CHAPTER 1: INTRODUCTION.

The desired objective of this thesis is the derivation of similarity solutions for one-dimensional coupled systems of reaction-diffusion equations and a one-dimensional tripled system as well as for a semi-linear system. These solutions are obtained by means of one-parameter group methods, employing various techniques from Hill [19]. The determination of these one-parameter groups is achieved by the classical and non-classical approaches, a description of which will be outlined briefly in Chapter 2. The one-parameter groups thus obtained are then used to identify similarity solutions of the aforementioned systems of equations.

The theory of transformation groups owes its very existence to the pioneering work of Marius Sophus Lie (1842-1899) whose interest was kindled by the principles of this theory in 1873. These transformation groups were to become his most renowned creation. Lie's theory of transformation groups was greatly developed from his two fundamental theorems and in the process, the underlying abstract group, known as the parameter group, was uncovered. The one-parameter transformation group preserves the invariance of differential equations, thereby enabling solutions of these differential equations to be obtained. It is found that attempting to solve differential equations via one-parameter group methods has the tendency to yield Abel equations of the second kind. Research in the 1930's disclosed the possibility of deriving the global form of the one-parameter group from its infinitesimal form. Investigations of differential invariants were made and tools of solution included automorphism groups of differential equations. Lie groups play an increasingly central role in quantum physics while the Helmholtz-Lie space problem is one of the most aesthetically appealing applications of Lie groups.

It must be acknowledged that the approach of solving differential equations by way of one-parameter groups does not consistently prove successful in the derivation of solutions. However, the one-parameter group method of solving differential equations has a wide applicability to differential equations of a linear as well as a semi-linear nature. The versatility of this approach also extends to deriving solutions of non-linear differential equations. The reader is referred to accounts of using one-parameter groups to solve ordinary differential equations in Bluman and Cole [9] and Dickson [13]. An illustration of the utility of this approach with partial differential equations is also given by Bluman and Cole [9].
Other treatments of recovering solutions are available in the literature, an example of which may be found in Olver [25]. The Bluman-Cole method was found to yield solutions which did not arise from Olver's approach. The differing treatments are also compared in a stimulating paper on the subject by Arrigo, Broadbridge and Hill [6], in which non-classical symmetry solutions of non-linear partial differential equations were considered in the light of the reduction methods of Bluman and Cole as well as Clarkson and Kruskal. In the same paper, it was discovered that exact solutions for Burger's equation could be obtained by the Bluman-Cole method but failed to be elicited from Clarkson and Kruskal's method. A considerable degree of interest has been evinced in the solution of systems of partial differential equations. An example of this is contained in Clarkson and Mansfield [10].

The first area of research in this thesis is a coupled system of diffusion equations which was obtained by Aifantis and Hill [5] for the existence of two distinct families of diffusion paths. The one-dimensional case of this system is given by

\[
\begin{align*}
\frac{\partial u}{\partial t} &= D_1 \frac{\partial^2 u}{\partial x^2} - k_1 u + k_2 v, \\
\frac{\partial v}{\partial t} &= D_2 \frac{\partial^2 v}{\partial x^2} - k_2 v + k_1 u;
\end{align*}
\]

(1.1)

where \(u(x, t)\) and \(v(x, t)\) denote the concentrations in paths 1 and 2 respectively, the self-diffusivities \(D_1\) and \(D_2\) are positive constants, while \(k_1\) and \(k_2\) also represent positive constants. The validity of the system (1.1) is preserved only when the distribution of defects or paths can be justifiably assumed continuous. Another application of the system (1.1) is that of a model for water transport through plant tissue, a subject intensively researched by Molz [23]. In this paper, Molz was able to derive a coupled system of equations, a specialisation of which reduces this system to the system (1.1).

Another of the many applications of the coupled system of diffusion equations for which the system (1.1) is a one-dimensional case includes heat conduction in heterogeneous media which formed a subject of rigorous investigation by Rubinstein [28]. In the same paper, heat exchange integrals were
instrumental to Rubinstein's derivation of the coupled system of equations of which equations (1.1) form a special one-dimensional case.

The equations (1.1) are also relevant to a reaction-diffusion system resulting from a simple model of a burning process, investigations of which were instituted by Forbes [15].

The coupled system (1.1) is considered in Chapter 2 whose main aim is that of constructing a one-parameter group for this coupled system. This one-parameter group is instrumental in deriving similarity solutions to the coupled system (1.1).

In preliminary work on the coupled system (1.1), a one-parameter group of transformations of the form

\[
\begin{align*}
    x_1 &= x_1(x, t, \varepsilon) = x + \varepsilon \xi(x, t) + O(\varepsilon^2), \\
    t_1 &= t_1(x, t, \varepsilon) = t + \varepsilon \eta(x, t) + O(\varepsilon^2), \\
    u_1 &= u_1(x, t, \varepsilon)u = u + \varepsilon \zeta(x, t)u + O(\varepsilon^2), \\
    v_1 &= v_1(x, t, \varepsilon)v = v + \varepsilon \chi(x, t)v + O(\varepsilon^2);
\end{align*}
\]

was used for (1.1), resulting in a one-parameter group consisting solely of constants. This constant group merely generated solutions \(u(x, t)\) and \(v(x, t)\) in the form of linear combinations of exponential terms of the form \(e^{(ax + bt)}\), similar to the solutions in Chapter 6. Some examples of these exponential-type solutions are presented in Chapter 6 which discusses the one-dimensional case of diffusion in the presence of three diffusion paths, using a transformation similar to (1.2). Chapter 2 employs a more general transformation of the following form

\[
\begin{align*}
    x_1 &= x_1(x, t, u, v, \varepsilon) = x + \varepsilon \xi(x, t, u, v) + O(\varepsilon^2), \\
    t_1 &= t_1(x, t, u, v, \varepsilon) = t + \varepsilon \eta(x, t, u, v) + O(\varepsilon^2), \\
    u_1 &= u_1(x, t, u, v, \varepsilon)u = u + \varepsilon \zeta(x, t, u, v) + O(\varepsilon^2), \\
    v_1 &= v_1(x, t, u, v, \varepsilon)v = v + \varepsilon \chi(x, t, u, v) + O(\varepsilon^2);
\end{align*}
\]
and in conjunction with the classical and non-classical methods discussed by Hill [19], the transformation (1.3) enables recovery of one-parameter groups for the system (1.1). Ultimately, similarity solutions are derived for the system (1.1).

In Chapter 3, system (1.1) is uncoupled using a special method of uncoupling (see Constanda [11]). The primary focus of Chapter 3 is the derivation of one-parameter groups for the uncoupled version of system (1.1) by means of the classical and non-classical methods involving a transformation of the form

\[ x_1 = x_1(x, t, y, \varepsilon) = x + \varepsilon v(x, t, y) + O(\varepsilon^2), \]
\[ t_1 = t_1(x, t, y, \varepsilon) = t + \varepsilon \tau(x, t, y) + O(\varepsilon^2), \]
\[ y_1 = y_1(x, t, y, \varepsilon) = y + \varepsilon \pi(x, t, y) + O(\varepsilon^2); \]

(1.4)

where \( y \) represents \( u, v \) or their sum. Consequently, further corresponding similarity solutions for the uncoupled version of system (1.1) are recovered.

Forbes [15] investigated the formation of stationary patterns of temperature and chemical concentration in a reaction-diffusion system arising from a simple model of a burning process which was first proposed by Sal'nikov [29] and modified slightly by Forbes [15]. This reaction-diffusion system of equations obtained by Forbes [15] is given by

\[ \frac{\partial C}{\partial t} = \sigma \nabla^2 C + \mu - Ce^{-1/T}, \]
\[ \frac{\partial T}{\partial t} = \alpha \nabla^2 T - \beta (T - \theta_a) + Ce^{-1/T}; \]

(1.5)

within some planar region \( \Omega \) with boundary \( \partial \Omega \). In the system (1.5), \( C \) denotes the intermediate chemical through which a substrate \( S \) (occupying \( \Omega \)) undergoes a two-stage decay process to form a final product, \( T \) represents absolute temperature, \( t \) is time, \( \sigma \) symbolises the dimensionless diffusion coefficient of the intermediate chemical \( C \), \( \mu \) is effectively a measure of substrate concentration, \( \alpha \) is a constant corresponding to the thermal conductivity of the medium, \( \beta \) corresponds...
to the coefficient of Newtonian cooling through the substrate surface, and the parameter $\theta_a$ measures the ambient temperature.

