ T + \frac{n^2 - 1}{4\pi} EE \right] \frac{1}{4\pi C} \frac{\partial}{\partial t} E \times H$$

and as seen from chapter two, as well, is the equivalent of the result of Landau and Lifshitz (1966). Their result coming from elastomechanics for a non-dispersive and non-dissipative fluid for low-frequency changes of the electromagnetic field. From the discussion of refractive index, allows an interpretation of the $EE$ term to be that proportional to the material density. That is to say that if no absorption is present, the force is all pondermotive. If collisions are present, non-pondermotive terms arise.
Equation 1.51 gives for a linearly polarised laser field (in x-direction).

\[ f_{Ne} = \frac{i_x}{8\pi} \frac{\partial}{\partial x} \left( B_x^2 + H_y^2 \right) \]  

-3.45

The above is the non-transient case. The incident E.M radiation was considered to be high intensity, the time dependence, was considered stationary and unchanged. The Poynting vector \( \mathbf{S} \) when compared to the non-linear force is less by the ratio of \( 2\pi/\omega \tau \cdot \tau \) being the rise time of the laser pulse; \( 10^{-12} \) see and \( \omega/2\pi \) for neodymium glass lasers. (Hora, 1991). Considering a perpendicularly incident plane wave, a collision-less situation; the equation 3.34; adding the \( \mathbf{j} \times \mathbf{H} \) term and involving the \( \partial n/\partial x \) in the Wentzel, Kramers, and Brillouin (WKB for short) solution for \( \mathbf{H} \), vis:-

\[
\mathbf{H} = \text{Re} \left[ \frac{i_x}{\omega} \frac{\partial n}{\partial x} \sin \left( \frac{\omega}{c} \int \Re(n) \, d\xi - \alpha x \right) \right] \exp(-\frac{\omega}{2} k x) \\
+ \text{Im} \left[ \frac{i_x}{\omega} \frac{\partial n}{\partial x} \cos \left( \frac{\omega}{c} \int \Re(n) \, d\xi - \alpha x \right) \right] \exp(-\frac{\omega}{2} k x)
\]

The term 3.45 can be neglected, and a derivation stemming from the above and refractive index, gives the following. Two terms involving refractive index and the WKB condition are

\[
\frac{\sqrt{3}}{2n} \left| \frac{\partial n}{\partial x} \right| \ll \frac{\omega}{c} |n| \quad \text{or} \quad \Theta = \frac{\sqrt{3}}{2} \frac{c}{\omega|n|^2} \left| \frac{\partial n}{\partial x} \right| \ll 1
\]

-3.46a

Using the second of 3.46a the electric and magnetic field lengths are

\[
\mathbf{E} = i_x \left| \frac{E_x}{n} \right| \exp i F_0 \exp \left[ + i \frac{k}{2} \left( x + \frac{\partial n}{\partial x} \right) \right]
\]

-3.46b

\[
\mathbf{H} = i_x \left| \frac{E_x}{n} \right| \exp i F_0 \exp \left[ + i \frac{k}{2} \left( x + \frac{\partial n}{\partial x} \right) \right]
\]

-3.46c

Where \( F_0 = \omega \left( t + \int (\Re n(\xi)/c) \, d\xi \right) \)

As terms involving derivative \( n \) in space give terms of second and higher order, they can be overlooked. Selecting the real part of 3.46b
\[ E'_{\gamma}^2 = E_{\gamma}^2 \left( \frac{1}{\text{Re}|n|^2} \right)^2 \cos^2 F_0 + \left( \frac{1}{|n|^2} \right)^2 \sin^2 F_0 \left[ \exp(-k(x)/\lambda) \right] - 3.46d \]

the time-averaged rule being:

\[ \overline{E'_{\gamma}}^2 = \frac{E_{\gamma}^2}{2|n|} \exp(\mp k(x)/\lambda) \]  

- 3.46e

Applying the same treatment to

\[ \overline{H_z^2} = \pm \lambda E_{\gamma}^2 |n| \exp(\mp k(x)/\lambda) \]  

- 3.46f

The time averaged non-linear force density, originally expressed as 3.45, becomes:

\[ \overline{f}_{NL} = -i_x \frac{E_{\gamma}^2}{16\pi} \frac{\partial}{\partial x} \left( \frac{1}{|n|} + |n| \right) \exp(\mp k(x)/\lambda) \]  

- 3.46g

or

\[ \overline{f}_{NL} = i_x \frac{E_{\gamma}^2}{16\pi} \left( \frac{1 - |n|^2}{|n|^2} \right) \exp(\mp k(x)/\lambda) \frac{\partial |n|}{\partial x} \]  

- 3.46h

\[ - i_x \frac{E_{\gamma}^2}{16\pi} \frac{1 + |n|^2}{|n|^2} \frac{2\omega}{c} \text{Im} \rho(n) \exp(\mp k(x)/\lambda) \]

As it is the collision-less case, k=0 and n becomes the out off density

\[ n_{\infty} = \frac{m \omega^2}{4\pi e^2} \]

the second term vanishes, resulting in

\[ \overline{f}_{NL} = i_x \frac{E_{\gamma}^2}{16\pi} \frac{\omega^2}{\rho^2} \frac{\partial |n|}{\partial x} \]  

- 3.47

This was originally derived by Hora in 1969, in "Physics of Fluids" 12, 182. It produced a collisional term and gave the approximation for the non-linear force in the WKB condition by

\[ \overline{f}_{NL} = -i_x \frac{E_{\gamma}^2}{16\pi} \frac{\partial}{\partial x} \left( \frac{1}{|n|} + |n| \right) \exp(+kx) \]  

- 3.48

100
In equation 3.46h \[ k \] being the absorption constant, sometimes expressed as \( K \), (Hora, 2000), the exponential factor is close to one in ranges, WKB solution gives a correct answer, provided \( \omega_p \leq \omega \). Also from this paper, a general equation evolves, which consists of the thermo-kinetic term, \( f_{nl} = f - f_{th} \).

\[
f_{nl} = \gamma_c \mathbf{j} \times \mathbf{H} + \frac{1}{4\pi} \nabla \cdot \mathbf{E} + \frac{1}{4\pi} \nabla \cdot (n^2 - 1) \mathbf{EE}
\]

As has been pointed out this is identical since formulation involving the Maxwellian stress tensor \( \mathbf{T} \), 3.44

\[
f_{nl} = \nabla \cdot \left( \mathbf{EE} + \mathbf{HH} - \gamma_c \left( \mathbf{E}^2 + \mathbf{H}^2 \right) \mathbf{I} + \left( n^2 - 1 \right) \mathbf{EE} \right) / 4\pi - \frac{1}{4\pi} \frac{\partial}{\partial t} \mathbf{E} \times \mathbf{H}
\]

This is also identical with Landau-Lifshitz expression on p242 of their "Electrodynamics of Continuous Media" (Pergamon, Oxford, 1966), proved the algebraic identity of equation 3.49 with equation 3.40. Equation 3.49 is therefore valid for the dispersive and dissipative plasma, while Landau and Lifshitz derivation was for nondispersive, nondissipative liquids only. The above also works for oblique incidence waves with collisionless plasma.

Using perpendicular incidence along the \( x \)-direction, so the forces are

\[
f_{nl} = \gamma_c \mathbf{j} \times \mathbf{H}
\]

From 3.40, or from 3.49

\[
f_{nl} = -\frac{\partial}{\partial x} \frac{\mathbf{E}^2 + \mathbf{H}^2}{8\pi}
\]

These are general density profiles. Simplification of the WKB approximation, the result becomes for collisional plasma

\[
f_{nl} = \int \frac{E_v^2}{16\pi} \frac{\omega_p^2}{\omega} \frac{1}{|n|^2} \frac{d|n|}{dx} + \int \frac{E_v^2}{16\pi} \frac{\omega_p^2}{\omega} \frac{2\omega c}{\omega}
\]

The second term is a non-pondermotive dissipative part of the non-linear force, (Hora, 1969; Stamper, 1977). Again using perpendicularly incidence plane waves, WKB type plasma, it being non-dissipative (collisionless), the non-linear force can be expressed as
\[ f_{NL} = -i_x \frac{E_x^2}{16\pi} \frac{\omega_p^2}{\omega^3} \frac{\partial}{\partial x} \psi = -i_x \frac{1}{16\pi} \frac{\omega_p^2}{\omega} \frac{\partial}{\partial x} E^2 \]  

WKB approximation giving \( \overline{E} = Ev/(n)^{\frac{3}{2}} \).

3.5 THE TRANSIENT NONLINEAR FORCE

First published by Klima and Pertzika in "Institute of Plasma Physics, Geochosl.Acad.Sci., Prague, 1972", was a formulation for momentum of quasi-monochromatic wave packets. Their formulation included both transverse and longitudinal waves, with terms up to the second order in the amplified of the electromagnetic field. They specifically dealt with the increase in momentum of the transverse wave upon entering the plasma. The plasma was assumed to be homogenous, the laser a time dependant pulse; as opposed to the fore-mentioned section where it was time constant. This led to many papers debating the issue, a very complex one when trying to deal with the general case; as opposed to Klima and Pertzika's perpendicular incidence. Hora in 1985 claimed to have developed the general transient non-linear force. One not dependent on very slowly varying intensity (Zeidler et al, 1995):

\[ f_{NL} = \frac{\gamma_e}{c} j \times H + \frac{1}{4\pi} \nabla \cdot E + \frac{1}{4\pi} \left( 1 + \frac{\gamma_e}{c} \frac{\partial}{\partial t} \right) \nabla \cdot E(n^2 - 1) \]

A very similar equation to 3.40; especially when compared to stress tensor formulation

\[ f_{NL} = \frac{1}{4\pi} \nabla \left[ EE + HH \frac{\gamma_e}{c} (E^2 + H^2) \right] + \left( 1 + \frac{\gamma_e}{c} \frac{\partial}{\partial t} \right) (n^2 - 1) \] \[ EE - \frac{1}{4\pi c} \frac{\partial}{\partial t} E \times H \]

The difference lying in the time derivative (bracketed) concerns the refractive index \( n \). The on/off behaviour of E.M waves derives from the transfer of E.M field energy to electrons which have been set oscillating. The formulations are Lorentz and gauge invariant (Hora,1991,2000), offered by Rowlands, in 1990. Such a general formulation in the invariant form, gives a magnetic susceptibility, \( \mu = 1 \). In addition to this the work of Chernikov et al in the Physics Review, (A40, 4070), lead to finding that plasmas go into chaos when exposed to very high laser intensity.
3.6 "GENUINE" TWO FLUID PLASMA MODEL

Since the Schlüter two fluid model, researchers such as Hora, Lalousis (especially in his Ph.D. thesis Dec, 1983), Goldsworthy, have made improvements to it. To fully understand the strong electric fields developed by laser produced plasma, these modifications were needed. When the electron cloud is pushed by the non-linear force; the ions are subsequently dragged on by virtue of electrostatic fields. Equations 3.28a and b, have to be coupled with the Maxwell equations; at the very least, the Poisson equation. The first thing to emerge from research is the fact plasmas are not free from internal electric fields at the high-frequency range of an irradiating laser. This was seen in studies of cosmic and geophysical plasma’s, by Alfven (1981) and Falthamma (1988). This led to a real time, collisional, non-linear generalisation for these high-intensity laser fields. Transfer of energy from electrons to ions, and back again, by an adiabatic compression, had to be included. Energy from laser to electrons to ions is a vital step if a process allowing for nuclear fusion is to be realised. The process is complex, some simplification such as line, or Kramers, or even backbody Planck type need to be omitted, except when fusion gain calculations are done. Here bremsstrahlung losses need to be included. The plasma under consideration is one-dimensional and of mainly deuterium. It includes collision produced viscosity and thermal conductivity; where Spitzer’s (L.Spitzer Jr, “Physics of Fully Ionised Gases, 2nd Ed, Wiley, 1962) value were corrected from his value by 50 to 100 for deuterium pellets. This agreed with the electric double layer preventing electrons carrying energy off. This energy transport ceased to be electrons and was only by ion thermal conductivity. The ratio between both conductivities is

\[
\frac{m_e}{m_l} = \frac{5.4858026 \times 10^{-4}}{2.014101795} = 2.723696793 \times 10^{-4} \quad \text{or}
\]

\[
= \frac{0.91095310 \times 10^{-3}}{3.344548} \left( \frac{m/10^{-27} \text{ kg}}{M_e} \right) = 2.7236965698 \times 10^{-4} \quad \text{or}
\]

\[
= \frac{0.5110034}{1876.139} \quad M_e V = 2.723332569 \times 10^{-4}
\]

Average value = 2.723575439 \times 10^{-4}

The square root of which is = 0.016530503
The thermal conductivity was reduced by a factor of 71 for deuterium plasma. (Fig 3.5)

Essentially the model involves seven unknowns,

$n_e$  electron densities  $n_i$  ion densities
$v_e$  electron velocity  $v_i$  ion velocity
$T_e$  electron temperature  $T_i$  ion temperature

and the longitudinal electric field $E$, corresponding to plasma oscillations. Expressed through seven equations.

i) Continuity for electrons and ions.

\[ \frac{\partial}{\partial t} (m_e n_e) + \frac{\partial}{\partial x} (m_e n_e v_e) = 0 \]  \hspace{1cm} -3.54a

\[ \frac{\partial}{\partial t} (m_i n_i) + \frac{\partial}{\partial x} (m_i n_i v_i) = 0 \]  \hspace{1cm} -3.54b

ii) Equation of Motion (Hora, 2000).

\[ \frac{\partial}{\partial t} (n_e m_e v_e) = -\frac{\partial}{\partial x} (n_e m_e v_e^2) - \frac{\partial}{\partial x} (n_e kT_e) - \frac{1}{8\pi} \frac{\partial}{\partial x} (E_e^2 + H_e^2) \]  \hspace{1cm} -3.55a

\[ -\frac{1}{4\pi} \frac{\partial}{\partial x} E - m_e n_e (v_e - v_i) v \lambda_d \frac{\partial}{\partial x} (n_e m_e v_e) = -\frac{\partial}{\partial x} (n_i m_i v_i^2) - \frac{\partial}{\partial x} (n_i kT_i) \]  \hspace{1cm} -3.55b

\[ + \frac{Z}{4\pi} \frac{\partial}{\partial x} E + m_i n_i (v_e - v_i v) \]
Expressed as conservation of momentum (Hora, 1991).

\[
\frac{\partial (n_e m_e v_e)}{\partial t} = -\frac{\partial (n_e m_e v_e^2)}{\partial x} - \frac{\partial p_e}{\partial x} - n_e e E - n_e m_e v (v_e - v) + f_{nl}
\]
\[
\frac{\partial (n_i m_i v_i)}{\partial t} = -\frac{\partial (n_i m_i v_i^2)}{\partial x} - \frac{\partial p_i}{\partial x} - n_i Z e E + n_i m_i v (v_i - v) + \frac{m_i}{m_e} f_{nl}
\]

iii) Conservation of Energy

\[
\frac{\partial}{\partial t} (n_e \frac{1}{2} k T_e) = -\frac{\partial}{\partial x} (n_e v_e \frac{1}{2} k T_e) - \frac{1}{2} k \frac{T_e}{m_e} n_e \frac{\partial}{\partial x} v_e + \frac{1}{m_e} \left( \frac{1}{k_e} \frac{\partial T_e}{\partial x} \right) - \frac{1}{2} k \frac{T_e}{m_e} n_e \frac{T_e - T_i}{\tau} + \frac{1}{m_e} W
\]  

-3.56a

\[
\frac{\partial}{\partial t} (n_i \frac{1}{2} k T_i) = -\frac{\partial}{\partial x} (n_i v_i \frac{1}{2} k T_i) - \frac{1}{2} k \frac{T_i}{m_i} n_i \frac{\partial}{\partial x} v_i + \frac{1}{m_i} \left( \frac{1}{k_i} \frac{\partial T_i}{\partial x} \right) - \frac{1}{2} k \frac{T_i}{m_i} n_i \frac{T_e - T_i}{\tau}
\]  

-3.56b

iv)

\[
\frac{\partial E}{\partial x} = -4 \pi e (n_e - Z n_i)
\]  

-3.57

Initial conditions and boundary conditions need to be put, which are time dependant with regards to the laser, i.e. pulsed. This has to be evaluated for each time step by the spatial dependence of W, the power density for energy transfer, as a solution of Maxwell’s equations. W can be viewed from the fluid mechanic point of view to be like the external heat deposited per unit volume; assuming no external forces. (Cairns, Sanderson, 1980) vis:-

\[
\frac{\partial}{\partial t} \left[ \rho (v + \frac{1}{2} v^2) \right] + \text{div} \left[ \rho (v h + \frac{1}{2} v^2) \right] + \text{div} q = W
\]

It includes all reflections and absorption properties belonging to the inhomogeneous plasma. All the equations and material to this point have been used to develop computer codes, of varying accuracy. Data gleaned from varying experiments, also adds to fill or test the accuracy of these codes. As stated in the introduction the cost of such experiments is prohibitive. While to power of the computers involved limits
access by many. To fully describe a plasma’s behaviour, like that of fluid, the thermodynamic state and velocity of flow are required (Cairns, Sanderson, 1980, Laloussis, 1983). The state variables, usually two such as pressure, density; plus three vector components of velocity are usually all that is required for the equation of state. The variables being independent of each other. They form the dependant variables against the independent variables of space and time.