In the same paper, Forbes considered the effects of thermal conduction in his derivation of the system (1.5). He assumed that the intermediate chemical C is able to diffuse through the decomposing substrate. The further assumption was made that the boundary $\partial \Omega$ of the region $\Omega$ is perfectly insulated and impervious to any of the reacting species. Therefore, the boundary conditions found appropriate by Forbes in this paper are the Neumann conditions,

$$\frac{\partial C}{\partial n} = \frac{\partial T}{\partial n} = 0; \quad (1.6)$$

on $\partial \Omega$.

In his paper, Forbes [15] outlined a numerical method for the solution of (1.5) and used a Fourier-series representation of the pattern of temperature and chemical concentration in the reaction-diffusion system in question resulting from the aforementioned model. In the course of his investigations, he derived a linearised solution of the governing equations for this system and represented it in Fourier-series form. While investigating patterns of small amplitude, a restriction arises, namely

$$\theta_a < \sqrt{\frac{\mu}{\beta}} - \frac{\mu}{\beta}; \quad (1.7)$$

indicating that pattern formation is only possible within this bounded interval of ambient temperatures $\theta_a$. The positivity of ambient temperature in (1.7) gives rise to the condition

$$\mu < \beta. \quad (1.8)$$

An alternative derivation of (1.7) and (1.8) can be made with reference to Lee and Hill [22]. It is observed later that for equation (1.16), the following inequality

$$(B_1E_2 + B_2E_1) < \frac{1}{2}(A_1 - A_2)(D_1 - D_2); \quad (1.9)$$
guarantees exponential decay of the source solutions with time. In the same paper, another inequality is stated which is a sufficient condition to obtain source solutions, namely

\[ B_1 B_2 > -\frac{1}{4} (A_1 - A_2)^2. \] (1.10)

The inequalities (1.9) and (1.10) can be used to derive (1.7) and (1.8). From the inequality (1.8), Forbes [15] observed that only if the ambient temperature \( \theta_a \) is not too high and Newtonian cooling of the burning surface occurs at a high enough rate can pattern formation take place.

The one-dimensional case of the system (1.5) is studied in Chapter 4 and is given by

\[
\frac{\partial C}{\partial t} = \sigma \frac{\partial^2 C}{\partial x^2} + \mu - Ce^{-1/T},
\]

\[
\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} - \beta (T - \theta_a) + Ce^{-1/T}.\] (1.11)

The primary concern of Chapter 4 is the derivation of solutions using one-parameter group methods. In this chapter, a one-parameter group is constructed for the coupled system (1.11), enabling similarity solutions for (1.11) to be determined. In conjunction with the classical and non-classical procedures, a transformation of the form

\[
x_1 = x_1(x, t, C, T, \varepsilon) = x + \varepsilon \xi(x, t, C, T) + O(\varepsilon^2),
\]

\[
t_1 = t_1(x, t, C, T, \varepsilon) = t + \varepsilon \eta(x, t, C, T) + O(\varepsilon^2),
\]

\[
C_1 = C_1(x, t, C, T, \varepsilon) = C + \varepsilon \zeta(x, t, C, T) + O(\varepsilon^2),
\]

\[
T_1 = T_1(x, t, C, T, \varepsilon) = T + \varepsilon \chi(x, t, C, T) + O(\varepsilon^2);
\] (1.12)

enables a one-parameter group to be deduced for the system (1.11).
Chapter 5 concerns the solutions of the semi-linear coupled equations, (see Hasimoto [16]), given by

\[ \frac{\partial u}{\partial t} + a_1 \frac{\partial u}{\partial x} = \lambda_1 uv, \]

\[ \frac{\partial v}{\partial t} + a_2 \frac{\partial v}{\partial x} = \lambda_2 uv; \]  

(1.13)

where \( u(t, x) \) denotes a wave propagating along the \( x \)-axis with a constant velocity of \( a_1 \) and \( v(t, x) \) represents a wave propagating along the \( x \)-axis with a constant velocity of \( a_2 \), while \( a_1, a_2 \) and the parameters \( \lambda_1 \) and \( \lambda_2 \) are assumed to be non-zero constants. Yoshikawa and Yamaguti [33] together with Hasimoto [16] and Yamaguti [32] made rigorous investigations of predator-prey systems with linear one-dimensional convective dispersal.

Results presented by Yoshikawa and Yamaguti [33] led to the suggestion that the interplay of non-linear terms and the space derivatives in the system (1.13) (with \( a_1 = \lambda_2 = -1 \) and \( a_2 = \lambda_1 = 1 \)) induced exponential growth of \( u(t, x) \) under suitable conditions. This suggestion was then confirmed in a proof by Yamaguti [32]. Yoshikawa and Yamaguti [33] provided proof of this exponential growth phenomenon of \( u(t, x) \) for the system (1.13) with \( a_1 = -p, a_2 = -q, \lambda_1 = 1 \) and \( \lambda_2 = -1 \) where \( p \neq q \).

Murray and Cohen [24] examined a model interactive predator-prey system in which non-linear convection indicating prey-dependent pursuit by the predator and predator-dependent flight by the prey dominates spatial dispersal. In the course of these investigations, Murray and Cohen [24] allowed for a linear density-independent component of motion which denotes prey-predator movement regardless of the presence of either predator or prey. From these researches, Murray and Cohen [24] demonstrated the existence of plane wave solutions (for the system examined under certain constraints) with amplitude and phase gradually changing with respect to space and time. These plane wave solutions were then shown to satisfy a non-linear Schrödinger equation.

In a study on conservation laws with Riemann initial data, Hsiao and de Mottoni [20] assumed (as did Murray and Cohen [24]) that the rate at which a biological population with density \( U(x, t) \) flees from a second biological population
with density $V(x, t)$ is proportional to the space gradient of the V's while the rate of movement of the V's in the direction of the U's is proportional to the space gradient of the U's. Murray and Cohen's equations then resulted without zeroth-order interaction.

The primary goal of Chapter 5 is that of obtaining similarity solutions for the coupled system (1.13) by two approaches using the one-parameter group method. Characteristic coordinates are introduced into the first method which involves a transformation similar to (1.3) while the second method uses only the transformation (1.3). Both approaches give rise to a one-parameter group which identifies the general similarity solution of the coupled system (1.13).

The theory of diffusion presented by Aifantis [3], [4] has been extended to systems in which more than two paths are present. The diffusion of ions and point defects in metals which are composed of a continuous distribution of high-diffusivity paths such as grain boundaries and dislocations was modelled by Aifantis [3], [4]. In the general theory, each point of the medium is assumed to be simultaneously occupied by $n$ diffusion paths and the concentrations in each diffusion path are assumed to be governed by a system of $n$ parabolic partial differential equations. Hill [17] developed a simple, discrete random walk model for diffusion in the presence of three diffusion paths. In this thesis, the one-dimensional form of Hill's model with modified constants is given by the following equations, namely

\[
\begin{align*}
\frac{\partial u}{\partial t} &= D_1 \frac{\partial^2 u}{\partial x^2} - a_{11} u + a_{12} v + a_{13} w, \\
\frac{\partial v}{\partial t} &= D_2 \frac{\partial^2 v}{\partial x^2} + a_{21} u - a_{22} v + a_{23} w, \\
\frac{\partial w}{\partial t} &= D_3 \frac{\partial^2 w}{\partial x^2} + a_{31} u + a_{32} v - a_{33} w,
\end{align*}
\]  

(1.14)

where $u(x, t)$, $v(x, t)$ and $w(x, t)$ correspond to non-negative concentrations. The self-diffusivities $D_1$, $D_2$ and $D_3$, the constants $a_{11}$, $a_{22}$, $a_{33}$ and the transition probabilities $a_{ij}$ ($i, j = 1, 2, 3; i \neq j$) in Hill's random walk model [17] are all assumed to be positive constants.

The one-dimensional case (1.14) of diffusion in the presence of three diffusion paths is discussed in Chapter 6. For the tripled system (1.14), a one-parameter group of transformations of the form
\[ x_1 = x_1(x, t, \varepsilon) = x + \varepsilon \xi(x, t) + O(\varepsilon^2), \]
\[ t_1 = t_1(x, t, \varepsilon) = t + \varepsilon \eta(x, t) + O(\varepsilon^2), \]
\[ u_1 = u_1(x, t, \varepsilon)u = u + \varepsilon \zeta(x, t)u + O(\varepsilon^2), \]
\[ v_1 = v_1(x, t, \varepsilon)v = v + \varepsilon \chi(x, t)v + O(\varepsilon^2), \]
\[ w_1 = w_1(x, t, \varepsilon)w = w + \varepsilon \lambda(x, t)w + O(\varepsilon^2); \]

is considered, thereby enabling similarity solutions for the tripled system (1.14) to be obtained.

The one-dimensional case of the general linear system of coupled diffusion equations with cross-effects is given by

\[ \frac{\partial u}{\partial t} = D_1 \frac{\partial^2 u}{\partial x^2} + E_1 \frac{\partial^2 v}{\partial x^2} - A_1 u + B_1 v, \]
\[ \frac{\partial v}{\partial t} = D_2 \frac{\partial^2 v}{\partial x^2} + E_2 \frac{\partial^2 u}{\partial x^2} - A_2 v + B_2 u; \]

for the non-negative concentrations \( u(x, t) \) and \( v(x, t) \). It is assumed that the constants \( A_i, B_i, D_i \) and \( E_i \) are positive \( \forall i \in \{1, 2\} \) and that \( D_1 D_2 - E_1 E_2 \neq 0 \) throughout this thesis. The terms \( A_i \) and \( B_i \) denote reaction coefficients while the \( D_i \) are self-diffusivities and the \( E_i \) represent cross-diffusivities. It is noted that the equations (1.1) form a special case of (1.16). Similarity solutions of the coupled system (1.16) are derived by means of a one-parameter group of transformations of the form (1.2) and these solutions are presented in Chapter 6.

A model of the human placenta developed by Aifantis [2] and a model of the geographical spread of epidemics (see Radcliffe [26] and [27]) are but two of the many varied applications of the general linear system of coupled diffusion equations with cross-effects of which equations (1.16) form a one-dimensional case.
CHAPTER 2: SOLUTIONS TO THE COUPLED SYSTEM OF REACTION-DIFFUSION EQUATIONS.

2.1: Introduction.

In this chapter, one-parameter groups leaving the system (1.1) invariant will be determined so as to obtain similarity solutions of (1.1). We therefore consider a general transformation of the form

\[ x_1 = x_1(x, t, u, v, \varepsilon) = x + \varepsilon \xi(x, t, u, v) + O(\varepsilon^2), \]
\[ t_1 = t_1(x, t, u, v, \varepsilon) = t + \varepsilon \eta(x, t, u, v) + O(\varepsilon^2), \]
\[ u_1 = u_1(x, t, u, v, \varepsilon) = u + \varepsilon \zeta(x, t, u, v) + O(\varepsilon^2), \]
\[ v_1 = v_1(x, t, u, v, \varepsilon) = v + \varepsilon \chi(x, t, u, v) + O(\varepsilon^2). \]

(2.1)

If the invariance of (1.1) is preserved by the transformation (2.1) and if \( u = \phi(x, t), v = \psi(x, t) \); then from \( u_1 = \phi(x_1, t_1), v_1 = \psi(x_1, t_1) \), equating terms of order \( \varepsilon \) gives

\[ \xi(x, t, u, v) \frac{\partial u}{\partial x} + \eta(x, t, u, v) \frac{\partial u}{\partial t} = \zeta(x, t, u, v), \]
\[ \xi(x, t, u, v) \frac{\partial v}{\partial x} + \eta(x, t, u, v) \frac{\partial v}{\partial t} = \chi(x, t, u, v). \]

(2.2)

The similarity variables \( u \) and \( v \) are obtained from the solutions of (2.2) which correspond to the functional forms of the similarity solutions of (1.1). We make use of two approaches, the classical and the non-classical, enabling the determination of the groups keeping (1.1) invariant. The classical method involves setting the infinitesimal version of the partial differential equations in question to zero without using equations (2.2) while the non-classical procedure resorts to using equations (2.2) and incorporates the classical groups as special cases. The reader is referred to the account given by Hill [19].
2.2: The Classical Procedure.

By results in Appendix I of this thesis and upon eliminating \( \frac{\partial^2 u}{\partial x^2} \) and \( \frac{\partial^2 v}{\partial x^2} \) by using (1.1), it is evident that

\[
\frac{\partial u}{\partial t} - D_1 \frac{\partial^2 u}{\partial x^2} + k_1 u - k_2 v = \frac{\partial u}{\partial t} - D_1 \frac{\partial^2 u}{\partial x^2} + k_1 u - k_2 v
\]

\[
+ \varepsilon [\xi_1 - D_1 \xi_{xx} + k_1 \xi - k_2 \chi + (2 \xi_x - \xi_u + \frac{D_1}{D_2} \xi_v)k_1 u - (2 \xi_x - \xi_u + \frac{D_1}{D_2} \xi_v)k_2 v
\]

\[
+ \{D_1 \eta_{xx} + 2 \xi_x - \eta_l + (\eta_u - \frac{D_1}{D_2} \eta_v)k_1 u - (\eta_u - \frac{D_1}{D_2} \eta_v)k_2 v\} \frac{\partial u}{\partial t} + D_1 \eta_{uu} \frac{\partial u}{\partial x} \frac{\partial u}{\partial x}
\]

\[
+ \{D_1 \xi_{xx} - 2D_1 \xi_{xx} - \xi_l + (3 \xi_x - \frac{D_1}{D_2} \xi_v)k_1 u - (3 \xi_x - \frac{D_1}{D_2} \xi_v)k_2 v\} \frac{\partial u}{\partial x} + (1 - \frac{D_1}{D_2}) \xi_v \frac{\partial v}{\partial t}
\]

\[
+ (2k_1 \xi_v u - 2k_2 \xi_v v - 2D_1 \xi_{xxv}) \frac{\partial v}{\partial x} + \frac{D_1}{D_2} (1 - 2) \eta_v \frac{\partial u}{\partial x} \frac{\partial v}{\partial t} + 2(\xi_u + D_1 \eta_{ux}) \frac{\partial u}{\partial x}
\]

\[
+ \frac{D_1}{D_2} (1 - 2) \xi_v \frac{\partial u}{\partial x} + 2(\xi_v + D_1 \eta_{vx}) \frac{\partial v}{\partial x} + D_1 (2 \xi_{ux} - \xi_{uv})(\frac{\partial u}{\partial x})^2 + 2D_1 (\xi_{xx} - \xi_{uv})(\frac{\partial u}{\partial x})^2,
\]

\[
\cdot \frac{\partial v}{\partial t} - D_2 \frac{\partial^2 v}{\partial x^2} + k_2 v - k_1 u = \frac{\partial v}{\partial t} - D_2 \frac{\partial^2 v}{\partial x^2} + k_2 v - k_1 u
\]

\[
+ \varepsilon [\chi_1 - D_2 \chi_{xx} + k_2 \chi - k_1 \varepsilon + (2 \chi_x - \chi_v + \frac{D_2}{D_1} \chi_u)k_2 v - (2 \chi_x - \chi_v + \frac{D_2}{D_1} \chi_u)k_1 u
\]

\[
+ \{D_2 \eta_{xx} + 2 \chi_x - \eta_l + (\eta_u - \frac{D_2}{D_1} \eta_v)k_2 v - (\eta_u - \frac{D_2}{D_1} \eta_v)k_1 u\} \frac{\partial v}{\partial t} + D_2 \eta_{uv} \frac{\partial v}{\partial x} \frac{\partial v}{\partial x}
\]

\[
+ \{D_2 \xi_{xx} - 2D_2 \chi_{xxv} - \xi_l + (3 \chi_x - \frac{D_2}{D_1} \xi_u)k_2 v - (3 \chi_x - \frac{D_2}{D_1} \xi_u)k_1 u\} \frac{\partial v}{\partial x} + (1 - \frac{D_2}{D_1}) \chi_u \frac{\partial u}{\partial x}
\]

\[
+ (2k_2 \chi_v u - 2k_1 \chi_v u - 2D_2 \chi_{xxu}) \frac{\partial u}{\partial x} + (\frac{D_2}{D_1} - 1) \eta_u \frac{\partial u}{\partial x} \frac{\partial v}{\partial t} + 2(\chi_u + D_2 \eta_{ux}) \frac{\partial u}{\partial x}
\]

\[
+ \frac{D_2}{D_1} (1 - 2) \chi_v \frac{\partial u}{\partial x} + 2(\chi_v + D_2 \eta_{vx}) \frac{\partial v}{\partial x} + D_2 (2 \chi_{ux} - \chi_{uv})(\frac{\partial v}{\partial x})^2 + 2D_2 (\chi_{xx} - \chi_{uv})(\frac{\partial v}{\partial x})^2,
\]

\[
\cdot \frac{\partial v}{\partial t} - D_2 \frac{\partial^2 v}{\partial x^2} + k_2 v - k_1 u = \frac{\partial v}{\partial t} - D_2 \frac{\partial^2 v}{\partial x^2} + k_2 v - k_1 u
\]
where the subscripts denote partial differentiation with \( x, t, u \) and \( v \) as independent variables (for example, \( \frac{\partial^2 \xi}{\partial x^2} = \xi_{xx} + \xi_{uu} \frac{\partial u}{\partial x} + \xi_{vv} \frac{\partial v}{\partial x} \)); and partial derivatives in the form \( \frac{\partial \xi}{\partial x}, \frac{\partial \xi}{\partial t} \) are used when \( u \) and \( v \) are considered to be dependent on \( x \) and \( t \). From (2.3), we find that (1.1) remains invariant under the transformation (2.1) provided the functions \( \xi(x, t, u, v), \eta(x, t, u, v), \chi(x, t, u, v) \) are such that the equations