Any plasma formed in code or reality expands, thus in one dimension there are as many negative values as positive values of velocity on the average. The magnitudes are usually of the same order. Taking a 1 keV deuterium plasma, starting at time zero and developing at density of near critical (10^{21} cm^{-3}) with a temperature of order 10^9 K for both ions and electrons. The laser intensity is taken to be that of a neo-dymium glass laser of 10^{16} W cm^{-1} (square pulse) with wavelength, 1.06 μm. The density being important, as it changes the refractive index and can result in total reflection of the source of the driving force. (Maddever et al, 1990).

To bring about fusion it is required that the nuclei of the deuterium are forced together so as to overcome the Coulomb force of repulsion. To bring them into proximity of a Fermi (10^{-15} metre). This means increasing the density of the ions, as it is the ions, which constitute the site of the nuclei. Examining some of the variables it can be seen the longitudinal electric field is zero at the outset, Figure 3.6. The electron fluid expands faster, thus developing an electric field. The electrons change direction, as discussed earlier, leading to an oscillation which appears as the plasma frequency. This being electron density dependant. Given enough time the field becomes a uniform internal one of 10^6 V cm^{-1} (Hora, 2000, Lalousis et al 1983). Spatially this occurs over 10^3 cm.

Once the laser is turned on at 0.6 psec, longitudinal Langmuir oscillation results. Their amplitude is about one-tenth the transverse laser field amplitude. Unfortunately non-conservative characteristics result when very strong oscillations occur due to the laser. This has been explored by workers such as Umstadter (1996), Lalousis (1983) and Eliezer (1998). Indeed Umstadter found using a 25TW-pulsed laser produced 30 MeV electrons in vast numbers and the plasma acceleration mechanism resulted. Lalousis et al, 1983, took the conservation equations and derived the following from the oscillation equation:
\[
\frac{\partial^2 E}{\partial t^2} + \nu \frac{\partial E}{\partial t} + \omega_{p0}^2 E = E_{s0} \omega_{p0}^2 + \frac{4\pi e}{Me} \frac{\partial}{\partial x} \frac{E_L^2 + H_L^2}{8\pi} + 4\pi e \nu (n_i v_i - Z_n e v_e)
\]

Where

\[
E_{s0} = \frac{4\pi e}{\omega_{p0}^2} \left[ \frac{\partial}{\partial x} \left( \frac{3n_i k T_i}{m_i} + Z n_i v_i^2 \right) - \frac{\partial}{\partial x} \left( \frac{3n_e k T_e}{m_e} + n_e v_e^2 \right) \right]
\]

\[
\omega_{p0}^2 = 4\pi e^2 \left( \frac{n_e}{m_e} + \frac{Z^2 n_i}{m_i} \right)
\]

The solution of which is the longitudinal electric fields, the electrostatic of Langmuir, termed \( E_s \)

\[
E_s = \frac{4\pi e}{W_p^2} \left[ \frac{\partial}{\partial x} \left( \frac{3n_i k T_i}{m_i} + Z n_i v_i^2 \right) - \frac{\partial}{\partial x} \left( \frac{3n_e k T_e}{m_e} + n_e v_e^2 \right) \right]
\]

\[
+ \frac{1}{m_e} \frac{\partial}{\partial x} \left( \frac{E_L^2 + H_L^2}{8\pi} \right) \left( 1 - \exp(-\nu t/2 \cos \omega_e t) \right) \]

\[
+ \frac{\omega_p^2 - 4\omega^2}{(\omega_p^2 - 4\omega^2)^2} \frac{4\pi e}{m_e} \frac{\partial}{\partial x} \left( E_L^2 + H_L^2 \right) \cos 2\omega t \]

\[
+ \frac{2\nu \omega}{(\omega_p^2 - 4\omega^2)^2} \frac{4\pi e}{m_i} \frac{\partial}{\partial x} \left( E_L^2 + H_L^2 \right) \sin 2\omega t
\]

Other forms of which appear in Goldsworthy, Hora, Stenning p35 1990. The original idea from workers like E.S Weibel, 1958. This would suggest one should see rippling through the small plasma sample (25\( \mu \)m), over very brief time periods (0.5 to 1.5 psec) once energy is being inputted from a laser.

This rippling should be evident from the bunching and dragging of the plasma. The density profile should reflect it, and hence phase reflection would be expected at these ripples, this in turn halting the acceleration, thus the hydrodynamic disappearing of the ripple within picoseconds. (Hora,2000,1988,1980, Eliezer,1988).
Figure 3.6 shows the zero internal field (given the accuracy of this work and not to undo the earlier comment on Alfven's work). At 1.1 picoseconds the field builds rapidly (over 0.2 picoseconds) and at the virtual surface of the tiny piece of plasma, that is within the first 8 micrometres. The first field peaks at approximately $10^8 \text{ Vcm}^{-1}$, the field dropping off from 20 micrometres. The movement of the electrons surrounding their "trailing behind" ions builds the field over this distance. In reality a very short distance, nearly half an interatomic distance. Clearly the graph shows a polarity reversal at 7.5 micrometres and after time 1.2 to 1.5 picoseconds. This is evidence for the direction undertaken by the electrons and hence the ions. This is a ripple effect as can be seen by the diminished, but still visible, ripple from 10 to 20 micrometres. By 25 micrometres there appears to be little field, but, a more sensitive probe should detect an ever-decreasing undulation of the field, given a larger piece of plasma.

The actual electromagnetic energy density of the laser field is shown in Figure 3.7, $(E^2 / 8\pi)$. This graph gives an indication of how much energy is actually deposited by the laser and where/when is it available. Pulsation is again detected both over time and through distance. It an be seen to be time dependant starting at 0.6 psec, dropping off and reforming at 0.7, and so on. The maxima would correspond to strong penetration. Each ripple corresponds to acceleration stopping (see Figure 3.8A and Figure 3.9). This is the stuttering effect reported by Maddever and also Hora. It shows the energy is not being evenly delivered over time and throughout the plasma. The peak is reached at around 0.9 picoseconds, dropping rapidly down by 1.1 picoseconds, then a smaller rise at 1.5 picoseconds. This same pulsation in or stuttering is evident through distance. The greatest density or deposit of energy being near the surface at 4-5 micrometres. Looking over time, the maximum densities occur at this distance. Further suggesting the energy does not penetrate very deeply and making pellet size a critical factor in obtaining fusion; a point that will be taken up in chapter four. The rippling shows the density changes, which result in reflection of the laser pulses. From the laser point of view a smoother delivery of energy is required. The field structures so far noted are results of the non-linear force. This same force will be the determining force for all the ion (and electron) movement determining graph shape, and are results derived from the 'genuine' two-fluid model.
The field density pulsation should also be seen in the ion density changing, thus effecting reflectivity. Low ion density should relate to higher laser field energy densities. By comparing Figure 3.7 and Figure 3.8A it can be seen the field values relate fairly well to the ‘dip’ in ion density at 5 or so micrometres. The wall of ions built up at the plasma surface accumulated over a few picoseconds. The movement to that position was against the direction of the laser beam. As noted a slight ripple is seen at a distance of 4-5 micrometres. Little effect is seen then until 12.5 or so micrometres on; an almost unperceivable ‘bump’. One would speculate that given more time and continuing irradiation this ion wall would migrate through the plasma. The pulse is better defined in Figure 3.8B.

It was suggested by Weibel in the early 1950’s, that this wall could be made to surf on the field wave. The idea being to have an input wave collide with a reflective wave and hence fuse the ions. Certainly, on first examination of this idea, it would seem feasible that if enough energy could be imparted to the two opposing waves that Coulomb force of repulsion could be overcome, the ions brought within a Fermi and then the nuclear force would take over fusing the nuclei. The time over which this event took would be crucial, and it is here the idea fails, (Hora, oral communication). One still feels the basic idea could be explored further.

Figure 3.9 shows the vector nature of ion velocity. Little movement until 0.6 picoseconds. Until that time, when the laser is activated the ions are uniformly travelling in all directions at relatively slow velocities. Taking one dimension they are travelling at \(-19000 \text{ cms}^{-1}\) and \(19000 \text{ cms}^{-1}\), uniform expansion of the 25-micrometre plasma. As the laser energy sets up the non-linear force the ions are greatly accumulated diminishing the original expansion velocities, even those quoted which occur on the periphery. For example, after 1.0 picoseconds and 2.5 micrometres the speed (magnitude of velocity) reaches \(9.0 \times 10^6 \text{ cms}^{-1}\), i.e around 9000 ms\(^{-1}\), 32,400 kilometres an hour or nearly three times the speed of sound in air. This expressed it in terms that the average person could appreciate; like "jet plane" speeds. At 1.1 picoseconds this has increased to \(9.5 \times 10^6 \text{ cms}^{-1}\) or 95 k\(\text{ms}^{-1}\) (342,000 km\(\text{h}^{-1}\)).

Clearly Figure 3.9 shows the oscillating nature if the values. From the surface rise over time to a high positive value, \(4.25 \times 10^5 \text{ cms}^{-1}\), the graph plummets, to high negative values \(-8 \times 10^6 \text{ cms}^{-1}\). This shows the ions have charged in one initial direction

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only to do an about face and charge in the other, through a distance of about 10 micrometres. In the next 5 or so micrometres the direction is again reversed, but, by 15-20 micrometres the forces seem to suffer a delay in their effect on the ions. Only by about 17.5 micrometres is the next undulation building and then not as dramatically. It can be seen that it takes time to build the moving walls of ions. In addition the largest proportion of effect is in area of immediate contact with the laser. It is further noted that the response time to cause a change in direction is very quick, a few picoseconds. Further the space required for the turn around is short, nearly half a Fermi; (less by an order of magnitude). A point of curiosity is the distance created between packets of undulation, some 5 micrometres. Given the Debye length is (Huba, 2000)

\[ \lambda_D = \left( \frac{kT}{4\pi ne^2} \right)^{1/2} \]
\[ = 7.43 \times 10^2 \ T^{1/2} \ n^{-1/2} \ cm, \ \text{sub in} \ 10^7 \ K. \]
\[ = 7.43 \times 10^2 \ (316.27766) N^{-1/2} \ cm \]
\[ = 2349572.301 \ n^{-1/2} \ cm \]

Taking \( n \) to be Avogadro number of ions

\[ = \frac{2349572.301}{7.760219069 \times 10^{11}} \]
\[ = 3.027713883 \times 10^{-6} \ cm \]
\[ \approx 3 \times 10^{-8} \ m \ \text{or} \ 0.03 \ \text{micrometres} \]

In a hot plasma \( (T, V \approx 10^2, n = 10^{14}) \), \( \lambda_D = 7 \times 10^{-4} \ cm \) for thermonuclear \( (T, V \approx 10^4, n = 10^{15}) \lambda_D = 2 \times 10^{-3} \ cm \) (Huba, 2000). For the sample chosen, at temperature chosen, \( n = 10^{21} \text{cm}^{-3} \), \( \lambda_D \) becomes \( \approx 0.743 \) micrometres, a reasonable value for Figure 3.9.

Given the theoretical value of thermal ion velocity is

\[ v_n = (kT/m_i)^{1/2} = 9.79 \times 10^5 \mu^{1/2}T_i^{1/2} \text{cms}^{-1}. \]

Where \( \mu = m_i / m_p \) which for \( ^1H \) is 1.0080602 approximately, while \( ^2H \) is 2.016992018.

Thus the velocity is for \( ^1H \): \( 3.071 \times 10^9 \text{cms}^1 \) and \( ^2H \): \( 1.5 \times 10^9 \text{cms}^1 \). These values are in reasonable agreement with those of maximum value for Figure 3.9. The discrepancy probably lying in the units used.
During the time that these graphs represent the temperature remains reasonably constant. This being in accord with the concept of adiabatic expansion. It is taken as \(1 \times 10^7\) K at the start, over the next time unit, the 0.6 picosecond it barely changes. Figure 3.10 shows a rise at 0.7 to 0.9 picoseconds on the surface. It is over the 0.7 to 1.0 picosecond the temperature rise for the electrons to migrate through the plasma. This creates a hot spot, a depth of 7.5 to 10 micrometres. This translates to maximum thermal movement of the electrons and corresponds the ‘calm period’ of ion velocity in Figure 3.9. This could be interpreted to illustrate a time delay in electron movement setting up a field to drag the ions. Certainly electron temperature is at base level when ion velocity is undergoing its greatest changes. Electrons being at least 2,000 times lighter than ions are more easily affected. Writers such as Hora and Lalousis, claim the effect on the ion is of the order of \(m_i/m_e\) less. The equi-partition time given by Spitzer (Lalousis, 1983) is around \(3 \times 10^{-10}\) seconds for the parameters used.