\[
\begin{align*}
\zeta_t - D_1 \zeta_{xx} + k_1 \zeta - k_2 \zeta + (2 \zeta_{xx} - \zeta_u + \frac{D_1}{D_2} \zeta_v)k_1u - (2 \zeta_{xx} - \zeta_u + \frac{D_1}{D_2} \zeta_v)k_2v \\
+ (D_1 \eta_{xx} + 2 \zeta_x - \eta_t + (\eta_u - \frac{D_1}{D_2} \eta_v)k_1u - (\eta_u - \frac{D_1}{D_2} \eta_v)k_2v) \frac{\partial u}{\partial t} + D_1 \eta_{uv} \frac{\partial u}{\partial t} \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} \\
+ (D_1 \eta_{xx} - 2D_1 \xi_{xx} - \xi_t + (3 \xi_u - \frac{D_1}{D_2} \xi_v)k_1u - (3 \xi_u - \frac{D_1}{D_2} \xi_v)k_2v) \frac{\partial u}{\partial x} + (1 - \frac{D_1}{D_2}) \xi_v \frac{\partial v}{\partial t} \\
+ (2k_1 \xi_v u - 2k_2 \xi_v v - 2D_1 \xi_{uv}) \frac{\partial v}{\partial x} + (\frac{D_1}{D_2} - 1) \xi_v \frac{\partial u}{\partial x} + 2(\xi_v + D_1 \eta_{uv}) \frac{\partial u}{\partial x} + 2D_1 (\xi_{uv} - \xi_{uu}) \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} + 2D_1 (\xi_{uv} - \xi_{uu}) \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} \\
- D_1 \xi_{uv} \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} + D_1 \eta_{uv} \frac{\partial v}{\partial x} \frac{\partial v}{\partial x} + 2D_1 \xi_{uv} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + D_1 \eta_{uv} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + D_1 \eta_{uv} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \\
+ 2D_1 \eta_{uv} \frac{\partial u}{\partial t} \frac{\partial u}{\partial x} + 2D_1 \eta_{uv} \frac{\partial v}{\partial t} \frac{\partial v}{\partial x} + 2D_1 \eta_{uv} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + 2D_1 \eta_{uv} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} = 0,
\end{align*}
\]

(2.4)

\[
\begin{align*}
\chi_t - D_2 \chi_{xx} + k_1 \chi + (2 \chi_{xx} - \chi_u + D_1 \eta_{xx} + D_1 \xi_{uv})k_2v - (2 \chi_{xx} - \chi_u + D_1 \eta_{xx} + D_1 \xi_{uv})k_1u \\
+ (D_2 \eta_{xx} + 2 \chi_x - \eta_t + (\eta_u - \frac{D_2}{D_1} \eta_v)k_2v - (\eta_u - \frac{D_2}{D_1} \eta_v)k_1u) \frac{\partial v}{\partial t} + D_2 \eta_{uv} \frac{\partial v}{\partial t} \frac{\partial v}{\partial x} \frac{\partial v}{\partial x} \\
+ (D_2 \eta_{xx} - 2D_2 \chi_{xx} - \chi_t + (3 \chi_u - \frac{D_2}{D_1} \chi_v)k_2v - (3 \chi_u - \frac{D_2}{D_1} \chi_v)k_1u) \frac{\partial v}{\partial x} + (1 - \frac{D_2}{D_1}) \chi_u \frac{\partial u}{\partial t} \\
+ (2k_2 \chi_v u - 2k_1 \chi_v v - 2D_2 \chi_{uv}) \frac{\partial u}{\partial x} + (\frac{D_2}{D_1} - 1) \chi_v \frac{\partial v}{\partial x} + 2(\chi_v + D_2 \eta_{uv}) \frac{\partial v}{\partial x} + 2D_2 (\chi_{uv} - \chi_{uu}) \frac{\partial v}{\partial x} \frac{\partial v}{\partial x} \\
- D_2 \chi_{uv} \frac{\partial v}{\partial x} \frac{\partial v}{\partial x} + D_2 \eta_{uv} \frac{\partial v}{\partial x} \frac{\partial v}{\partial x} + 2D_2 \chi_{uv} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + D_2 \eta_{uv} \frac{\partial v}{\partial x} \frac{\partial v}{\partial x} + D_2 \eta_{uv} \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} \\
+ 2D_2 \eta_{uv} \frac{\partial v}{\partial t} \frac{\partial v}{\partial x} + 2D_2 \chi_{uv} \frac{\partial v}{\partial x} \frac{\partial v}{\partial x} + 2D_2 \eta_{uv} \frac{\partial v}{\partial x} \frac{\partial v}{\partial x} + 2D_2 \eta_{uv} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} = 0,
\end{align*}
\]

(2.4)
are satisfied identically. The constants $D_1, k_i$ are positive for all $i \in \{1, 2\}$ and they are to remain unrestricted.

Upon equating to zero the coefficients of $\frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial t}, \frac{\partial^2 u}{\partial x \partial t}$ and $\frac{\partial v}{\partial x} \frac{\partial^2 u}{\partial x \partial t}$ in (2.4) and the coefficients of $\frac{\partial v}{\partial x} \frac{\partial^2 v}{\partial x \partial t}, \frac{\partial^2 v}{\partial x \partial t}$ and $\frac{\partial u}{\partial x} \frac{\partial^2 v}{\partial x \partial t}$ in (2.4), it is evident that

$$\eta(x, t, u, v) = \eta(t).$$

(2.5)

In view of (2.5), inspection of the coefficients of $\frac{\partial u}{\partial x} \frac{\partial u}{\partial t}$ and $\frac{\partial u}{\partial t} \frac{\partial v}{\partial x}$ in (2.4) and the coefficients of $\frac{\partial v}{\partial x} \frac{\partial v}{\partial t}$ and $\frac{\partial v}{\partial t} \frac{\partial u}{\partial x}$ in (2.4) gives

$$\xi(x, t, u, v) = \xi(x, t).$$

(2.6)

The result (2.5) implies that the coefficients of $\frac{\partial u}{\partial t}$ and $\frac{\partial v}{\partial t}$ in (2.4) and (2.4) respectively, reduce to yield $\xi_x = \eta(t)$ and hence, by (2.6) we have

$$\xi(x, t) = \frac{1}{2} \eta'(t)x + a(t);$$

(2.7)

where $a$ denotes an arbitrary function.

Since $D_1$ and $D_2$ are required to remain unrestricted, the coefficient of $\frac{\partial v}{\partial t}$ in equation (2.4) implies $\zeta_v = 0$ and so,

$$\zeta(x, t, u, v) = \zeta(x, t, u).$$

(2.8)

In view of (2.7), the coefficient of $\left(\frac{\partial u}{\partial x}\right)^2$ in equation (2.4) results in $\zeta_{uu} = 0$ and in conjunction with (2.8), we find that

$$\zeta(x, t, u) = \rho(x, t)u + \sigma(x, t);$$

(2.9)

where $\rho$ and $\sigma$ represent arbitrary functions.

Via the results (2.7) and (2.9), it may be deduced from the coefficient of $\frac{\partial u}{\partial x}$ in (2.4) that $\xi_t + 2D_1\zeta_{xu} = 0$ and thus,
\[ \frac{1}{2} \eta''(t)x + a'(t) + 2D_1 \rho_x = 0. \]  
(2.10)

From (2.6) and (2.9), the terms in (2.41) not involving derivatives of \( u \) or \( v \) simplify to give

\[ \{ \rho_t - D_1 \rho_{xx} + k_1 \eta'(t) \} u + \{ \rho - \eta'(t) \} k_2 v + \sigma_t - D_1 \sigma_{xx} + k_1 \sigma - k_2 \chi = 0. \]  
(2.11)

As \( D_1 \) and \( D_2 \) are required to remain unrestricted, the coefficient of \( \frac{\partial u}{\partial t} \) in (2.42) implies that \( \chi_u = 0 \), resulting in

\[ \chi(x, t, u, v) = \chi(x, t, v). \]  
(2.12)

By (2.5), (2.6), (2.7), (2.8) and (2.12), the coefficients of \( \frac{\partial u}{\partial t} \), \( \frac{\partial u}{\partial x} \), \( \frac{\partial u}{\partial x} \), \( \frac{\partial u}{\partial x} \), \( \frac{\partial u}{\partial x} \), \( \frac{\partial u}{\partial x} \), \( \frac{\partial u}{\partial x} \), \( \frac{\partial v}{\partial x} \), \( \frac{\partial v}{\partial x} \), \( \frac{\partial v}{\partial x} \), and \( \frac{\partial v}{\partial x} \) in (2.41) and the coefficients of \( \frac{\partial v}{\partial t} \), \( \frac{\partial v}{\partial x} \), \( \frac{\partial v}{\partial x} \), \( \frac{\partial v}{\partial x} \), \( \frac{\partial v}{\partial x} \), \( \frac{\partial v}{\partial x} \), \( \frac{\partial v}{\partial x} \), \( \frac{\partial v}{\partial x} \), \( \frac{\partial u}{\partial x} \), \( \frac{\partial u}{\partial x} \), \( \frac{\partial u}{\partial x} \), and \( \frac{\partial u}{\partial x} \) in (2.42) yield no additional information.