The variables were chosen from the two fluid model, as expressed in the continuity equations (3.54a and 3.54b); and the equations of motion (Hora, 1991). Due to the very limited computing power available, numerically discrete variables were chosen based on earlier computerised models using the step Lax-Wendroff method, (Lalousis, 1983). In these the stepwise grid point formula, coupled with boundary conditions, could be used to find values for \(f_{NL}\) in equations (3.45), (3.46g), (3.46h), (3.47). The electric field, changing over time, derives from the Poisson’s equation, after equation 3.47.

\[
\frac{\partial E}{\partial t} = 4\pi e (n_e v_e - Zn_i v_i)
\]

These values were simulated using the SPLUS package, and each set against discrete time units. This was done to better highlight the rippling in the plasma as discussed by many previous authors (Hora and Lalousis). Many of these values agreed with these previous workers, but the idea here was to view all of them over time, howbeit, very short units of it.
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Figure 3.6 Electric field inside the plasma changing over time. The field being generated around the focus of the laser beam in the plasma.

Figure 3.7 Electric field density of the laser. The energy concentration is in Joules per cubic centimetre. A caviton can be seen at about 0.4 picosecond and at a distance of 3.5 micrometre.
Figure 3.8A  
Ion density, per cubic centimetre. The number of ions being taken in order of $10^{18}$.

Figure 3.8B  
Ion number density shows the rippling effect better than ion density.
Figure 3.9  Ion velocity in centimetres, illustrating the surface effect of the initial activity with “calm period” leading to next rippling effect.

Figure 3.10  Electron temperature in units of $10^7$ degrees Kelvin.
CHAPTER 4

NUCLEAR FUSION AND ENERGY GAINS

4.1  NUCLEAR FUSION – INTRODUCTION

A process involving the nuclei of atoms. A process in which two small nuclei (below iron, element 26 in periodic table) join together to produce a larger nucleus, with an increase in binding energy and consequently a release of energy. The process has been known about since the 1920’s. Rutherford, Oliphant and Harteeck (Gross, 1984) performed collision experiments with deuterons in 1934. These experiments mark the beginning of the study of nuclear fusion. Indeed during Project Manhattan, the building of the fission A-bomb, Edward Teller urged J. Robert Oppenheimer to let him pursue the "super bomb" or H-bomb.

The energy is obtained by a conversion of mass into energy, and hence, a decrease in the equivalent mass. The amount can be calculated using Einstein’s famous $E = \Delta mc^2$, $\Delta m$ the change in mass from constituent parts to total mass. By a choice of MeV units for energy and not Joules, as well as binding energy (B.E) per nucleon; the calculation is simplified. For example, taking the final step in a series of fusion reactions which occurs in stars (one could have taken any of 51 listed by Gross in 1984)

$$^3_1H + ^2_1H \rightarrow ^4_2He + ^1_0n + 17.6 \text{ MeV}$$

<table>
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<tr>
<th>ELEMENT</th>
<th>AVERAGE BINDING ENERGY PER NUCLEON (MeV)</th>
<th>TOTAL (B.E) (MeV)</th>
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<td>TOTAL BINDING ENERGY OF REACTANTS</td>
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<tr>
<td>$^4_1He$</td>
<td>4×7.07=28.28</td>
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Increase in Binding Energy = (28.28-10.71) MeV≈17.6 MeV. Often this energy is given off as γ rays.

There is more energy to be harvested via fusion than fission. The problem is one of controlling the process and using it with temperatures that are obtainable on Earth. The reaction cited above takes place at 10⁸ K. How do you hold a substance at that temperature, even ignoring the outward blasting energy? Before that question comes a more fundamental one; if nuclei are overall positive, like charges repel, how do you overcome the Coulomb (electrostatic) forces of repulsion? The nucleons have to be within the operating distance for the strong nuclear forces to bind them. Fission began with a reactor, Fermis' atomic pile under a baseball stadium in Chicago. Then came the bomb, Fusion began with the bomb, Bikini Atoll in the early 1950's, now must come the reactor!

Unfortunately, due to vested interests, or ignorance, the first effort focused on the problem of confinement. Only with time and the painstaking work of researchers such as Professor H. Hora, J. Nuckolls etc, has the idea of inertial confinement fusion been allowed to mature. The theories, techniques and work came too close to weapons research for some governments to allow it a free reign. The invention of the “solution looking for a problem”, the laser opened new visors for ICF. The idea of using small pellets of plasma, being hit so quickly that their own inertia keep them in place while fusion was allowed to occur. It meant nuclear energy without radioactivity if the right reaction and conditions were used. That has been a large part of the problem, just what are those? Throughout the decades many predictions of how many years would be required to develop a reactor. Most have been far too optimistic (no authors are deliberately cited here!). Certainly with intense effort such as it took to develop the atomic bomb, or land a man on the Moon, a ten to twenty year period is viable. Probably more realistic would be Hora’s 30 years and 80 years for really financially viable ie cheaper than light water reactors.

The physics is in place, and many nations (Japan, France and U.S.A) are pursuing the engineering. The basic difficulties now lie in two areas (Hora, 1991). One is the requirement for very high-energy pulses of laser energy. The other problem, is the interaction of the laser light with the plasma; as Hora (1991) puts it on p330 of his
1991 monograph “the plasma ‘does everything’ to prevent the input of energy from lasers.” One needs to get the plasma to absorb the energy before it moves!

Bigger and better lasers are always being built; there is a great deal of commercial and military impetus there. Hence, the focus of this writing has been the plasma, and what conditions are the best for fusion, The next obvious question is what energy gains can be hoped for? This is energy gained over and above that used by the laser in initiating and maintaining the process. It is no use having a reaction that produces just enough energy to sustain itself.

4.2 NUCLEAR FUSION – REACTIONS

The stars exhibit two basic reactions, the carbon-nitrogen cycle and the proton-proton cycle. These are confined by massive gravitational forces. These are out of our reach.

Oliphant, Harteck and Rutherford developed in 1934,

\[ D + D \rightarrow 50\% \ 3^7He + n + 3.27 \text{ MeV} \]  
\[ D + D \rightarrow 50\% \ 3^7T + p + 4.03 \text{ MeV} \]  

And

\[ D + T \rightarrow 4^2He + n + 17.6 \text{ MeV} \]  

The tritium being bred from \(^7Li\) by neutrons from (b) or (a). (Hora, 1991). It is noted that magnetic confinement uses reaction (b). This leads to reactor damage caused by the neutrons; it is hence radioactive. The containment vessel wall can lose a centimetre a day (Vieider et al, 1989). The exposure of this information caused problems and goes to illustrate the problem of vested interests hampering scientific progress. Among the many possible fusion reactions, those utilising the heavy hydrogen isotope, deuterium (D) combined with the even heavier isotope of hydrogen, the radioactive tritium (T). Restating it (Stenning et al, 1992)

\[ D + T \rightarrow n\left(14.1 \text{ MeV}\right) + \frac{3}{2}He\left(3.5 \text{ MeV}\right) \]
This one is marked for best economic fusion energy, but has this high energy, compared to $\alpha$ particle, radioactive neutron. With inertial confinement fusion a better reaction to use is

$$H^4 + ^{11}_5 B \rightarrow ^{4}_2 He(2.88 \text{ MeV})$$  \hspace{1cm} (D)

(B.C. Maglich of Princeton quotes the energy for this reaction at 8 MeV, in his editorial preface to Weibel’s paper, 1958, and Hora 1991 as 8.9 MeV. Stenning’s figure is taken to mean per $^3_2 He$; giving a total of 8.64 MeV).

The burning of coal releases more radioactivity per unit of energy via the amount of uranium released (Stenning et al, 1992; Hoyle and Hoyle, 1980). This is still under investigation. In the same vein the reaction

$$D + ^3_2 H_e \rightarrow H(14.7 \text{ MeV}) + ^4_2 He(3.6 \text{ MeV})$$  \hspace{1cm} (E)

Is a candidate for clean electrical energy production. The 50% each, competing reactions (detail of a))

$$D + D \rightarrow n(2.4 \text{ MeV}) + ^3_2 He(0.8 \text{ MeV})$$
$$D + D \rightarrow H(3.0 \text{ MeV}) + T(1.0 \text{ MeV})$$  \hspace{1cm} (F)

are viable but radioactive. Further, nuclear spin polarisation can suppress this reaction. This aside, fusion gain calculations with $D-T$ reaction are pursued; as are the $D^4_2 He$ reactions, and $H^{11}_5 B$. The reactions are to be ideal, with adiabatic compression. The plasma composition will vary according to three types listed. The parameters will be spherical volume $V_0$, hence radius $R_0$, $T_0$ velocity $v_0$ of pellet radius, with a linear velocity profile into the centre of the pellet. From this $G$, or fusion gain can be calculated, given as

$$G = \frac{\text{Reaction energy}}{\text{Input energy}} = \frac{\varepsilon_{xy}}{E_0} \int dt \int dr \frac{\rho_i^2}{A} <\sigma v> \hspace{1cm} -4.1$$

$x, y$ being the constituents of the plasma. $<\sigma v>$ velocity averaged fusion cross-section. $A=4$ for binary reaction, or 2 for cases like $DD$. (Stenning, et al, 1992).
It needs to be noted that reactions take time to run their course, so this often the independent variable against which temperature or alike is plotted. It takes time to start, and after a certain time it is finished; hence a range is used. At the beginning of this chapter the point was made about the high temperatures involved. This causes the constituents to be moving very fast, so efforts are made to keep the particles together to fuse. For example, a $DT$ plasma, needs a density $n \approx 10^{16} \text{cm}^{-3}$, at $T = 10^9 \text{K}$; a sufficiently long time $\tau$ is required within set volume within the thermonuclear reactor. The time involved in this case is determined by the energy content $Q$ of the plasma, energy losses $W$ (this energy lost through heating the walls, electron bremsstrahlung, neutron emission, etc).

$$\tau = \frac{Q}{W}$$  \hfill (4.2)

The smaller the losses, allows a longer confinement time. The reaction’s intensity is $n \tau$ given by the confinement parameters like higher density; shorter the time to required for a given number of nuclei to interact. For example, $n \tau = 10^{14} \text{scm}^{-3}$ for temperature near $10^8$, using minimum time and density (Hora, 1991). For energy released in a thermonuclear reactor to exceed energy consumed, the criterion above must be satisfied. This is known as Lawson’s criterion (published in 1957). It involves a definite combination of the confinement parameter $n \tau$ and temperature $T$. (Gross, 1984; Dolan, 1982).

For ICF the time it takes for the pellet to fly apart, $\tau_D$ "disassembly time" is important, it relates to the time it takes for a sound wave to traverse the pellet. So the speed of sound in a $10keV$, $D$-$T$ plasma is $10^8 \text{ms}^{-1}$, $\tau_D \approx 1\text{ns}$ hence fuel density in excess of $n = \frac{10^{14}}{\tau_D} \sim 10^{23} \text{cm}^{-3}$ about liquid state density. If a small, liquid density $D$-$T$ pellet can be heated to thermonuclear temperature before it can expand, $\approx 10^{-9}$s, $E \approx 1MJ$ or $0.28kWh$, (enough to run the television for the night) is delivered in $1\text{ns}$; power level reaches $10^6/10^{-9} = 10^{15}W$. So the energy of fusion can be viewed as (Dunderstadt and Moses, 1982)
\[ E_{\text{fusion}} = E_{\text{thermal}} + E_{\text{radiation}} \]

\[ E_{\text{fusion}} = n_0 n_\tau < \nu \sigma > W_\tau = \frac{n^2}{4} < \nu \sigma > W_\tau \]  \hspace{1cm} \text{-4.3}  

\(< \nu \sigma > = \text{Maxwellian averaged reaction rate parameter}\)

\[ W = \text{energy released per fusion reaction, eg for DT, 17.6 MeV}\]

\[ \tau = \text{Confinement time}\]

\[ n = \text{ion density number } n_0 - n_\tau = \frac{n}{2} \]

If ideal gas behaviour

\[ E_{\text{th}} = \frac{1}{2} nkT_i + \frac{1}{2} nkT_e = 3nkT, T_i = T_e \]  \hspace{1cm} \text{-4.4a}  

Taking \( E_{\text{rad}} \approx 0, T = 4keV \), fusion energy release exceeds bremsstrahlung radiation loss, for ICF 20 to 100 keV; magnetic fields can be ignored, cyclotron radiation is of little concern.

\[ \frac{E_{\text{fusion}} \approx E_{\text{thermal}}}{\frac{n^2}{4} < \nu \sigma > W_\tau = 3nkT} \]  \hspace{1cm} \text{4.4b}  

\[ n \tau > \frac{12kT}{< \nu \sigma > W} \]  \hspace{1cm} \text{-4.5}  

at 10keV DT, 100keV DD – Lawson’s criteria urges DT reaction.

Vital for ICF is an alternative to Lawson’s criteria. Here the interest is in fuel density \( \rho \), which is radius \( R \) dependant, rather than \( n \tau \). The disassembly time is not as the pellet radius \( R \) divided by the speed of sound, as stated before, from exterior to interior centre, needs to be established:

\[ \text{Fuel disassembly time } = \tau_d \sim \frac{R}{V_s} \]  \hspace{1cm} \text{-4.6a}  

128
The thermonuclear reaction time is not as inverse of the reaction rate can be offered as:

\[
\text{Thermonuclear reaction time} = \tau_b \sim \left( \frac{\rho}{m_i} \right) \frac{1}{\langle v \sigma \rangle} \]

-4.7

These two rates offer an estimate of the efficiency of thermonuclear burn

\[
f_b = f_\delta = \left( \frac{\langle v \sigma \rangle}{m_i v_s} \right) \rho R \]

-4.8

This gives an idea of the fraction of the fuel consumed.

\[
v_s = \left( \frac{kT}{m_i} \right)^{1/2} \sim T^{1/2} \]

-4.9

As expressed DT fuel operates at efficient temperatures of 20 to 80 keV (Duderstadt and Moses, 1982), this gives:

\[
\frac{\langle v \sigma \rangle}{m_i v_s} \sim const \sim 1 \]

-4.10

The burn fraction can be expressed as:

\[
f_b \sim \rho R \text{ in grams per square centimetre} \]

-4.11

As stated in reaction (C), the alpha particle gets 3.5 MeV, hence the fuel rise must exceed the alpha particle. This gives for DT, a 20 keV, \( \rho R \approx 0.5 \text{ g cm}^{-2} \), efficient self-heating could occur.

In fusion reaction binary collisions occur, the fusion reaction itself plus collisions which slow down charge particle production and energy deposition (or self-heating). Such processes depend on \( \rho^2 \); so to increase the density by \( 10^3 \), decreases the collision rate by \( 10^6 \).

\[
\text{Rate of \{thermonuclear burn, energy deposition by charged particles, electron-\ion energy exchange\} } \sim \rho^2
\]

129
As inertial confinement time ~R, each major process for ICF can be expressed per unit mass. This includes burn efficiency, self-heating and burn propagation; all ~ \rho R. For \[ DT \quad \rho R \approx 3 g \text{ cm}^{-2} \], as will be used a bit further on; Lawson criterion gives \[ n \tau > 10^{14} s \text{ cm}^{-3} \], a new aim for ICF is \[ \rho R > 3 g \text{ cm}^{-2} \].

Briefly the relationship between the \( \rho R \) and \( n \tau \) criteria will be examined. (Gross, 1984; Dolan, 1982; Dunderstadt and Moses, 1982).

Taking a freely expanding sphere of R

\[
\tau_d \sim \frac{R}{4v_s}
\]  

-4.6b

In a spherical fuel pellet, half the mass is beyond 80% of the radius.

Number density \( n = \frac{\rho}{m_i} \)  

-4.12

\[ n \tau = \frac{\rho R}{4v_s m_i} \]  

-4.13

Using some appropriate numbers

\[ \rho R = 3 g \text{ cm}^{-2} \Rightarrow n \tau = 2 \times 10^{15} s \text{ cm}^{-3} \]

For an efficient thermonuclear burn, \( n \tau \) must be well in excess of Lawson criterion, \( 10^{14} s\text{ cm}^{-3} \). It is noted here that MCF, working close to this burns a small fraction of the fuel. This is not good. Looking at depletion effects as accounted for by burn fraction \( f_b \). The rate equation for tritium fuel density.

\[
\frac{dn_t}{dt} = -n_D n_T < v \sigma > \quad \sigma \text{ being for DT} 
\]  

-4.14a

Using equimolar densities \( n_D = n_T = \frac{\gamma}{2} \)

\[
\frac{dn}{dt} = -\frac{n^2}{2} < v \sigma > 
\]  

-4.14b

Integrating from \( t=0 \) to \( t = \tau_d \)
\[ \frac{\gamma_n - \gamma_0}{\gamma_0} = \frac{1}{2} < \nu \sigma > \tau_d \quad n_0 = \text{Initial fuel number density.} \]

Defining the burn fraction \( f_b \) now as

\[ f_b = \frac{n_0 - n}{n_0} = 1 - \frac{n}{n_0} \quad 4.15 \]

Taking as a disassembly time \( \tau_d = \frac{R}{4v_s} \) and \( \rho = nm_i \)

\[ \rho R = \left( \frac{8m_i v_s}{< \nu \sigma >} \right) \frac{f_b}{1 - f_b} \quad 4.16 \]

Or

\[ f_b = \frac{\rho R}{(8m_i / < \nu \sigma >) + \rho R} \]

Evaluating the denominator for \( DT \) fuel are \( 20keV \)

\[ \frac{8m_i v_s}{< \nu \sigma >} \sim 6.3 g cm^{-2} \]

Allowing for fuel depletion, \( f_b \) for \( DT \) becomes

\[ f_b = \frac{\rho R}{6.3 + \rho R} \quad 4.17 \]

\( \rho R \sim 3 g cm^{-2} \) gives burn fraction of \( f_b = 0.30 \) that is a burn of some 30% of the fuel.

Now it is required that an explanation of the \( < \nu \sigma > \) be offered, in terms of a fusion reaction. The theory is best defined in the literature via weak interactions – (decay, reactions such as the emission of E.M radiation from an excited atom), and quantum electro dynamics using scattering theory of Quantum Mechanics, QM

Two bodies collide:

\[ A + b \rightarrow C + d \quad \text{or} \quad A(b, d)c \quad (i) \]

It could also occur

\[ A + b \rightarrow \begin{cases} Z + c \\ W + d \end{cases} \quad (ii) \]
This is what in QM is referred to as scattering and leave the same species as entered it. A nuclear reaction is then a collision process where the species of nuclei undergo changes, and all this governed by nuclear forces. As was stated earlier in the chapter, the Coulomb force of repulsion, or Coulomb barrier has to be overcome, it is expressed as

\[ E_c = \frac{Z_1 Z_2 e^2}{R_0} \]

-4.18

\( Z_1 \) and \( Z_2 \) being the charges and \( R \) being the nucleus, which is given by:

\[ R = \frac{\hbar}{m_x c} A^{1/3} \]

-4.19

The kinetic energy of the nuclei has to exceed the \( E_c \) value in order to get close enough for the reaction to occur. The fusion reaction is one where the rest mass of the product is smaller than the rest mass of reactants. The difference is what is derived as energy. Fission of course is in the same category. (i) now is written

\[ A + B \rightarrow C + D + Q \]  

(iii)

\( Q \) represents the energy

\[ Q = [(M_A + M_B) - (M_C + M_D)]c^2 \]

(iv)

The \( M_D \) representing the respective rest masses. (This neglects \( \gamma \) ray involvement.) \( Q \) is kinetic energy, that shared between \( C \) and \( D \). The total energy depends on the rate of the reaction, known as the cross section, \( \sigma \). This can be defined as the transition probability between initial and final states of reaction, according to QM theory. (Ray, 1977).

From the viewpoint of elementary particle physics in an invariant form, forms as per: 2 particles, \( A \) and \( B \), four momenta \( p_a \text{ and } p_b \), \( n \) particles produced with four momenta \( p_1, p_2 \ldots p_n \), which are summed vectorally for a total. Q.M views this, in Dirac notation, a state undergoing a transition from initial state \( |i\rangle \) formed by 2 momenta \( p_a \text{ and } p_b \); to a final state \( \langle f | \). The final state is defined by momenta
variables \( p_1, p_2 \ldots p_n \). Heisenberg developed the concept of the S-Matrix to express this. The interaction of wave fields whose quanta are the particles are the objects considered. The ‘kets’ \(|i\rangle\) and ‘bra’ \langle f |) transition is the QM transition amplitude. The reaction obeys the conservation laws for energy and momentum, so the matrix element is:

\[
< f | s | i >= \delta_{f,i} + i(2\pi)^4 \frac{d^4}{dx^4} \left( p_a + p_b - \sum_{i=1}^{n} p_i \right) N(f | T | i) \tag{4.20}
\]

\( \delta_{f,i} \) being the Kronecker delta
\[
\delta_{f,i} = 0 \quad \text{if} \quad |i\rangle \neq \langle f |
\]
\[
\delta_{f,i} = 1 \quad \text{if} \quad |i\rangle = \langle f |
\]

\( \delta^4 \left( p_a + p_b - \sum_{i=1}^{n} p_i \right) \) being Dirac’s delta function in 4 dimensions.

\( N \) is a normalising constant, cancels out in final cross section expression.

\( T = “\text{transition matrix}”. \) (Ray, 1977).

From elementary particle theory, the cross-section \( \sigma \) describes the transition, probability per unit time

\[
\sigma = \frac{1}{2V\lambda(s, m_a^2, m_b^2)} \frac{1}{(2\pi)^{3-d}} \int \frac{d^3p_i}{2E_i} \delta^4 \left( p_a + p_b - \sum_{i=1}^{n} p_i \right) \left| (f | T | i) \right|^2 \tag{4.21}
\]

\( E_i \) being the energy of the \( i \)th particle for the final product of reaction, re four momenta \( p_i \). The integration is over the space part of \( p_i \). Variable \( s \) presents the square of the total centre of mass energy in the initial state:

\[
s = \left( E_a^{cm} + E_b^{cm} \right)^2 \quad \text{(Mandelstam variable)} \tag{4.22}
\]

\( E_a^{cm}, E_b^{cm} \) being energies of \( A \) and \( B \) respectively in the centre of mass frame.

\[
\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yx - 2zx \tag{4.23}
\]

\( \sigma \) is obtained by theoretically calculating the matrix element \( \langle f | T | i \rangle \). As was stated at the beginning of ‘subsection’, it pertains to Q.E.D and weak interactions, when it can be explicitly calculated. This has not been done so far for nuclear forces, the
strong interaction. Assuming a non-relativistic scattering theory approach from QM, nuclear physics adopts Bohr's "compound nucleus" model. Sometimes referred to as "liquid-drop model" in fusion explanations. Essentially a two step approach. The first step is colliding nuclei to form a compound one in the excited state. The second step sees the decay, but there may be a few possible modes of decay, or "channels". It is as stated like the emission of E.M. radiation from an excited state in QM

\[ A + d \rightarrow C^* \rightarrow D + e \]  

\( C^* \) = Excited compound nucleus, which lasts only long enough so that the mode of formation is forgotten by the compound nucleus. This excludes total energy, parity, spin, and so on, thus the mode of decay is independent to its formation.

The cross-section is written:

\[ \sigma(d,e) = \sigma_i \frac{W_d}{W} \text{ where } W = \sum W_d \]

\( W_d \) being the decay probability of a particular channel to get \( W \). \( \sigma_i \) is the cross-section for formation of the compound nucleus. This compound model fits experiment reasonably well.

Definition of rate \( R \) gives the cross-section multiplied by the relative flux, for a unit volume

\[ R = \int n_1(v_1)n_2(v_2)|v_1 - v_2|\sigma(|v_1 - v_2|)d^3v_1d^3v_2 \]

\[ n_1(v_1) \]

\[ n_2(v_2) \]

number of respective nuclei present with velocities \( v_1 \) and \( v_2 \).

Along the assumption has been made these follow a Maxwell-Boltzmann distribution at temperature \( T \). (\( k \) is Boltzmann constant \( m_t \) the masses, \( n_e \) the densities.)

\[ R = n_1n_2\left(\frac{m_1}{2kT}\right)^\frac{1}{2}\left(\frac{m_2}{2kT}\right)^\frac{1}{2}\int d^3v_1d^3v_2|v_1 - v_2|\sigma \exp\left(-\frac{m_1v_1^2 + m_2v_2^2}{2kT}\right) \]

Defining new variables
\[ \mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2 \] 

\[ \mathbf{v} = \frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{m_1 + m_2} \] 

Using

\[ R = n_1 n_2 < \sigma v > \] 

\[ < \sigma v > = \left( \frac{m_1}{2\pi kT} \right)^{\frac{3}{2}} \left( \frac{m_2}{2\pi kT} \right)^{\frac{1}{2}} \int \int d^3 v_1 d^3 v_2 \sigma(v) v \exp \left( -\frac{MV^2 + \mu v^2}{2kT} \right) \] 

\[ = \frac{2}{\sqrt{\mu}} \left( \frac{1}{2kT} \right)^{\frac{1}{2}} \left( \frac{\mu}{m_1} \right)^2 \int_0^{\infty} E_{\kappa} \sigma(E_{\kappa}) \exp \left( -\frac{\mu E_{\kappa}}{m_2 kT} \right) dE_{\kappa} \] 

\[ \mu = \frac{m_1 m_2}{m_1 + m_2} \] the reduced mass

\[ E_{\kappa} = \frac{1}{2} m_1 v^2 \] bombarding energy of particle one for the reaction.

The value \( \sigma(E_{\kappa}) \) can be found experimentally or theoretically, then \( < \sigma v > \) can be calculated. For low energies the approximate formula

\[ \sigma(E_{\kappa}) = \frac{A}{E} \exp \left( -\frac{B}{\sqrt{E_{\kappa}}} \right) \] 

is used; \( A \) and \( B \) being constants. Various authors have made graphs for values of \( < \sigma v > \) having dimensions \( cm^3 \text{ sec}^{-1} \).

There are variations on the \( < \sigma v > \) formula. For example Gross, (1984):

\[ < \sigma v > = \frac{4}{(2\pi \mu)^{\frac{3}{2}} (kT)^{\frac{1}{2}}} \int_0^{\infty} \sigma(E_r) E_r x p - E_r / kT dE_r \] 

\[ E_r = \frac{1}{2} \mu v_r^2, v_r \] being the relative velocity

Stenning, \( et \ al \) (1992):
\[<\sigma v> = \frac{\sqrt{m_i}}{\sqrt{\pi (kT)^3}} \int_0^\infty \frac{m_i}{2} v^2 \sigma(v) \exp \left( \frac{m_i v^2}{kT} \right) dv^2 \quad -4.33\]

In terms of ICF.

\[<\rho R> = \rho \rho \int R = \frac{R_0 \rho_0(0) f(B)}{1 - \tau^2} \quad -4.34\]

4.3 NUMERICAL RESULTS OF THE ENERGIES (INPUT AND OVERALL GAIN) OF FUSION REACTION.

This work now intends to explore, by computer simulation (see appendix), the energy of the three reactions, \(D^3He\), \(H^{11}B\) and \(DT\). The implications of equation 4.1 are employed. The changes in volume, and density, as well as energy over time are examined. This is against the background of the material in sections 4.1 and 4.2.

In the case of \(D^3He\) especially the algorithm allows for four temperature changing effects:

\(T_{ad}:\) Adiabatic cooling, a thermodynamic expansion process

\(T_{\alpha}:\) \(\alpha\) particle reheat\(\) \(T_p:\) proton reheat \(\)Due to coulomb collisions with electrons of the plasma.

\(T_{br}:\) Bremsstrahlung

The full dynamic temperature becomes:

\[\Delta T = -T_d + T \nabla A + T_p - T_{br} \quad -4.35\]

The inertial temperature for each time step:

\[T_{n+1} = T_n + \Delta T, \quad T_0 = T_0 \left( \frac{E_0}{2 n_0 v_0 k_B} \right) \quad -4.36\]

Again, a non-linear mathematical description.

The overall process involves high compression involving
(a) Shaping in time by the input energy of the driver
(b) "Clever" pellet design  

(Eliezer et al, 1984; Hora, 1985)

The sequence is:

Energy absorption (of driver) → Energy transport → Compression → Nuclear fusion.

Following is a series of simulations of fusion gains verses input energy. The first fuel examined is $D^3He$. It is examined over a range of densities, which are expressed as so many times solid state; and a range of volumes. It will be seen that sometimes the density is fixed and the volume allowed to vary; while at others the volume will be fixed and the density varied. The energy is taken to be that of a 10% efficient laser. The energy offered is therefore the energy actually captured by the plasma, and is examined over a range of values. The reactions are viewed first omitting energy loss via bremsstrahlung (when later it was included it showed to have little effect at the parameters chosen). The number of points plotted to define the graph ranges from 1,000 to 10,000. The choice of values is based upon common values used for ICF by authors such as Stenning. The concept here being to examine values further a field purely to identify any trends. It is acknowledged that the plasma is usually expanding, so there is a time constraint on the time plasma was at the stated volume. Clearly, it is taken that fusion has occurred and the program gives what gain one could expect. Later temperature changes over time will be considered.

The first graphs to consider are Figure 4.1 and Figure 4.2. (The data for which is found in Table 4.1.) The volume is fixed at $1\times10^{-1}\text{cm}^3$ for while the density is explored over $4\times10^{-3}$ to $4\times10^{-10}$ and $4\times10^3$ to $4\times10^{10}n_e$, thus offering less solid state to ten times a giga solid state. The idea behind "4×" is to place the sample in the middle of the range; ie use intermediate values.

From Figure 4.1 it can be seen that small energy values maintain the reaction, but the gains are very small. Certainly not worth pursuing. Overall though an upward trend in the tangent to curves at maximum value is evident. Higher initial values are required but the gain trend is moving in the desired direction, in a beautiful uniform manner. The leap to tens, millions times solid state density, Figure 4.2 shows positive returns. These are still noted as very modest and not economically viable. As stated from the
chapters before, density is a vital factor. Further it is noted the degree of gain, compared to input energy drops off. $10^{16.1} J$ at $10^3 n_z$ gives ‘nearly’ the gain of $10^{2.2}, 10^{17.5} J$ (approx) at $10^{10} n_z$ gives $10^{3.2}$. Thus indicating it may not be worth trying to increase the density further.