Via the result (2.6), the coefficient of \( \frac{\partial v}{\partial x} \) in equation (2.42) leads to \( \chi_{vv} = 0 \) and together with (2.12), it is observed that

\[ \chi(x, t, v) = \alpha(x, t) v + \beta(x, t); \]  
(2.13)

where \( \alpha \) and \( \beta \) are arbitrary functions. From (2.7) and (2.13), it may be deduced from the coefficient of \( \frac{\partial v}{\partial x} \) in (2.42) that \( \xi_t + 2D_2 \chi_{vv} = 0 \) and thus,

\[ \frac{1}{2} \eta''(t)x + a'(t) + 2D_2 \chi_x = 0. \]  
(2.14)

By making use of (2.7) and (2.13), the remaining terms in (2.42) simplify to yield

\[ \{ \alpha_t - D_2 \alpha_{xx} + k_2 \eta'(t) \} v + \{ \alpha - \eta'(t) \} k_1 u + \beta_t - D_2 \beta_{xx} + k_2 \beta - k_1 \varsigma = 0. \]  
(2.15)
Thus, the results obtained from equations (2.4) are summarised as follows:

\[ \eta(x, t, u, v) = \eta(t), \]

\[ \xi(x, t, u, v) = \frac{1}{2} \eta'(t)x + a(t), \]

\[ \zeta(x, t, u, v) = \rho(x, t)u + \sigma(x, t), \]

\[ \chi(x, t, u, v) = \alpha(x, t)v + \beta(x, t), \]

\[ \frac{1}{2} \eta''(t)x + a'(t) + 2D_1 \rho_x = 0, \]

\[ \frac{1}{2} \eta''(t)x + a'(t) + 2D_2 \alpha_x = 0, \]

\[ k_2 \chi(x, t, u, v) = \{\rho_t - D_1 \rho_{xx} + k_1 \eta'(t)\}u + \{\rho - \eta'(t)\}k_2 v + \sigma_t - D_1 \sigma_{xx} + k_1 \sigma, \]

\[ k_1 \zeta(x, t, u, v) = \{ \alpha_t - D_2 \alpha_{xx} + k_2 \eta'(t)\}v + \{ \alpha - \eta'(t)\}k_1 u + \beta_t - D_2 \beta_{xx} + k_2 \beta. \]

In order to reconcile (2.165) and (2.166), it is required that

\[ D_1 \rho_x - D_2 \alpha_x = 0. \]

Furthermore, (2.165) implies that

\[ \rho(x, t) = -\frac{1}{8D_1} \eta''(t)x^2 - \frac{1}{2D_1} a'(t)x - \frac{1}{2D_1} b(t); \]

where \( b \) denotes an arbitrary function.

Similarly, from the result (2.166), it is evident that

\[ \alpha(x, t) = -\frac{1}{8D_2} \eta''(t)x^2 - \frac{1}{2D_2} a'(t)x - \frac{1}{2D_2} c(t); \]

where \( c \) is another arbitrary function.

Substituting (2.164) into (2.167) yields

\[ \{\rho_t - D_1 \rho_{xx} + k_1 \eta'(t)\}u + \{\rho - \eta'(t) - \alpha\}k_2 v + \sigma_t - D_1 \sigma_{xx} + k_1 \sigma - k_2 \beta = 0. \]

(2.20)
Since none of the coefficients of \( u, v \) or \( u^0v^0 \) depend on \( u \) or \( v \), these coefficients must be identically zero. Hence, from (2.20) it is required that

\[
\rho_t - D_1 \rho_{xx} + k_1 \eta(t) = 0, \\
\rho - \alpha = \eta(t), \\
\sigma_t = D_1 \sigma_{xx} - k_1 \sigma + k_2 \beta.
\]  

(2.21)

By making use of (2.18) and (2.19), equation (2.21) yields

\[
\left( \frac{1}{8D_2} - \frac{1}{8D_1} \right) \eta''(t)x^2 + \left( \frac{1}{2D_2} - \frac{1}{2D_1} \right) a(t)x + \frac{1}{2D_2} c(t) - \frac{1}{2D_1} b(t) - \eta(t) = 0.
\]  

(2.22)

The constants \( D_1 \) and \( D_2 \) are required to remain unrestricted. Thus, from the coefficients of \( x^2 \) and \( x \) in (2.22), we obtain \( \eta''(t) = 0 \) and \( a'(t) = 0 \), respectively giving

\[
\eta(t) = c_1 t + c_2, \quad a(t) = c_3;
\]  

(2.23)

where \( c_1, c_2 \) and \( c_3 \) denote three arbitrary constants.

From (2.23) and the coefficient of \( u^0v^0 \) in (2.22), we also obtain the result

\[
\frac{1}{2D_2} c(t) - \frac{1}{2D_1} b(t) = c_1.
\]  

(2.24)

The results listed in (2.23) imply that (2.16) yields

\[
\xi(x, t, u, v) = \frac{1}{2} c_1 x + c_3.
\]  

(2.25)

Substituting the result (2.23) into (2.18) yields

\[
\rho(x, t) = -\frac{1}{2D_1} b(t).
\]  

(2.26)

From (2.26), the equation (2.21) simplifies, giving \(-\frac{1}{2D_1} b'(t) + k_1 c_1 = 0\) which implies that
\[ b(t) = 2D_1 k_1 c_1 t + c_4 ; \]  

(2.27)

where \( c_4 \) denotes a fourth arbitrary constant. By (2.27) therefore, (2.26) yields

\[ \rho(x, t) = -k_1 c_1 t - \frac{c_4}{2D_1} . \]  

(2.28)

By making use of (2.27), it is evident from (2.24) that

\[ \frac{1}{2D_2} c(t) = c_1 + \frac{c_4}{2D_1} + k_1 c_1 t . \]  

(2.29)

From (2.23) and (2.29), the expression for \( \alpha(x, t) \) in (2.19) becomes

\[ \alpha(x, t) = -k_1 c_1 t - (c_1 + \frac{c_4}{2D_1}) . \]  

(2.30)

The results (2.28) and (2.30) satisfy the equation (2.17) identically and imply that equations (2.16\_3) and (2.16\_4) become

\[ \zeta(x, t, u, v) = -(k_1 c_1 t + \frac{c_4}{2D_1})u + \sigma(x, t) , \]  

(2.31)

\[ \chi(x, t, u, v) = -(k_1 c_1 t + c_1 + \frac{c_4}{2D_1})v + \beta(x, t) . \]

Substituting the results (2.23\_1), (2.30) and (2.31) into (2.16\_8) yields

\[ (k_2 - k_1)c_1 v - 2k_1 c_1 u + \beta_1 - D_2 \beta_{xx} + k_2 \beta - k_1 \sigma = 0 . \]  

(2.32)

The coefficients of \( u \) and \( v \) in (2.32) compel \( c_1 = 0 \), reducing (2.32) to

\[ \beta_1 = D_2 \beta_{xx} - k_2 \beta + k_1 \sigma . \]  

(2.33)

Since \( c_1 = 0 \) and we assign \( d_1 = c_3 \), \( d_2 = c_2 \) and \( d_3 = -\frac{c_4}{2D_1} \), it may be concluded from the results obtained thus far that the classical group derived for the coupled system (1.1) under the transformation (2.1) is given by
\[ \xi(x, t, u, v) = d_1, \]
\[ \eta(x, t, u, v) = d_2, \]
\[ \zeta(x, t, u, v) = d_3 u + \sigma(x, t), \]
\[ \chi(x, t, u, v) = d_3 v + \beta(x, t); \]

such that

\[ \frac{\partial \sigma}{\partial t} = D_1 \frac{\partial^2 \sigma}{\partial x^2} - k_1 \sigma + k_2 \beta, \]
\[ \frac{\partial \beta}{\partial t} = D_2 \frac{\partial^2 \beta}{\partial x^2} - k_2 \beta + k_1 \sigma. \]

It should be mentioned that equations (2.34) and (2.35) are the key results in this chapter. It is interesting to note that (2.35) is equivalent to (1.1) with \( \sigma \) and \( \beta \) replacing \( u \) and \( v \) respectively. The functions \( \sigma(x, t) \) and \( \beta(x, t) \) are solutions to the original coupled system of equations (1.1). Each solution to (2.35) generates a classical group for (1.1) thereby possibly enabling an infinite variety of similarity solutions to (1.1) to be obtained. Some similarity solutions for (1.1) corresponding to the classical group (2.34) will now be derived. If for example, \( d_1 = d_2 = d_3 = 1 \), the classical group (2.34) will yield

\[ \xi(x, t, u, v) = 1, \]
\[ \eta(x, t, u, v) = 1, \]
\[ \zeta(x, t, u, v) = \sigma(x, t) + u, \]
\[ \chi(x, t, u, v) = \beta(x, t) + v. \]

From equations (2.2) and the group (2.36), the following equations are obtained, namely,
\[
\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = \sigma(x, t) + u, \\
\frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} = \beta(x, t) + v.
\] (2.37)

Similarity solutions to (2.37) will be recovered by means of Lagrange's method (see [8]). Commencing the use of this technique, equations (2.37) may be rewritten as

\[
P_1(x, t, u) \frac{\partial u}{\partial x} + Q_1(x, t, u) \frac{\partial u}{\partial t} = R_1(x, t, u),
\]

\[
P_2(x, t, v) \frac{\partial v}{\partial x} + Q_2(x, t, v) \frac{\partial v}{\partial t} = R_2(x, t, v);
\] (2.38)

where \( P_1(x, t, u) = P_2(x, t, v) = Q_1(x, t, u) = Q_2(x, t, v) = 1 \), \( R_1(x, t, u) = \sigma(x, t) + u \) and \( R_2(x, t, v) = \beta(x, t) + v \).

The subsidiary equations associated with equations (2.38) are given by

\[
\frac{dx}{l} = \frac{dt}{l} = \frac{du}{\sigma(x, t) + u},
\]

\[
\frac{dx}{l} = \frac{dt}{l} = \frac{dv}{\beta(x, t) + v}.
\] (2.39)

Since \( \frac{dx}{l} = \frac{dt}{l} \) in (2.39), integration gives

\[
t = x + c_1',
\] (2.40)

where \( c_1' \) is an arbitrary constant. The first integral of (2.39) is then

\[
r_1(x, t, u) = x - t = e_1;
\] (2.41)

where \( e_1 = - c_1' \).
From (2.41), it is evident that \( \frac{dt}{l} = \frac{du}{\sigma(x, t) + u} \) is equivalent to \( \frac{du}{dt} = \sigma(t + e_1, t) + u \) and so,

\[
\frac{du}{dt} - u = \sigma(t + e_1, t). 
\] (2.42)

An integrating factor for (2.42) is given by

\[
I(t) = e^{-t}. 
\] (2.43)

By the integrating factor method and using (2.43), the equation (2.42) may be written as

\[
\frac{d}{dt}(e^t u) = e^t \sigma(t + e_1, t); 
\] (2.44)

integrating which yields

\[
u = e^t \int e^{-\omega} \sigma(\omega + e_1, \omega) d\omega; 
\] (2.45)

where \( c \) is an arbitrary constant. From (2.45) therefore,

\[
u = e^t \{K(t, e_1) + e_2\}; 
\] (2.46)

where \( e_2 \) represents an arbitrary constant and \( K(t, e_1) \) is defined by

\[
K(t, e_1) = \int e^{-\omega} \sigma(\omega + e_1, \omega) d\omega - e_2. 
\] (2.47)

By the results (2.41) and (2.46), the second integral of (2.391) is

\[
s_1(x, t, u) = u e^{-t} - K(t, x - t) = e_2. 
\] (2.48)

As the Jacobians \( \frac{\partial(r_1, s_1)}{\partial(x, t)} \), \( \frac{\partial(r_1, s_1)}{\partial(x, u)} \), \( \frac{\partial(r_1, s_1)}{\partial(t, u)} \) are all non-zero when \( K_t \neq -ue^{-t} \), hence in any region of space in which \( K_t \neq -u e^{-t} \), the general solution of (2.371) is given by \( F(r_1, s_1) = 0 \), or equivalently by (2.41) and (2.48),

\[
F(x - t, ue^{-t} - K(t, x - t)) = 0; 
\] (2.49)

where \( F \) is arbitrary.
The result (2.49) may be alternatively expressed as

\[ u(x, t) = e^t K(t, x - t) + e^t f(x - t); \]
(2.50)

where \( K(t, x - t) = \int e^{-\omega} \sigma(\omega + x - t, \omega) d\omega \), (the right-hand side of this expression for \( K(t, x - t) \) incorporates \( e_2 \)) and \( f \) denotes an arbitrary function. In a derivation similar to that for (2.37), in any region of space for which \( L_t \neq -ve^{-t} \), the general solution of (2.37) is given by

\[ v(x, t) = e^t L(t, x - t) + e^t g(x - t); \]
(2.51)

where \( L(t, x - t) = \int e^{\omega} \beta(\omega + x - t, \omega) d\omega \), \( d \) is an arbitrary constant and \( g \) denotes an arbitrary function.

Substituting the general solutions (2.50) and (2.51) back into (2.37) confirms their validity.

To summarise, in any region of space for which \( (\int e^{\omega} \sigma(\omega + x - t, \omega) d\omega)_t \neq -ue^{-t} \), \( d \)

and \( (\int e^{\omega} \beta(\omega + x - t, \omega) d\omega)_t \neq -ve^{-t} \), the general solutions to equations (2.37) are given by

\[ u(x, t) = e^t f(x - t) + e^t \int e^{-\omega} \sigma(\omega + x - t, \omega) d\omega, \]
(2.52)

\[ v(x, t) = e^t g(x - t) + e^t \int e^{\omega} \beta(\omega + x - t, \omega) d\omega; \]

where \( f \) and \( g \) represent arbitrary functions, \( c \) and \( d \) are arbitrary constants, while the functions \( \sigma \) and \( \beta \) satisfy the coupled system (1.1). For \( f \) and \( g \) to be
ascertained, the results (2.52) are then substituted into (1.1) to obtain the defining equations for \( f(x - t) \) and \( g(x - t) \).

**Example 1:**

We shall consider one of the simplest pairs of solutions for (1.1), namely

\[
\sigma(x, t) = \frac{1}{k_1 + k_2} \exp[-(k_1 + k_2)t], \quad \beta(x, t) = -(\frac{1}{k_1 + k_2}) \exp[-(k_1 + k_2)t].
\]

Using (2.53) in the classical group (2.36) yields

\[
\xi(x, t, u, v) = 1, \quad \eta(x, t, u, v) = 1,
\]

\[
\zeta(x, t, u, v) = \frac{1}{k_1 + k_2} \exp[-(k_1 + k_2)t] + u,
\]

\[
\chi(x, t, u, v) = -(\frac{1}{k_1 + k_2}) \exp[-(k_1 + k_2)t] + v.
\]

The group (2.54) yields solutions of the form (2.52), namely

\[
u(x, t) = e^{\lambda f(x - t)} - \frac{1}{(k_1 + k_2)(k_1 + k_2 + 1)} \exp[-(k_1 + k_2)t] + d_4 e^\lambda,
\]

\[
v(x, t) = e^{\lambda g(x - t)} + \frac{1}{(k_1 + k_2)(k_1 + k_2 + 1)} \exp[-(k_1 + k_2)t] + d_5 e^\lambda;
\]

where \( f \) and \( g \) are arbitrary functions, while \( d_4 \) and \( d_5 \) are arbitrary constants. To determine \( f \) and \( g \), we let \( w = x - t \) before substituting the solutions (2.55) into (1.1) to yield

\[
D_1 f''(w) + f'(w) - (1 + k_1) f(w) + k_2 g(w) + k_2 d_5 - (1 + k_1) d_4 = 0,
\]

\[
D_2 g''(w) + g'(w) - (1 + k_2) g(w) + k_1 f(w) + k_1 d_4 - (1 + k_2) d_5 = 0.
\]