In Figure 4.3A density range is examined against a much smaller volume. Yet again the low densities of $4 \times 10^{-10}$ to $4 \times 10^{-3} n_z$ while requiring much lower energy input, do not produce a gain worthy of pursuing. It is noted the gain increases linearly, but the energy input leaps ahead. Figure 4.4 gives ideal values with $4 \times 10^3$ to $4 \times 10^{10} n_z$ densities, operating on extremely small volumes. Gains of nearly 1,000 times ($10^{2.8}$) start to appear at $10^5 n_z$ approx. The input energies are within the range of the latest Terawatt lasers. Even at 1% a more realistic figure for today, $10^{10} J$ is available. This value matches up with densities of $10^7 n_z$. The low volume value could be matched to plasma fuel volume that a “spray” would be. Hence instead of injecting pellets, some sort of “atomiser” could be used to feed the chamber. The energy gains are worthy of pursuit.

Figure 4.5 follows this further. By using high density, $4 \times 10^{10} n_z$ fuel, still $D^3 He$ and varying the volume right down to $10^{-8} cm^3$. The gains do not greatly increase, only 1,000 times for a starting input energy of $1.4 \times 10^6 J$, with maximum gain being reached at around $10^{10.7} J$ input energy. Larger volumes require much larger energy inputs, which go outside the range of present and foreseeable future lasers.

Figure 4.6 shows that the requirement of very high densities to be relaxed to $4 \times 10^3 n_z$, but a range of droplet size volumes used the gain drops down to 251 times from Figure 4.5 the volume does not have to be so infinitesimal. Indeed the best returns are on $10^{-1} cm^3$ for $4 \times 10^3 n_z$ giving around 200 times gains for an energy investment starting at $1.4 \times 10^6 J$ and giving maximum returns at $6 \times 10^{10} J$. This is again the range of the larger lasers, especially at 1-% efficiency.

Testing the small volumes against the very small density of $4 \times 10^{-10} n_z$, Figure 4.7 shows the expected trend of very low energy inputs. There is, however, an indication
that increasing volume gives slight increase in gain. On closer examination this can be
seen to be offset by persistent, extremely low gains.

In summary, $D^3He$ show its optimal values for maximum gain of about 1,000 to be a
volume of $10^{-7} cm^3$ and a density of $4 \times 10^6 n_s$. A more realistic (as in obtainable now)
density would be $4 \times 10^3 n_s$, at a volume of $10^{-1} cm^3$, giving gains around 250 times.
The energy-input requirement is also much lower.

The next fuel to be considered is HB; starting with graph Figure 4.8 (the data coming
from Table 4.2): density is allowed to vary against a very small volume. Modest gains
of $10^{1.3}$, ie 19.95 are obtained from density $4 \times 10^5 n_s$, using a volume of $10^{-7} cm^3$. The
range of input is quite broad; $3.61 \times 10^4$ too $3.61 \times 10^{11}$. The peak occurs at $10^{8.1} J$, or
$1.26 \times 10^8 J$; again within the range available.

The next graph Figure 4.9 looks at the question of density being low against the same
tiny volume of Figure 4.8. It can clearly be seen the same trend as with $D^3He$ holds,
as the theory indicated, density has to be at a compression value of many times (in the
thousands) state for viable gains. Figure 4.9 shows minute returns.

Figure 4.9 only serves to further the view. Very small densities may only require small
input energies, but there is virtually no gain, for example $10^{-6.7}$ ie $1.995 \times 10^{-7}$

At this stage $D^3He$ appears to be a better fuel, purely on a gains basis. HB
at $4 \times 10^5 n_s$, volume $1 \times 10^{-7} cm^3$ give a gain of $10^{1.2}$ for an input energy of $10^{8.1} J$, as
opposed to for the same parameters $D^3He$ gives $10^{2.3}$, the input energy being $10^{7.1} J$.
The better $D^3He$ may actually be $4 \times 10^3 n_s$, volume of $10^{-1} cm^3$ and a gain of $10^{2.4}$.
The input energy is higher at $10^{8.1} J$, though, as opposed to $10^{7.1} J$ for $4 \times 10^5 n_s$. It is
the more realistic density value that highlights its choice. No matter which one is used,$D^3He$ is proving to be the better fuel, on a return for energy basis, against parameters
required to set it up. The role of bremsstrahlung so far does not appear to be major.
The scenarios were run including it and little change in the print out appeared. This
may not be the case when considering temperature effects. Further, these simulations
do not extend to hybrid reactions, such as $DD$ with $D^3He$. The work of Stenning (92) would show a slight fusion gain increase for such reactions.

**TABLE 4.1 GAINS VERSUS ENERGY FOR D$^3$He**

<table>
<thead>
<tr>
<th>DENSITY (Solid State Density)</th>
<th>VOLUME (cm$^3$)</th>
<th>INPUT ENERGY (Joules)</th>
<th>GAIN</th>
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TABLE 4.2  GAINS VERSUS ENERGY FOR HB

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<td>10^{1.3}</td>
</tr>
</tbody>
</table>

Turning now to the third fuel, DT, deuterium-tritium and using densities and volume relevant to the “successful” parameters for the other fuels. (See Table 4.3).

Figure 4.10, Figure 4.11 shows that at the smallest volumes and reasonable solid state densities, fusion gain is good. Taking values, volumes of 10^{-7} cm³, 10^{-5} cm³ and densities of 4×10^{4} n_e, 4×10^{5} n_e; fusion gains are 3.4 for input energy of 10^{5.1} J; 3.5 for 10^{6.2} J, respectively. Straight away this fuel is appearing superior to the other two.

Examining the trends Figure 4.12 gives for volume of 10^{-1} cm³, the higher density 4×10^{5} n_e gives a 3.3 gain for input E of 10^{12.2}. This is on the borderline for current energy delivery capabilities. Figure 4.13 further examines the density effect, using the same volume range for Figure 4.12, Figure 4.13 gives 4×10^{3} n_e, a middle range for Figure 4.12, a gain of 3.5 for an input of 10^{10.1}. A value within the energy range of today. By increasing the density to 4×10^{6} n_e, the return is the same but the energy requirement is much higher at 10^{15.1}, outside a realistic value for today. It can be seen that there is a suggested upper limit on density for gain. Variation in volume from
$10^{-1} \text{ cm}^3$ down to $10^{-7} \text{ cm}^3$ (Figures 4.14 and 4.15) reveals a real decrease in input for marginal increases in gain. See Table 4.3.

Volume reduction lowers the input energy while increasing the gain. Low densities matched against low volumes does not give even reasonable gains. A density of $4 \times 10^1 n_x$, volume $10^{-1} \text{ cm}^3$ gives a gain of $10^{2.5}$, for an input energy of $10^{8.5}$, while the same density for $10^{-5} \text{ cm}^3$ has a gain of only 0.2, for $10^5 J$! Clearly it is not volume alone, but a clever matching to right density. Re-examining density trends, in Table 4.3.

Increasing density increases the input energy right out the bounds of present day capabilities. By choosing the best return for lowest density, $4 \times 10^3 n_x$ giving $10^{3.5}$ and looking at that density for lower volume, $10^{-7} \text{ cm}^3$ the gain is only $10^{2.5}$. That is 3,162.3 to 316.2, respectively; the larger volume gives a thousand times improvement for increase of input energy of 398,107 times! $10^{4.5}$ verses $10^{10.1}$. Note though this is still within current technology. This trend can be seen by choosing the highest gain values from Table 3. From this table, a density of $4 \times 10^4 n_x$, with a volume of $10^{-7} \text{ cm}^3$ giving a gain of $10^{3.4}$ for an input energy of, $10^{5.1} J$ stands out as the best compromise. Especially appealing is the low input energy, coupled with a ‘reasonable’ density. The volume as for the other fuels suggest some sort of ‘atomiser’ approach to fuel pellet size.

**TABLE 4.3  GAINS VERSUS ENERGY FOR HB**

<table>
<thead>
<tr>
<th>DENSITY (Solid State Density)</th>
<th>VOLUME (cm$^3$)</th>
<th>INPUT ENERGY (Joules)</th>
<th>GAIN</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4 \times 10^1$</td>
<td>$10^{-1}$</td>
<td>$10^{8.5}$</td>
<td>$10^{2.5}$</td>
</tr>
<tr>
<td>$4 \times 10^1$</td>
<td>$10^{-5}$</td>
<td>$10^{5.0}$</td>
<td>$10^{0.2}$</td>
</tr>
<tr>
<td>$4 \times 10^3$</td>
<td>$10^{-1}$</td>
<td>$10^{10.1}$</td>
<td>$10^{3.5}$</td>
</tr>
<tr>
<td>$4 \times 10^3$</td>
<td>$10^{-7}$</td>
<td>$10^{4.5}$</td>
<td>$10^{2.5}$</td>
</tr>
<tr>
<td>$4 \times 10^4$</td>
<td>$10^{-7}$</td>
<td>$10^{5.1}$</td>
<td>$10^{3.4}$</td>
</tr>
<tr>
<td>$4 \times 10^4$</td>
<td>$10^{-1}$</td>
<td>$10^{12.2}$</td>
<td>$10^{5.3}$</td>
</tr>
<tr>
<td>$4 \times 10^5$</td>
<td>$10^{-5}$</td>
<td>$10^{8.2}$</td>
<td>$10^{1.5}$</td>
</tr>
<tr>
<td>$4 \times 10^5$</td>
<td>$10^{-7}$</td>
<td>$10^{6.2}$</td>
<td>$10^{3.4}$</td>
</tr>
<tr>
<td>$4 \times 10^8$</td>
<td>$10^{-1}$</td>
<td>$10^{15.1}$</td>
<td>$10^{3.5}$</td>
</tr>
</tbody>
</table>
Reviewing the best candidates from the three fuels, a simple table can be constructed, Table 4.4.

**TABLE 4.4 BEST VALUES OF GAINS VERSUS ENERGY**

<table>
<thead>
<tr>
<th>FUEL</th>
<th>DENSITY (Solid State Density)</th>
<th>VOLUME (cm$^3$)</th>
<th>INPUT ENERGY (Joules)</th>
<th>GAIN</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D^3He$</td>
<td>$4 \times 10^3$</td>
<td>$10^{-1}$</td>
<td>$10^{10.8}$</td>
<td>$10^{2.4}$</td>
</tr>
<tr>
<td></td>
<td>$4 \times 10^5$</td>
<td>$10^{-2}$</td>
<td>$10^{7.1}$</td>
<td>$10^{2.3}$</td>
</tr>
<tr>
<td>$H_2B$</td>
<td>$4 \times 10^3$</td>
<td>$10^{-7}$</td>
<td>$10^{8.1}$</td>
<td>$10^{1.3}$</td>
</tr>
<tr>
<td>$DT$</td>
<td>$4 \times 10^4$</td>
<td>$10^{-7}$</td>
<td>$10^{5.1}$</td>
<td>$10^{3.4}$</td>
</tr>
</tbody>
</table>

Clearly the outstanding fuel is $DT$ at density $4 \times 10^4 n_e$, volume $10^{-7} cm^3$ for an energy input of $10^{5.1} J$; the gain is $10^{3.4}$. The density and volume are within present capabilities, as is input energy; the return is starting to look like it is worth doing; 2,511.89 times.
4.4 NUMERICAL RESULTS OF TEMPERATURE TIME RELATIONSHIP OF THE PLASMA IN FUSION REACTIONS.

It is now time to look at the role of temperature. Starting with Figure 4.16 and Table 4.5, temperature-time dependence for three densities of HB at a fixed volume of $10^{-7} \text{cm}^3$. The energy input required is raised with increasing density. For example $4 \times 10^3 n_s$ is at $1.5 \times 10^9 J$ while $4 \times 10^5 n_s$ is at $2.5 \times 10^9 J$. The temperature is lowered by lowering the density; $4 \times 10^3 n_s, T = 7.6 eV$, while $4 \times 10^5 n_s, T = 10^{6.8} keV$. After $10^{-12}$ seconds the temperature drops off with adiabatic cooling for all densities Figure 4.16. Density $4 \times 10^5 n_s$ reached a maximum temperature of $105.9 \text{ keV}$. The optimum temperature for initial adiabatic burning during compression and then expansion was given by Stenning et al (1992) as $110 \text{ keV}$ or $1.1 \times 10^5 eV$. He arrives at the same conclusion that increased densities (especially $10^5 n_s$) along with reheat and self-absorption of bremsstrahlung reduces this value, he gives $20 eV, 2 \times 10^5 eV$.

To check this claim, this simulation was required to increase the volume of the plasma from $10^{-7} \text{cm}$ to $10^{-3} \text{cm}$. Figures 4.17 to 4.18, at that volume and $4 \times 10^5$ (keeping input energy constant) the temperature was reduced to $10^{1.8} eV$, well below his (Fig 4.17). A change in density to $4 \times 10^4$ lead to a temperature max of $10^{5.3}$ or $10^{2.2} eV$. This was the only graph showing the “peak” prior to rapid temperature decline.

Figure 4.17A, was a ‘test’ graph where the program was to reduce the density to $1 \times 10^5 n_s$ while maintaining the volume of from Figure 4.17, $1 \times 10^5 \text{cm}^{-3}$. The real test was to ‘spread’ the input energy base. This approach gave two ‘peaks’. This is virtually Stenning’s approach to illustrate the temperature spikes associated with volume ignition. The only other figure showing a trace of this is Figure 4.19 for DT, with $4 \times 10^3 n_s, 1 \times 10^4 \text{cm}^3$ at $3.2 \times 10^{10} J$ giving a ‘spike’ optimum temperature of $10^{5.6} eV$.

By comparing Figure 4.16 to Figure 4.18, it can be seen the time increases for increasing volume, while it also increases ever so slightly for decreasing density Figure 4.19 and Table 4.6 is for DT and by using the same range of densities and
concentrating on the larger volumes. Firstly, the only values showing the characteristic “peak” is on Figure 4.19, as stated. It is density $4 \times 10^5 n_z$, volume $10^{-1} \text{cm}^3$, maximum temperature of $10^{5.6} \text{eV}$ or $10^{2.5} \text{keV}$. The higher volumes, of DT involve lower temperature, eg $4 \times 10^5 n_z$ at $10^{-5} \text{cm}$ gives a maximum temperature of $10^{7.5} \text{eV}$. Figure 4.20 and Figure 4.21. DT seems to follow the same pattern as HB, raising the volume and the temperature is lowered quite considerably.