As an illustration, with the constraints \( D_1 D_2 = D_1 k_2 + D_2 k_1 \) and \( d_4 = d_5 = 0 \), a special solution of (2.56) is given by \( f(w) = \frac{k_2}{k_1 - D_1} e^w \), \( g(w) = e^w \). Consequently from (2.55), we obtain a further solution of (1.1) namely,
\[ u(x, t) = \frac{k_2}{k_1 - D_1} e^x - \frac{1}{(k_1 + k_2)(k_1 + k_2 + 1)} \exp[-(k_1 + k_2)t], \]

\[ v(x, t) = e^x + \frac{1}{(k_1 + k_2)(k_1 + k_2 + 1)} \exp[-(k_1 + k_2)t]. \]

(2.57)

**Example 2:**

Another pair of solutions for (1.1) is given by

\[ \sigma(x, t) = \exp[-k_1 t] + \frac{\sqrt{k_1 k_2}}{(D_1 - D_2)} e^{\delta t} \int \exp[-\gamma^* \tau]\sqrt{\frac{\tau - D_2 t}{D_1 t - \tau}} I_1(\eta^*) d\tau, \]

\[ \frac{D_1 t}{D_2 t} \int \exp[-\gamma^* \tau] I_0(\eta^*) d\tau; \]

(2.58)

\[ \beta(x, t) = \frac{k_1}{(D_1 - D_2)} e^{\delta t} \int \exp[-\gamma^* \tau] I_0(\eta^*) d\tau; \]

where \( \delta, \gamma^* \) and \( \eta^* \) are defined by

\[ \delta = \frac{D_2 k_1 - D_1 k_2}{D_1 - D_2}, \quad \gamma^* = \frac{k_1 - k_2}{D_1 - D_2}, \]

\[ \eta^* = 2 \frac{\sqrt{k_1 k_2}}{(D_1 - D_2)} \sqrt{D_1 t - \tau}(\tau - D_2 t); \]

(2.59)

and \( I_0 \) and \( I_1 \) are the modified Bessel functions of order zero and one, respectively. Choosing \( d_1 = 1 \), \( d_2 = d_3 = 0 \) in the group (2.34) and using (2.2) results in a further solution of (1.1) given by

\[ u(x, t) = \frac{(k_1 b_1 - k_2 b_2)}{(k_1 + k_2)} \exp[-(k_1 + k_2)t] + \frac{(b_1 + b_2)k_2}{(k_1 + k_2)} \exp[-k_1 t] \]

\[ + \frac{\sqrt{k_1 k_2}}{(D_1 - D_2)} e^{\delta t} \int \exp[-\gamma^* \tau] \sqrt{\frac{\tau - D_2 t}{D_1 t - \tau}} I_1(\eta^*) d\tau, \]

\[ v(x, t) = \frac{(k_2 b_2 - k_1 b_1)}{(k_1 + k_2)} \exp[-(k_1 + k_2)t] + \frac{(b_1 + b_2)k_1}{(k_1 + k_2)} \exp[-k_1 t] \]

\[ + \frac{k_1}{(D_1 - D_2)} e^{\delta t} \int \exp[-\gamma^* \tau] I_0(\eta^*) d\tau; \]

(2.60)
where \( b_1 \) and \( b_2 \) denote arbitrary constants. Further solutions of (1.1) may be obtained by these methods. Solutions similar to (2.58) are considered in Chapter 4.

### 2.3: The Non-Classical Procedure

For the one-parameter group of transformations described in (2.1), the terms \( A(x, t, u, v) \), \( B(x, t, u, v) \) and \( C(x, t, u, v) \) are introduced and defined by

\[
A(x, t, u, v) = \frac{\zeta(x, t, u, v)}{\eta(x, t, u, v)},
\]

\[
B(x, t, u, v) = \frac{\xi(x, t, u, v)}{\eta(x, t, u, v)},
\]

\[
C(x, t, u, v) = \frac{\chi(x, t, u, v)}{\eta(x, t, u, v)};
\]

so that (2.2) becomes

\[
\frac{\partial u}{\partial t} = -B \frac{\partial u}{\partial x},
\]

\[
\frac{\partial v}{\partial t} = C - B \frac{\partial v}{\partial x}.
\]

Differentiating (2.62) partially with respect to \( x \) and making use of (1.1), (2.62); we deduce that

\[
\frac{\partial^2 u}{\partial x \partial t} = A_x - \frac{1}{D_1} AB - \frac{k_1}{D_1} Bu + \frac{k_2}{D_1} Bv + (A_u - B_x + \frac{1}{D_1} B^2) \frac{\partial u}{\partial x} + A_v \frac{\partial v}{\partial x} - B_v \frac{\partial v}{\partial x} \cdot \frac{\partial u}{\partial x}^2,
\]

\[
\frac{\partial^2 v}{\partial x \partial t} = C_x - \frac{1}{D_2} CB - \frac{k_2}{D_2} Bv + \frac{k_1}{D_2} Bu + (C_v - B_x + \frac{1}{D_2} B^2) \frac{\partial v}{\partial x} + C_u \frac{\partial u}{\partial x} - B_u \frac{\partial u}{\partial x} - B_v \frac{\partial v}{\partial x}^2.
\]
As before, \( D_1, D_2, k_1 \) and \( k_2 \) are positive and are to remain unrestricted as far as possible. In order for non-trivial transformations to be obtained, it is required that \( \eta(x, t, u, v) \neq 0 \). Then (2.62) and (2.63) are substituted into (2.4). Hence, equating to zero the coefficients of \( \frac{\partial u}{\partial x}, \frac{\partial \nu}{\partial x}, \frac{\partial u}{\partial x}, \frac{\partial \nu}{\partial x}, \left(\frac{\partial u}{\partial x}\right)^2, \left(\frac{\partial \nu}{\partial x}\right)^2, \left(\frac{\partial u}{\partial x}\right)^3, \left(\frac{\partial \nu}{\partial x}\right)^3 \) and the remaining terms yields the following simplified equations, namely

\[
A_t = D_1 A_{xx} - 2AB_x + \left(\frac{D_1}{D_2} - 1\right)CA_v - k_1A + k_2 C + (A_u - \frac{D_1}{D_2} A_v - 2B_x)(k_1u - k_2v),
\]

\[
2D_1 A_{xx} = \left(\frac{D_1}{D_2} - 1\right)BA_v + 2(A + k_1u - k_2v)B_v,
\]

\[
B_t = D_1 B_{xx} - 2BB_x - 2D_1 A_{xu} + 2AB_u + \left(\frac{D_1}{D_2} - 1\right)CB_v + \left(\frac{D_1}{D_2} B_v - 3B_u\right)(k_2v - k_1u),
\]

\[
B_{uu} = 0,
\]

\[
2D_1 A_{uv} = 2D_1 B_{xv} - \left(1 + \frac{D_1}{D_2}\right)BB_v,
\]

\[
A_{vv} = 0,
\]

\[
D_1 A_{uu} = 2D_1 B_{xu} - 2BB_u,
\]

\[
B_{uv} = 0,
\]

\[
B_{vv} = 0,
\]

\[
C_t = D_2 C_{xx} - 2CB_x + \left(\frac{D_2}{D_1} - 1\right)AC_u - k_2 C + k_1 A + (C_v - \frac{D_2}{D_1} C_u - 2B_x)(k_2v - k_1u),
\]

\[
B_t = D_2 B_{xx} - 2BB_x - 2D_2 C_{xv} + 2CB_v + \left(\frac{D_2}{D_1} - 1\right)AB_u + \left(\frac{D_2}{D_1} B_u - 3B_v\right)(k_1u - k_2v),
\]

\[
2D_2 C_{xu} = \left(\frac{D_2}{D_1} - 1\right)BC_u + 2(C + k_2v - k_1u)B_u,
\]

\[
2D_2 C_{uv} = 2D_2 B_{xv} - \left(1 + \frac{D_2}{D_1}\right)BB_u,
\]

\[
D_2 C_{vv} = 2D_2 B_{xv} - 2BB_v,
\]

\[
C_{uu} = 0.
\]

The system of equations (2.64) determines the non-classical group of (1.1).
From (2.64) and (2.64b), it is evident that $B_u = a(x, t)$ which implies that

$$B(x, t, u, v) = a(x, t)u + \gamma(x, t, v);$$  \hspace{1cm} (2.65)

where $a$ and $\gamma$ denote arbitrary functions.

By (2.65), examination of (2.64b) yields $\gamma_{vv} = 0$, leading to

$$\gamma(x, t, v) = b(x, t)v + c(x, t);$$  \hspace{1cm} (2.66)

where $b$ and $c$ are arbitrary functions. Thus, from (2.65) and (2.66), we find that

$$B(x, t, u, v) = a(x, t)u + b(x, t)v + c(x, t).$$  \hspace{1cm} (2.67)

By making use of (2.67) in considering (2.64a), it is implied that

$$A_v = -\frac{1}{4} \left( D_1 + D_2 \right) b a u^2 + \left[ b_x - \frac{1}{2} \left( D_1 + D_2 \right) \left( b^2 v + b c \right) \right] u + k(x, t, v);$$  \hspace{1cm} (2.68)

where $k$ is an arbitrary function.