**TABLE 4.5 TEMPERATURE VERSUS TIME FOR HB**

<table>
<thead>
<tr>
<th>DENSITY (Solid State Density)</th>
<th>VOLUME (cm$^3$)</th>
<th>INPUT ENERGY (Joules)</th>
<th>OPTIMUM TEMPERATURE (eV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4 \times 10^3$</td>
<td>$10^{-3}$</td>
<td>$1.5 \times 10^9$</td>
<td>$10^{3.6}$</td>
</tr>
<tr>
<td>$4 \times 10^3$</td>
<td>$10^{-5}$</td>
<td>$1.5 \times 10^9$</td>
<td>$10^{5.6}$</td>
</tr>
<tr>
<td>$4 \times 10^3$</td>
<td>$10^{-7}$</td>
<td>$1.5 \times 10^9$</td>
<td>$10^{7.6}$</td>
</tr>
<tr>
<td>$4 \times 10^4$</td>
<td>$10^{-3}$</td>
<td>$2.0 \times 10^9$</td>
<td>$10^{2.7}$</td>
</tr>
<tr>
<td>$4 \times 10^4$</td>
<td>$10^{-5}$</td>
<td>$2.0 \times 10^9$</td>
<td>$10^{4.3}$</td>
</tr>
<tr>
<td>$4 \times 10^4$</td>
<td>$10^{-7}$</td>
<td>$2.0 \times 10^9$</td>
<td>$10^{6.8}$</td>
</tr>
<tr>
<td>$4 \times 10^5$</td>
<td>$10^{-5}$</td>
<td>$1.5 \times 10^9$</td>
<td>$10^{5.5}$</td>
</tr>
<tr>
<td>$4 \times 10^5$</td>
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<td>$2.0 \times 10^9$</td>
<td>$10^{5.5}$</td>
</tr>
<tr>
<td>$4 \times 10^2$</td>
<td>$10^{-5}$</td>
<td>$2.5 \times 10^9$</td>
<td>$10^{4.3}$</td>
</tr>
<tr>
<td>$4 \times 10^2$</td>
<td>$10^{-3}$</td>
<td>$2.5 \times 10^9$</td>
<td>$10^{1.5}$</td>
</tr>
<tr>
<td>$4 \times 10^5$</td>
<td>$10^{-5}$</td>
<td>$2.5 \times 10^9$</td>
<td>$10^{3.5}$</td>
</tr>
<tr>
<td>$4 \times 10^5$</td>
<td>$10^{-7}$</td>
<td>$2.5 \times 10^9$</td>
<td>$10^{4.9}$</td>
</tr>
</tbody>
</table>

**TABLE 4.6 TEMPERATURE VERSUS TIME FOR DT**

<table>
<thead>
<tr>
<th>DENSITY (Solid State Density)</th>
<th>VOLUME (cm$^3$)</th>
<th>INPUT ENERGY (Joules)</th>
<th>OPTIMUM TEMPERATURE (eV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4 \times 10^3$</td>
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<td>$3.2 \times 10^{10}$</td>
<td>$10^{3.6}$</td>
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<tr>
<td>$4 \times 10^3$</td>
<td>$10^{-3}$</td>
<td>$3.2 \times 10^{10}$</td>
<td>$10^{3.5}$</td>
</tr>
<tr>
<td>$4 \times 10^4$</td>
<td>$10^{-1}$</td>
<td>$3.2 \times 10^{10}$</td>
<td>$10^{2.5}$</td>
</tr>
<tr>
<td>$4 \times 10^4$</td>
<td>$10^{-5}$</td>
<td>$3.2 \times 10^{10}$</td>
<td>$10^{6.4}$</td>
</tr>
<tr>
<td>$4 \times 10^5$</td>
<td>$10^{-1}$</td>
<td>$3.2 \times 10^{10}$</td>
<td>$10^{4.5}$</td>
</tr>
<tr>
<td>$4 \times 10^5$</td>
<td>$10^{-5}$</td>
<td>$3.2 \times 10^{10}$</td>
<td>$10^{5.5}$</td>
</tr>
<tr>
<td>$4 \times 10^5$</td>
<td>$10^{-7}$</td>
<td>$3.2 \times 10^{10}$</td>
<td>$10^{7.5}$</td>
</tr>
</tbody>
</table>

On the basis of lowest optimum temperature HB at $4 \times 10^5 n_z$, volume $10^{-3} \text{cm}^3$ is the most manageable fuel, with a temperature of $10^{1.8} \text{eV}$. Against this DT has $4 \times 10^5 n_z$, volume $10^{-1} \text{cm}^3$, having a temperature of $10^{1.5} \text{eV}$.
Considering next the third fuel $D^3He$, Table 4.8, Figure 4.22 to Figure 4.24 shows that as the density is raised from $10^{10}n_e$ (unreasonable values for the sake of this argument) to $10^{12}n_e$, but volume is maintained at $10^{-8}cm^3$, the optimum temperature drops dramatically, $10^{3.4}$ to $10^{1.4}eV$. Figure 4.24 helps make the point that as volume increases, from $10^{-7}$ to $10^{-1}cm^3$, the temperature drops from $10^{7.4}$ too $10^{1.4}eV$ respectively. Now, having established the trend for $D^3He$, the best values of Figure 4.23 need to be ignored as too unrealistic. Clearly $4 \times 10^5n_e$ has emerged as the best density and when coupled with a volume of $10^{-1}cm^3$ gives a low temperature of $10^{1.4}eV$.

<table>
<thead>
<tr>
<th>DENSITY (Solid State Density)</th>
<th>VOLUME (cm$^3$)</th>
<th>INPUT ENERGY (Joules)</th>
<th>OPTIMUM TEMPERATURE (eV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4 \times 10^3$</td>
<td>$10^{-1}$</td>
<td>$3.2 \times 10^{10}$</td>
<td>$10^{3.4}$</td>
</tr>
<tr>
<td>$4 \times 10^3$</td>
<td>$10^{-3}$</td>
<td>$3.2 \times 10^{10}$</td>
<td>$10^{7.4}$</td>
</tr>
<tr>
<td>$4 \times 10^4$</td>
<td>$10^{-5}$</td>
<td>$3.2 \times 10^{10}$</td>
<td>$10^{5.2}$</td>
</tr>
<tr>
<td>$4 \times 10^4$</td>
<td>$10^{-6}$</td>
<td>$3.2 \times 10^{10}$</td>
<td>$10^{7.4}$</td>
</tr>
<tr>
<td>$4 \times 10^5$</td>
<td>$10^{-1}$</td>
<td>$3.2 \times 10^{10}$</td>
<td>$10^{1.4}$</td>
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<tr>
<td>$4 \times 10^5$</td>
<td>$10^{-6}$</td>
<td>$3.2 \times 10^{10}$</td>
<td>$10^{7.4}$</td>
</tr>
<tr>
<td>$4 \times 10^5$</td>
<td>$10^{-7}$</td>
<td>$3.2 \times 10^{10}$</td>
<td>$10^{7.4}$</td>
</tr>
<tr>
<td>$4 \times 10^{10}$</td>
<td>$10^{-1}$</td>
<td>$3.2 \times 10^{10}$</td>
<td>$10^{3.4}$</td>
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<tr>
<td>$4 \times 10^{11}$</td>
<td>$10^{-8}$</td>
<td>$3.2 \times 10^{10}$</td>
<td>$10^{2.4}$</td>
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<tr>
<td>$4 \times 10^{12}$</td>
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</tbody>
</table>

Comparing the fuels on the basis of lowest optimum temperature, Table 4.8 presents itself.

<table>
<thead>
<tr>
<th>FUEL</th>
<th>DENSITY (Solid State Density)</th>
<th>VOLUME (cm$^3$)</th>
<th>OPTIMUM TEMPERATURE (eV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D^3He$</td>
<td>$4 \times 10^5$</td>
<td>$10^{-1}$</td>
<td>$10^{1.4}$</td>
</tr>
<tr>
<td>$H B$</td>
<td>$4 \times 10^5$</td>
<td>$10^{-3}$</td>
<td>$10^{1.8}$</td>
</tr>
<tr>
<td>$D T$</td>
<td>$4 \times 10^4$</td>
<td>$10^{-1}$</td>
<td>$10^{1.5}$</td>
</tr>
</tbody>
</table>

At this point, temperature is dictating a density of $4 \times 10^5n_e$ and larger volumes than the straight gain graphs obtained in section 4.3. The temperature is crucial, as it has to
be one manageable with materials of this planet. In the Sun and in thermonuclear weapons it is not an issue, but when trying to confine, control and indeed ignite the reaction, the lower the better. MCF gained early support because the temperatures were going to be too great for any material to contain it. ICF has to then show it can take place at more ‘moderate’ (to use Stenning et al (92) word), than those. Heat is the thing that is produced to generate electricity, but it has to be within the constraints of the technology available. Table 4.9 represents a summary of the best candidates available for the optimum fuel to use. This is done across sections 4.3 and 4.4, given the limitation of data available to be collected.

### TABLE 4.9  BEST VALUES OF GAINS VERSUS ENERGY (CONSIDERING TEMPERATURE)

<table>
<thead>
<tr>
<th>FUEL</th>
<th>DENSITY (Solid State Density)</th>
<th>VOLUME (cm³)</th>
<th>GAIN</th>
<th>INPUT ENERGY (Joules)</th>
<th>OPTIMUM TEMP (eV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(D^3He)</td>
<td>(4 \times 10^3)</td>
<td>(10^{-1})</td>
<td>(10^{2.4})</td>
<td>From Figures 1-16</td>
<td>(10^{10.8})</td>
</tr>
<tr>
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<td>(10^{2.4})</td>
<td>17-24</td>
<td>(3.2 \times 10^{10})</td>
</tr>
<tr>
<td></td>
<td>(4 \times 10^5)</td>
<td>(10^{-7})</td>
<td>(10^{2.3})</td>
<td></td>
<td>(10^{7.1})</td>
</tr>
<tr>
<td>(H^3B)</td>
<td>(4 \times 10^5)</td>
<td>(10^{-3})</td>
<td>N/A</td>
<td>(2.5 \times 10^9)</td>
<td>(10^{7.4})</td>
</tr>
<tr>
<td></td>
<td>(4 \times 10^5)</td>
<td>(10^{-7})</td>
<td>(10^{1.2})</td>
<td></td>
<td>(10^{8.1})</td>
</tr>
<tr>
<td>(D^2T)</td>
<td>(4 \times 10^4)</td>
<td>(10^{-7})</td>
<td>(10^{3.4})</td>
<td>(N/A)</td>
<td>(10^{5.1})</td>
</tr>
<tr>
<td></td>
<td>(4 \times 10^5)</td>
<td>(10^{-1})</td>
<td>(10^{3.3})</td>
<td>(2.5 \times 10^9)</td>
<td>(10^{1.5})</td>
</tr>
</tbody>
</table>

*Based on trend and Fig 4.20 and 4.21*
D3He: Gain VS Energy

D3He: Gain versus Energy

<table>
<thead>
<tr>
<th>#</th>
<th>Density</th>
<th>Volume</th>
<th>begin(E)</th>
<th>end(E)</th>
<th>B</th>
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<th>NINT</th>
<th>Key</th>
</tr>
</thead>
<tbody>
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<td>+1.392E+08</td>
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<td>N</td>
<td>1000</td>
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<tr>
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<td>+1.392E+01</td>
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<td>+1.392E+02</td>
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<td>N</td>
<td>1000</td>
<td></td>
</tr>
</tbody>
</table>

Figure 4.1 D3He Gain versus Energy
**Figure 4.2** D3He Gain versus Energy
Figure 4.3  D3He Gain versus Energy
D3He: Gain versus Energy

<table>
<thead>
<tr>
<th>#</th>
<th>Density</th>
<th>Volume</th>
<th>begin(E)</th>
<th>end(E)</th>
<th>B</th>
<th>T</th>
<th>NINT</th>
<th>Key</th>
</tr>
</thead>
<tbody>
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<td>+1.000E-07</td>
<td>+1.392E+02</td>
<td>+1.392E+09</td>
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<td>N</td>
<td>10000</td>
<td></td>
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<tr>
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<td>+1.000E-07</td>
<td>+1.392E+03</td>
<td>+1.392E+10</td>
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**Figure 4.4** D3He Gain versus Energy
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**Figure 4.5** D3He Gain versus Energy
D3He: Gain VS Energy

![Graph showing Gain VS Energy](image)

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**Figure 4.6** D3He Gain versus Energy
D3He: Gain versus Energy

Figure 4.7  D3He Gain versus Energy
HB: Gain VS Energy

![Graph showing HB: Gain VS Energy](image)

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**Figure 4.8** HB Gain versus Energy
**Figure 4.9** HB Gain versus Energy
Figure 4.10 DT Gain versus Energy
Figure 4.11 DT Gain versus Energy
DT: Gain VS Energy

![Graph showing DT Gain versus Energy](image)

### DT: Gain versus Energy

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**Figure 4.12 DT Gain versus Energy**
DT: Gain VS Energy

DT: Gain versus Energy

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Figure 4.13 DT Gain versus Energy
DT: Gain VS Energy

![Graph showing DT: Gain VS Energy]

DT: Gain versus Energy

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Figure 4.14 DT Gain versus Energy
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**Figure 4.16** HB Temperature versus Time
Figure 4.17 HB Temperature versus Time
HB: Temperature VS Time

Figure 4.18 HB Temperature versus Time
DT: Temperature vs Time

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**Figure 4.19** DT Temperature versus Time
### DT: Temperature vs Time

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**Figure 4.20** DT Temperature versus Time
DT: Temperature VS Time

DT: Temperature vs Time

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Figure 4.21 DT Temperature versus Time
**Figure 4.22** D3He Temperature versus Time
**Figure 4.23** D3He Temperature versus Time

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**Figure 4.24 D3He Temperature versus Time**
4.5 CONCLUSION

The process of identifying the appropriate fuel and the correct parameters for its operational success is a careful balancing act. The density, volume and input energy required to give a worthwhile return are mediated by the temperature produced. Given the limitations of the model used and the limited scope of factors that can be manipulated; the question as to which fuel, in what form is a likely candidate could be answered.

Clearly with all three fuels increasing density, decreases temperature. Further, the three also would indicate increasing volume decrease temperature. This set of trends is going to work against some of the candidates from section 4.3, see Table 4.9.

$DT$ emerges as the ‘best’ fuel. The most desirable parameters for it being: density $4 \times 10^5 \text{ n}_s$, volume $10^{-1} \text{ cm}^3$ using input energy from $3.2 \times 10^{10} J$ to $10^{12.2} J$ (depending on graph/model used.) The last vital statistic is the optimum temperature and it comes in a ‘cool’ $10^{1.5} \text{ eV}$. These parameters, given good fuel pellet quality, best engineering, should give a return of $10^{3.3}$. 
CONCLUSION

This thesis set out to overview the areas of controlled fusion that deal with energy deposition in a plasma, the movements within the plasma and identify the variables so concerned. Finally it explored some current fuel candidates and what, given the current state of knowledge, maybe expected from them in terms of fusion energy gains.

Chapter one overviewed the mathematical framework through which the bulk of the subjects operates. It concluded by examining the tensor’s role and the role of non-linear models. A framework was suggested via Lie algebra, to bring together the elements of field theory. The chapter as a whole was used to look at the mathematical tools that could, and for the most part are, employed in the forth-coming chapters.

Chapter two set out to examine the kinetic nature of the plasma, starting with Boltzmann’s equation and Maxwellian distributions. It moved into the area of macroscopic equations, to describe the plasma as a whole. The microscopic to macroscopic relationship was developed.

This chapter then dealt, from Maxwell’s equations, with electromagnetic fields involved. Indeed, by the end of this thesis, papers were emerging dealing more with the role of the magnetic fields alone. The mathematical tools of chapter one were employed to develop many ideas; the main one being that of field energy and the pondermotive force. Out of this role of the refractive index of a plasma and its density emerged. Plasmas were seen to have wave properties such as a frequency. The chapter concluded with the wave properties and absorption characteristics of plasmas and how these along with temperature are going to be involved in the next chapter on the actual nature of a plasma.

Chapter three dealt with the nature and properties so revealed by this mathematical approach. It focused in on the movement of electrons and ions in a fully ionised plasma. Electrostatic forces were added to the list of forces involved. The non-
neutrality of an in-homogenous plasma was examined. The distances between the particles and the effect of collisions between was viewed in terms of electric currents being evident in any plasma. Models, such as Schlüters two fluid one were examined along with the non-linear force. Some equivalences were discussed between models, and this gave a more detailed view of the non-linear force. Out of this emerged the “genuine” two fluid plasma model. The chapter concluded by using this two fluid model to examine internal electric fields, the plasma frequency and its relation to a laser’s frequency. Also emerging from this model use was the nature of the surface of a plasma as it expands outline; the idea of the double layer between electrons and ions. Finally the hydrodynamics of the plasma was discussed. The chapter then proceeded to analyse the key variables of ion density, velocity, electric field densities and temperature over time. These variables were also looked at with regards to the depth of penetration of the laser beam. This was viewed against the background of the oscillations set up in plasmas by the ions pursing the moving electrons.

Chapter four followed on from this by dealing with the nuclear reaction details. This ranged from overall reaction processes to Lawson’s criteria as adopted to inertial confinement fusion. These details lead to setting up the criteria for fusion gain. The three fuel D.T., D³He, and HB were analysed to see which would be capable of giving the best energy return. The variables of density, volume and temperature were manipulated via the computer model to get an idea of what combination of these would be desirable for each fuel. The conclusion was reached as to which fuel and what conditions would make it, re the model used, the best fuel.

It is again acknowledged that all is set against the limitations of the model used. It can be concluded that around 10⁴, 10⁵ times solid state density and volumes of around 10⁻¹ to 10⁻³ cm³ are the most desirable for any of the three fuels. The fuel that emerged the most worthwhile was deuterium-tritium.

The very least that has emerged from this and the studies reviewed for it, is that an actual answer to the questions of fuel type and conditions for best gains is possible. All such studies urge the furthering of commitment to research of all aspects of this achievable form of energy.
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# APPENDIX 1

## SYMBOLS AND ASSOCIATED MEANINGS

<table>
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<th>SYMBOL</th>
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<tr>
<td>$\bar{V}$ =</td>
<td>Potential difference, bar above meaning average</td>
</tr>
<tr>
<td>$E =$</td>
<td>Electric intensity $\left(-\frac{\partial V}{\partial r}\right)$, the electric field due to charge separation.</td>
</tr>
<tr>
<td>$n =$</td>
<td>Density, occasionally used as “number of”.</td>
</tr>
<tr>
<td>$n_i =$</td>
<td>Ion density - usually taken per cm$^3$</td>
</tr>
<tr>
<td>$n_e =$</td>
<td>Electron density - usually taken per cm$^3$</td>
</tr>
<tr>
<td>$n_{el} =$</td>
<td>Combined density of electrons and ions.</td>
</tr>
<tr>
<td>$m_i =$</td>
<td>Mass of ions</td>
</tr>
<tr>
<td>$m_e =$</td>
<td>Mass of electrons</td>
</tr>
<tr>
<td>$m_c =$</td>
<td>Combined mass of ions and electrons, sometimes $m_{el}$.</td>
</tr>
<tr>
<td>$\mu = m_i / m_p =$</td>
<td>Ratio of ion mass to proton mass</td>
</tr>
<tr>
<td>$M_\pi =$</td>
<td>Mass of meson.</td>
</tr>
<tr>
<td>$\lambda_D = \left(\frac{k T}{4 \pi e^2}\right)^{1/2}$</td>
<td>Debye Length in centimetres.</td>
</tr>
<tr>
<td>$\nu_{Te} = \left(\frac{k T_e}{m_e}\right)^{1/2}$</td>
<td>Electron thermal velocity.</td>
</tr>
<tr>
<td>$\nu_{Ti} = \left(\frac{k T_i}{m_i}\right)^{1/2}$</td>
<td>Ion thermal velocity</td>
</tr>
<tr>
<td>$A =$</td>
<td>Mass number</td>
</tr>
<tr>
<td>$Z =$</td>
<td>Number of charge carriers.</td>
</tr>
<tr>
<td>$n =$</td>
<td>Refractive index</td>
</tr>
<tr>
<td>$f_{LN} =$</td>
<td>Non-linear force.</td>
</tr>
</tbody>
</table>
\( \rho = \) General symbol for an expression of total density, \( \rho_0 \) being initial density.

\( \rho_e, \rho_i = \) Mass density of electron fluid \((n_e, m_e)\), ion fluid \((n_i, m_i)\)

\( \varepsilon_0 = \) Electric constant, permittivity of vacuum \(F.m^{-1} - A.s.V^{-1}.m^{-1}\)

\( Q = \) Electric charge, volume density of charge \(C.m^{-3}\)

\( E_z = \) Electric field of laser radiation, sometimes \( E_v \) if in a vacuum.

\( T_e = \) Temperature of electrons

\( T_i = \) Temperature of ions

Temperatures are usually given in eV units for plasmas. This relationship to degree Kelvin is given by

\[
\begin{align*}
kT &= 1.0eV = 1.6030 \times 10^{-19} J \\
k &= 1.38054 \times 10^{-23} \frac{J}{K} \\
\frac{e}{k} &= 11,600 \frac{K}{V} \\
T(eV) &= 8.61 \times 10^{-3} T(K)
\end{align*}
\]

Note relativistic effects become significant about 25keV for electron gas, and 50MeV for protons.

\( k = \) Boltzmann’s constant,
\( 1.38066 \times 10^{-23} \frac{J.K^{-1}}{\text{or}} 1.38 \times 10^{-16} \text{cm}^2 \text{sec}^{-2} \)

\( \omega = \) frequency, as in \( \omega_p \) – natural plasma frequency.

\( c = \) Speed of light, but may be used to specify, in some texts, the speed of sounds (phonon) in media. Avoided where possible in this work.

\( p_e, p_i = \) Momentum of electrons/ions

\( Vei = \) Electron, ion collision frequency (phenomenological)

\( \wedge = \) Coulomb logarithm, as per \( \log \wedge \)

\( h = \) Planck’s constant,
\( 6.62618 \times 10^{-34} \text{ J.Hz}^{-1}, h = h/2\pi = 1.054589 \times 10^{-34} J.S \)

\( \sigma = \) Electrical conductivity
\( A_c \) = Coulomb cross section

\( S = \) Poynting vector W.m\(^{-2}\) \( \mathbf{S} = \mathbf{E} \times \mathbf{H} \), the energy flux density for electromagnetic radiation.

\( D = \) Electric flux density, displacement C.m\(^{-2}\), \( \text{div } \mathbf{D} = \) density charge.

\( H = \) Magnetic field strength A.m\(^{-1}\). \( \nabla \times \mathbf{H} = \mathbf{J} + \partial \mathbf{D} / \partial t \)

\( e = \) electronic charge of a proton (+) or electron (-)

\( I = \) Intensity of laser beam.

\( I = \) Current Amp (A)

\( J = \) Electric current density (S.I) Am\(^{-2}\) or (Gaussian), \( 3 \times 10^5 \text{ statamp cm}^{-2} \)

\( n = \) Refractive index

\( P_e = \) Thermo-kinetic electron pressure = \( n_e kT_e \)

\( P_i = \) Thermo-kinetic ion pressure = \( n_i kT_i \)

\( t = \) Discretisation in the time continuum

\( T_{\text{eff}} = \) Effective temperature

\( T_{\text{th}} = \) Thermal temperature

\( u_e = \) Electron fluid velocity

\( u_i = \) Ion fluid velocity

\( \gamma = \) Macroscopic adiabatic constant

\( \mathcal{K}_e, \mathcal{K}_i = \) Thermal conductivity coefficient of electrons, ions.

\( \tau = \) Equipartition time between electrons and ions

\( \omega = \) Radian frequency of laser light

\( \omega_p = \) Plasma frequency

Often the unit of a plasma period is used (Laloussis and Hora for example). It is given as \( \frac{2\pi}{\omega_p \text{ min}} \), 'min' meaning minimum, and being in "lots of" \( 5 \times 10^{15} \text{ sec} \), 5 fsec. From this the frequency
can be calculated as \[ \frac{2\pi}{\omega} = 5 \times 10^{-15} \]
\[ \frac{\omega}{2\pi} = \frac{1}{5 \times 10^{-15}} = 0.2 \times 10^{15} \text{ Hz}. \]

Lalousis for example axis label in units 1, 2 etc for plasma periods \( \frac{2\pi}{\omega} \). Hence a 0.5 becomes \( 0.5 \times 5 \times 10^{-15} = 2.5 \times 10^{-15} \text{ sec.} \)

\[ \therefore \frac{2\pi}{\omega} = 2.5 \times 10^{-15} \]
\[ \therefore \omega = 2.5133 \times 10^{15} \text{ Hz}. \]

Later this becomes \( 2.0 \times 5 \times 10^{-15} = 10 \times 10^{-15} \text{ or } 10^{-14} \text{ sec.} \)

The \( \omega \) “drop off”. Even at fixed density and temperature.

42 plasma periods: \( 42 \times 5 \times 10^{-15} = 210 \times 10^{-15} \text{ or } 2.1 \times 10^{-13} \text{ sec.} \)

\[ \frac{2\pi}{\omega} = 2.1 \times 10^{-13} \text{ or } \omega = \frac{2\pi}{2.1 \times 10^{-13}} = 2.99 \times 10^{13} \text{ Hz}. \]

The frequency declines.

\[ q = \]
heat flux vector due to thermal conduction, radiation transport etc.

\[ q = \]
Coulomb charge – general, no specific unit charge as in Coulomb’s law.

\[ R = \]
Used as radius, e.g. of fuel pellet or as rate of say fusion occurring. Both are in chapter four and each should be clear from context.

\[ eV \text{ or } MeV = \]
Electronvolt, or megaelectronvolt. It is the work done, energy expended in moving a charge equal to that of the electron \( 1.602 \times 10^{-19} C \) through a potential difference of 1 Volt. The energy gained by one electron is moved through a P.D. of IV.

Work Done = energy gained = charged moved × potential
difference.

\[ 1 eV = 1.6 \times 10^{-19} \, C \times 1 \, J \text{C}^{-1}\; \text{as} \; 1 \, \text{Volt} = 1 \, J \text{C}^{-1} \]

\[ \therefore 1 eV = 1.6 \times 10^{-19} \, J \]

or \[ 1 J = \frac{1}{1.6 \times 10^{-19}} \]

\[ = 6.25 \times 10^{19} \, \text{eV} \]

In terms of mass: \( 1 \, \text{amu} = 1.655 \times 10^{-27} \, \text{kg} \) so by \( E = mc^2 \)

\[ 1 \, \text{amu} = 1.655 \times 10^{-27} \times \left(3 \times 10^8\right)^2 \, J \]

\[ = \frac{1.655 \times 9 \times 10^{-11}}{1.6 \times 10^{-19}} \, eV \]

\[ = 931 \times 10^6 \, eV \]

\[ = 931 \, MeV \]

**Nuclide** = A species of atom containing the name numbers of both protons and neutrons. Protons and neutrons can be considered versions of the same particle, the nucleon.

**Binding Energy** = (of a nuclide) is the energy that was converted from mass and then released when a nuclide is formed from its constituent nucleons, all initially in the free state. As a result the binding energy of a nuclide is also equivalent to the minimum energy required to completely separate the constituents of that nuclide.

\( T_{ad} = \) Adiabatic cooling - temperature

\( T_a = \) \( \alpha \) particle reheat - temperature

\( T_p = \) proton reheat - temperature

\( T_{br} = \) bremsstrahlung - temperature

\(< \sigma v > = \) velocity averaged fusion cross-section

\( \sigma = \) scattering cross section

\( v = \) velocity
Self Similarity Model = Deals with the energy transfer for a fully symmetric spatially constant power density. The expansion is taken to be into a vacuum, the radius \( R \) of the expanding pellet is a function of time. During expansion adiabatic transfer of thermal energy into kinetic energy of expansion will occur. Complete radial symmetric hydrodynamic calculations start with initial radial velocity profile \( v(r, t = 0) \), ion density profile \( n_i(r, t = 0) \) ends after sometime \( t \) into a solution where \( n_i \) becomes a Gaussian density profile, velocity is linear, and temperature as dropped adiabatically. The fact that density and velocity profile remain similar is the reason for using ‘self-similar’.

Thermonuclear (Gross, R.A, 84) =

In low-temperature plasmas, particles in the high-energy tail of the Maxwell Boltzmann distribution will make the greatest contribution to the fusion reaction. For instance in a \( DD \) plasma at \( 10^7 \)K the greatest contribution to the rate of the thermonuclear reaction comes from the nuclear collisions between the deuterons and energy of almost \( 6keV \). Now of the total number of particles there is only a small fraction of the particles with this energy. The particles in the Maxwellian tail reacts (a reference to a distribution curve shape), the depleted part of the velocity distributions is repopulated by collisions, allowing the reaction to continue. The significance of this thermal repopulating process is noted by adjectival prefix to fusion of ‘thermonuclear’.

\[ \text{Kronecker delta} = \delta_{ij} \text{ where } 2 \text{ variables take the value 1 where equal, } i=j, \text{ or zero otherwise. Named after Leopold Kronecker (1823-91)}/ \]

\[ \text{Dirac delta function} = \delta(x), \text{ defined to be zero for all non-zero real x, and infinite for } x=0. \]
NUCLEAR FUSION PROGRAM

#include "fusion.h"
#include <windows.h>
#include <math.h>
#include <time.h>
#include <ctype.h>
#include <stdio.h>
#include <string.h>
#include <stdlib.h>
#include <commdlg.h>
#include <sys/timeb.h>

#define PRN_MENU "\FUSION.PRN"
#define PTS_MENU "\FUSION.PTS"
#define NUC_MENU "\FUSION.MNU"
#define ALL_DATA "\CURVE_%i.ALL"
#define END_DATA "\CURVE_%i.END"

#define VIDEO 100
#define PRINTER 200

#define MAX_CURVES 12
#define DHe 0
#define HB 1
#define DT 2

#define MENU_ITEMS 5
#define MaxSW 70

#define EM 5.4550989E-04

// M_PI = 3.1416...
// about twenty decimals
//
#define END (M_PI / 2.0)

#define LOG_THRESHOLD 1.0E-30

// IF THE energy OR THE gain OR THE temperature
// OR THE time IS VERY SMALL THEN IT IS BEST TO
// DEEM THE DATA AS INVALID IN ORDER TO AVOID
// ROUN丁NG PROBLEMS.
#define APP_NAME "FUSION"
#define APP_GRAPH "GRAPH"

#define LINEAR_SCALE TRUE
#define LOG_SCALE FALSE

#define LEN 81
#define RING 0.2

// NOTE: All messages below 0x0400 are RESERVED by Windows,
// #define WM_USER 0x0400 and above.

#define WM_GRAPH_XFER_STATUS_PTR 0x4010

struct demo_tag{
    // density_GvsE AND density_TvsT
    // ARE RELATIVE.

    long int nint_GvsE,
        nint_TvsT;

double density_GvsE,
    volume_GvsE,
    energy_begin_GvsE,
    energy_end_GvsE,
    //
    density_TvsT,
    volume_TvsT,
    energy_TvsT,
    minimum_time_TvsT,
    maximum_time_TvsT;

    BOOL brems_GvsE,
    two_comp_GvsE,
    //
    brems_TvsT,
    two_comp_TvsT;
};

struct nucleus_tag{
    short int ncl_np,
    //
    ncl_curve, // INTERNAL STORAGE ONLY.
    ncl_mode; // TO BE USER FRIENDLY.

double ncl_ef,
    ncl_f1,
    ncl_f2,
    ncl_z,
ncl_z2,
ncl mh,
ncl mh2,
ncl ea,
ncl ea2,

//
ncl am_OneComp,
ncl am_TwoComp,
//
ncl xm,
ncl zia,
ncl d0,
ncl a,
//
array t[MaxSW],
array sw[MaxSW];

char nuclei_title[LEN];

struct demo_tag demo[MAX_CURVES];
);

struct gain_tag{
    long int ia_number;
    double ia_time,
    ia temp,
    ia depfg,
    ia dnshe,
    ia prions,
    iadeptot,
    ia depfr,
    ia fg;
};

struct display_tag{
    double xx_value,
    yy_value;
    BOOL hide_last_line;
};

struct error_tag{
    long int wrong_prions,
    wrong dpions,
    left extrapolate;
};

struct prn_tag{
    int col_mode,
pen_mode,
    font_mode;
};

struct points_tag{
    int grid,
    last_grid,
    steps_GvsE,
    trays_TvsT;
};

struct status_tag{
    char st_window_text[LEN];

    int st_mode,
    st_grid,
    st_last_grid,
    st_num_curves;

    BOOL st_zero_graph,
    st_old_graph;
};

long FAR PASCAL _export WndProc_Main(HWND, UINT, UINT, LONG);
long FAR PASCAL _export WndProc_Graph(HWND, UINT, UINT, LONG);

BOOL FAR PASCAL _export SetUp_GvsE(HWND hdlg, UINT messg,
    WPARAM wParam, LPARAM lParam);

BOOL FAR PASCAL _export SetUp_TvsT(HWND hdlg, UINT messg,
    WPARAM wParam, LPARAM lParam);

void Menu_Bar_Graying(HWND mbg_hwnd, struct prn_tag mbg_prn,
    struct points_tag mbg_points,
    int mbg_nuc,
    int mbg_mode);

void Set_Font(LOGFONT *If, int lf_width, int lf_height);

HDC Get_Printer_DC(void);

int Printer_Graph(HDC print_hdc,
    struct prn_tag prn,
    struct status_tag *graph_status,
    int nom_size_xx,
    int nom_size_yy);

void Set_Status(struct status_tag *set_status, BOOL zero, BOOL old);
void Msg_To_Graph(struct status_tag *mtg_status);
void Display_Summary_GvsE(HDC, hdc, 
    struct nucleus_tag menu, 
    int       display, 
    int       nom_size_xx, 
    int       nom_size_yy, 
    int       x_shift, 
    int       y_shift);

void Display_Summary_TvsT(HDC, hdc, 
    struct nucleus_tag menu, 
    int       display, 
    int       nom_size_xx, 
    int       nom_size_yy, 
    int       x_shift, 
    int       y_shift);

COLORREF Colour(int col);

int Inside_Box(HWND ib_hwnd, 
    POINT ib_point, 
    // 
    int ib_left_xx, 
    int ib_right_xx, 
    int ib_bottom_yy, 
    int ib_top_yy);

void Device_To_Real(HWND dtr_hwnd, 
    POINT dtr_pt, 
    double *out_xx, 
    double *out_yy, 
    int     initial_position_xx, 
    int     final_position_xx, 
    int     initial_position_yy, 
    int     final_position_yy, 
    double initial_value_xx, 
    double final_value_xx, 
    double initial_value_yy, 
    double final_value_yy);

void Pop_File_Init(HWND hwnd, OPENFILENAME *ofn_ptr, char *data_file);
void Gen_Data_File_Name(char *name);
void Close_File(FILE *cf_file, short int *cf_error);
void Erase_End_Files(int valid_curves);

int Double_To_Exp_String(char *target, 
    double source, 
    int     pre_dec, 
    int     post_dec);

int D_To_One_Dec_String(char *target, double source);
void Delay(double wait);
void Bell(double wait);

double Sweeper(int counter,
               int max_limit,
               double start_s,
               double fin_s,
               BOOL linear_log);

void Min_Max(FILE *file, double *min, double *max);

int Gain(long int nint,
         // double rd,    // RELATIVE DENSITY rd
         double v0,     // VOLUME v0
         double e0,     // ENERGY e0
         // struct nucleus_tag ncl,
         // BOOL is_two_comp,
         BOOL do_brems,
         // FILE *stream,
         short int *file_error,
         // double nom_min_time,
         double nom_max_time,
         // double *rtn_g,
         double *rtn_depg,
         // struct error_tag *error_detection);

HWND global_hwnd_graph = NULL;

#pragma argsused

int PASCAL WinMain(HWND hWnd, HINSTANCE hInstance, HINSTANCE hPrevInstance,
                   LPSTR lpszCmdParam, int nCmdShow)
{
  int character_width,
  character_height,
  // initial_graph_xx,
  initial_graph_yy,
  initial_graph_width,
  initial_graph_height;

  HDC hdc_info;
HWND hwnd_main;

POINT info_zero;

MSG msg;

WNDCLASS wndclass_main,
     wndclass_graph;

TEXTMETRIC tm_info;

character_width = 0;
character_height = 0;
//
initial_graph_xx = 0;
initial_graph_yy = 0;
initial_graph_width = 0;
initial_graph_height = 0;

hdc_info = (HDC)(0);

hwnd_main = NULL;

memset(&tm_info, 0, sizeof(TEXTMETRIC));

//+++++

if(hPrevInstance == NULL)
{
}
else
{
    Bell(RING);
    MessageBox(hwnd_main, "PROGRAM RUNNING ALREADY!", APP_NAME,
               MB_ICONSTOP | MB_OK);
    return(0);
}

wndclass_main.style = CS_HREDRAW | CS_VREDRAW;
wndclass_main.lpfnWndProc = WndProc_Main;
wndclass_main.cbClsExtra = 1;
wndclass_main.cbWndExtra = 0;
wndclass_main.hInstance = hInstance;
wndclass_main.hIcon = LoadIcon(hInstance, "Nuclear_Fusion");
wndclass_main.hCursor = LoadCursor(NULL, IDC_ARROW);
wndclass_main.hbrBackground = (HBRUSH)(GetStockObject(WHITE_BRUSH));
wndclass_main.lpszMenuName = "Nuclear_Fusion";
wndclass_main.lpszClassName = APP_NAME;
RegisterClass(&wndclass_main);

wndclass_graph.style = CS_HREDRAW | CS_VREDRAW;
wndclass_graph.lpfnWndProc = WndProc_Graph;
wndclass_graph.cbClsExtra = 0;
wndclass_graph.cbWndExtra = 0;
wndclass_graph.hInstance = hInstance;
wndclass_graph.hIcon = NULL;
wndclass_graph.hCursor = LoadCursor(NULL, IDC_ARROW);
wndclass_graph.hbrBackground = (HBRUSH)(GetStockObject(LTGRAY_BRUSH));
wndclass_graph.lpszMenuName = NULL;
wndclass_graph.lpszClassName = APP_GRAPH;

RegisterClass(&wndclass_graph);

// hdc_info = GetDC(hwnd_main);
SelectObject(hdc_info, GetStockObject(SYSTEM_FIXED_FONT));
GetTextMetrics(hdc_info, &tm_info);
character_width = (tm_info.tmAveCharWidth);
character_height = (tm_info.tmHeight + tm_info.tmExternalLeading);
ReleaseDC(hwnd_main, hdc_info);

hwnd_main = CreateWindow(APP_NAME, "Hydrogen Fusion", WS_OVERLAPPEDWINDOW, 0, 0, 78.0 * character_width, 28.5 * character_height, NULL, NULL, hInstance, NULL);

info_zero.x = 0;
info_zero.y = 0;

ClientToScreen(hwnd_main, &info_zero);

// GET CO-ORDINATES FOR GRAPH.
initial_graph_xx = (int)(2.0 * (double)(character_width)) + info_zero.x;
initial_graph_yy = (int)(1.5 * (double)(character_height)) + info_zero.y;

initial_graph_width = (int)(73.00 * (double)(character_width));
initial_graph_height = (int)(23.25 * (double)(character_height));

// END OF GETTING CO-ORDINATES.
LASER PLASMA INTERACTION
FOR
APPLICATION TO FUSION ENERGY

by

Peter John Evans

A thesis submitted as part of the requirements
for the degree of Master of Science (Honours)

University of Western Sydney

August, 2002

© P Evans August 2002
PLEASE NOTE

The greatest amount of care has been taken while scanning this thesis,

and the best possible result has been obtained.
ACKNOWLEDGEMENT

I would like to thank my supervisor Dr Frederick Osman for his complete dedication to this thesis. He was an invaluable guiding force; he believed in me, that I might believe in myself. I would also like to thank the “typist”, Lisa Dalmer, for her many hours of work, and tolerance.

DEDICATION

This whole effort is dedicated to those who kept me in physics, and showed the way: David, Fred, George and Reynaldo.
STATEMENT OF AUTHENTICATION

The work presented in this thesis is to the best of my knowledge and belief, original except as acknowledged in the text. I hereby declare that I have not submitted this material, either in whole or in part, for a degree at this or any other institution.

......................
Peter Evans
August 2002
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The thesis presents an investigation into inertial confinement fusion through mathematical models and computer simulations. It is stressed that this area of research is vital to further energy needs of the planet, and hence, its survival. The thesis identifies the salient parameters affecting fusion; in both energy absorption and fusion gains. This is achieved by first familiarising the reader with the mathematical tools required. These are then applied to a directed investigation into plasma structure. Parameters such as these involved in electromagnetic energy absorption are identified first; for example, the role of refractive index. The next step is to model the immediate response of the plasma to this energy input, with a view to how this may be advantageous to initiating fusion.

Models are developed that best suit plasma behaviour. Special attention is paid to the "Genuine" Two Fluid Model and the pondermotive and transient forces. This material is translated into a numerical study of the parameters, such as the electric field density and temperature distributions once electromagnetic energy is supplied to a plasma. The parameters are presented graphically against time and distance into a small plasma fuel pellet. It is noted how field density and ions form undulations through the plasma. Types of plasma fuels are discussed with regards to their key parameters, such as density, volume and temperature. Computations are performed using the laser driven inertial fusion energy option based on volume ignition with the natural adiabatic self-similarity compression and expansion hydrodynamics (J. Plasma Physics 60, 743 (1999)). The numerical work uses a reaction code to study the fusion reaction of $DT$, $D^3He$ and $HB$. The relative merits of each fuel are discussed against the parameters of density, volume and energy input verses fusion gains. This is mediated by the temperature rises expected using the recent new results on "One Step Laser Fusion" (paper 525 by H.Hora, P.Toups, P.Evans, F.Osman, R.Castillo, K.Mima, M.Murakami, S.Nakai, K.Nishihare, C.Yamanaka, and T.Yamanaka, at the "Inertial Fusion Science and Applications conference; Kyoto 9-14 September, 2001.") This is in the new direction of fusion gains for the low temperature high-density large laser pulse ignition. The data is presented in both graphical and table form.
INTRODUCTION

This work touches on a vast body of knowledge and research dealing with the goal of producing controlled fusion. The purpose of such a quest is to meet the world's ever growing hunger for energy, while sources of energy are actually dwindling.

Nuclear fission as a form of energy exists in both bomb and reactor form. Nuclear fusion found only in the bomb on this planet. Since its discovery many researchers have turned to the peaceful use of such a powerful energy source. Any such understanding must begin with an understanding of the fuel, the plasma.

The centre of this work is inertial confinement fusion, as opposed to magnetic confinement fusion. I.C.F requires a mechanism for imparting the energy to the plasma in order to compress it and fuse it. The device to compress it and fuse it. The device looked at for this, is the laser. Thus this work will focus on the deposition by electromagnetic radiation of energy to a plasma. To the plasma creating forces capable of moving the components of the plasma, the electrons and ions. Moving them with velocities capable of overcoming repulsive forces and ramming the nuclei together, to fuse them.

This is a very complex field of study, involving many fields of understanding, electromagnetic field theory, atomic structure and quantum mechanics (noticeably little used or mentioned in the literature.) It deals with the mathematics of gradient fields, the resulting forces, moving charges and the theories of thermo-kinetic movement and especially employs non-linear mathematics. It is an overview at the beginning, of the forces involved, and the models of plasma structure. It is too vast an area of knowledge to plough into at this level, without showing such an overview. Coupled with the plasma structure are the values of temperature, energies and densities, etc involved. For example some numerical work is offered in chapter three dealing with the non-neutrality of plasmas. Specially, the internal 'churning' that goes on inside a plasma, before any external fields, such as lasers supplied electromagnetic fields are offered to them.
There is not as much experimental data available as would be desired due to:

a) classified work connected with thermonuclear devices;
b) little done due to expense of research;
c) split ideas with experiment into magnetic and inertial confinement.

To quote John M. McMahon on p578 of his paper “Fusion Laser Technology Revisited”:

“Even if unlimited resources were available, it does not appear that we have the understanding of all the details and technical issues necessary to define the ultimate system now.”

From the very vast, complex area of plasma structure, electromagnetic interaction and associated effects; a numerical study is undertaken of the effects. The effects being, ion velocities, temperature density changes and at what depths these effects are apparent.

Given the vastness of the subject, an effort is made to examine the possible gains in energy that would be made. This is assuming a fusion burn has been successfully undertaken. The questions are examined as to what fuel type and under what conditions will the best gain be achieved. This point has to be balanced against the temperatures needed to be examined in the light of the Earth’s materials ability to contain them.

Essentially the situation will be shown to be an educated balancing act. The variables being juggled via computed simulations derived from the equations. As late as 1988, Professor H. Hora stated “…the poor and underdeveloped state of plasma theory…” (p8 “Nonlinear Effects and Nonthermal Plasmas”; 1988). There has been some clearing up of the role of the variables in theory since then. The models have been refined, but it rests on the basic direction operating when the statement was made.
To further strongly emphasise the state of knowledge that this work was drawn from, Hora’s reference to Edward Teller’s view (made in the same paper) is vital. Teller believed it might not be a plasma physicist who achieves controlled fusion energy production. Teller went on, as Hora reports, when dealing with there being any plasma experts “…as a person who knows about all mistakes one can do in a field”. Teller went on, in the presence of Werner Heisenberg and some 300 odd theoreticians; “…you are not an expert”.

It is a field requiring much better planned and financed research if the necessary goal is achieved.

The following presentation sums up what are the key areas of theory for both energy capture and energy gain. It will deal with electromagnetic radiation theory and basic nuclear theory where required. It seeks to offer a review of some key research and build upon it to examine some key values for fuels, given the models employed. Actual details of the technologies required would be the subjects for many an engineering thesis.

The overall aim will be to numerically examine the way electric fields operate within plasmas, the densities and movement of the all-important ions, especially density variations. Further to this will be a numerical review of the fuel candidates, such as D.T., D³He, HB and what conditions of density and volume would best serve their use. Indeed which one emerges as the ‘best’ candidate fuel given, depends on the limitations of the model employed. These points are run against the temperature generated by the fusion burns involved.

The overall point to be made is that such values can be generated from reasonable computer simulations. If that is no, then time and money should justifiably be put into actually developing a working inertial confinement fusion situation. The world’s energy demands demand it.