Substituting (2.68) into (2.64c) compels $b(x, t) = 0$ and $k(x, t, v) = k(x, t)$. Thus, the results (2.67) and (2.68) are simplified to yield

$$B(x, t, u, v) = a(x, t)u + c(x, t),$$

$$A(x, t, u, v) = k(x, t)v + \kappa(x, t, u);$$  \hspace{1cm} (2.69)

where $\kappa$ is an arbitrary function. From (2.69), it is observed that (2.647) gives

$$\kappa(x, t, u) = -\frac{1}{3D_1} a^2 u^3 + (a_x - \frac{1}{D_1} a c) u^2 + d(x, t)u + e(x, t);$$  \hspace{1cm} (2.70)

where $d$ and $e$ denote arbitrary functions. It follows from (2.692) and (2.70) that

$$A(x, t, u, v) = k(x, t)v - \frac{1}{3D_1} a^2 u^3 + (a_x - \frac{1}{D_1} a c) u^2 + d(x, t)u + e(x, t).$$  \hspace{1cm} (2.71)

Employing (2.691) as well as (2.71) in (2.643) leads to
\[-\frac{2}{3D_1} a^3 u^3 + (4a_{xx} \frac{2}{D_1} c a^2) u^2 + (-3D_1 a_{xx} - a_t + 2ac_\xi + 2ca_\xi + 3k_1a + 2ad)u
+ (2k - 3k_2)av + (D_1 c_{xx} - c_t - 2cc_\xi - 2D_1 d_\xi + 2ae) = 0. \tag{2.72}\]

The coefficient of \(u^3\) in (2.72) compels \(a(x, t) = 0\) and by setting \(c(x, t) = B(x, t)\), the expressions (2.71), (2.69) and (2.72) simplify to give

\[A(x, t, u, v) = d(x, t)u + k(x, t)v + e(x, t),\]
\[B(x, t, u, v) = B(x, t),\tag{2.73}\]
\[d_x = \frac{1}{2} B_{xx} - \frac{1}{2D_1} B_t - \frac{1}{D_1} B B_x.\]

Substituting (2.73) into (2.64) yields

\[B(x, t) = 2 \left( \frac{D_1 D_2}{D_1 - D_2} \right) \frac{k_x}{k}. \tag{2.74}\]

The results (2.73) and (2.74) satisfy equations (2.64) identically.

From (2.73), (2.74), (2.64) and (2.64), it is evident that \(C_u = j(x, t)\) and so,

\[C(x, t, u, v) = j(x, t)u + \varphi(x, t, v); \tag{2.75}\]

where \(j\) and \(\varphi\) represent arbitrary functions.

By (2.73), (2.74) and (2.75), examination of (2.64) gives \(\varphi_{vv} = 0\) and therefore,

\[\varphi(x, t, v) = r(x, t)v + m(x, t); \tag{2.76}\]

where \(r\) and \(m\) denote arbitrary functions. From (2.75) and (2.76), we have

\[C(x, t, u, v) = j(x, t)u + r(x, t)v + m(x, t). \tag{2.77}\]

Substituting (2.73) and (2.77) into (2.64) results in

\[B(x, t) = -2 \frac{D_1 D_2}{D_1 - D_2} \frac{i_x}{j}. \tag{2.78}\]
Reconciling (2.732), (2.74) and (2.78) requires \( \frac{k_x}{k} = -\frac{i_x}{j} \) or \((kj)_x = 0\), implying that

\[
k(x, t)j(x, t) = n(t) ;
\]

where \(n\) represents an arbitrary function.

Upon using (2.732) and (2.77), equation (2.64_{11}) leads to

\[
r_x = \frac{1}{2} B_{xx} - \frac{1}{2D_2} B_1 - \frac{1}{D_2} B B_x .
\]

By the results (2.73_1), (2.73_2) and (2.77), equation (2.64_{1}) gives rise to

\[
[D_1 d_{xx} - d_t - 2dB_x + \frac{(D_1 - D_2)}{D_2} jk + \{d - 2B_x - \frac{D_1}{D_2} k\}k_1 + k_2 j - k_1 d]u
\]

\[+ [D_1 k_{xx} - k_t - 2kB_x + \frac{D_1 - D_2}{D_2} rk - \{d - 2B_x - \frac{D_1}{D_2} k\}k_2 + k_2 r - k_1 k]v
\]

\[+ [D_1 e_{xx} - e_t - 2eB_x + \frac{(D_1 - D_2)}{D_2} mk + k_2 m - k_1 e]u^0 v^0 = 0 ;
\]

respectively yielding

\[d_t = D_1 d_{xx} - 2dB_x + \frac{D_1}{D_2} jk - k_1 d + k_2 j + (d - 2B_x - \frac{D_1}{D_2} k)k_1 ,
\]

\[k_t = D_1 k_{xx} - 2kB_x + \frac{D_1}{D_2} rk - k_1 k + k_2 r - (d - 2B_x - \frac{D_1}{D_2} k)k_2 ,
\]

\[e_t = D_1 e_{xx} - 2eB_x + \frac{D_1}{D_2} mk - k_1 e + k_2 m .
\]

Then substituting (2.73_1), (2.73_2) and (2.77) into (2.64_{10}) leads to
\[
[D_2 j_{xx} - j_t - 2 j B_x + \left( \frac{D_2 - D_1}{D_1} \right) j d - k_2 j + k_1 d - (r - 2 B_x - \frac{D_2}{D_1} j) k_1] u
\]
\[
+ [D_2 r_{xx} - r_t - 2 r B_x + \left( \frac{D_2 - D_1}{D_1} \right) j k - k_2 r + k_1 k + (r - 2 B_x - \frac{D_2}{D_1} j) k_2] v
\]
\[
+ [D_2 m_{xx} - m_t - 2 m B_x + \left( \frac{D_2 - D_1}{D_1} \right) j e - k_2 m + k_1 e] u \psi = 0 ; \quad (2.83)
\]

respectively yielding

\[
j_t = D_2 j_{xx} - 2 j B_x + \left( \frac{D_2}{D_1} - 1 \right) d j - k_2 j + k_1 d - (r - 2 B_x - \frac{D_2}{D_1} j) k_1 ,
\]
\[
r_t = D_2 r_{xx} - 2 r B_x + \left( \frac{D_2}{D_1} - 1 \right) k j - k_2 r + k_1 k + (r - 2 B_x - \frac{D_2}{D_1} j) k_2 , \quad (2.84)
\]
\[
m_t = D_2 m_{xx} - 2 m B_x + \left( \frac{D_2}{D_1} - 1 \right) e j - k_2 m + k_1 e .
\]

Substituting (2.732) and (2.74) into (2.643) results in

\[
\left( \frac{k_x}{k} \right)_t = D_1 \left( \frac{k_x}{k} \right)_{xx} - 4 \frac{D_1 D_2}{(D_1 - D_2)} \left( \frac{k_x}{k} \right)_{x} + \left( 1 - \frac{D_1}{D_2} \right) d x . \quad (2.85)
\]

From (2.732), (2.78) and (2.6411), it is evident that

\[
\left( \frac{j_x}{j} \right)_t = D_2 \left( \frac{j_x}{j} \right)_{xx} + 4 \frac{D_1 D_2}{(D_1 - D_2)} \left( \frac{j_x}{j} \right)_{x} + \left( 1 - \frac{D_2}{D_1} \right) r x . \quad (2.86)
\]

We observe (2.733) and (2.80) to identically satisfy (2.85) and (2.86) respectively. Summarising briefly from the results obtained thus far, the non-classical group derived for the coupled system (1.1) under the transformation (2.1) is given by

\[
A(x, t, u, v) = d(x, t) u + k(x, t) v + e(x, t) ,
\]
\[
B(x, t, u, v) = B(x, t) , \quad (2.87)
\]
\[
C(x, t, u, v) = j(x, t) u + r(x, t) v + m(x, t) ;
\]
where the functions \( d, k, e, j, r \) and \( m \) satisfy (2.82), (2.84), (2.85) and (2.86) and the functions \( d, j, k, r \) and \( B \) are related by

\[
k(x, t)j(x, t) = n(t),
\]

\[
B(x, t) = 2 \left( \frac{D_1 D_2}{D_1 - D_2} \right) \frac{k_x}{k} = -2 \frac{D_1 D_2}{(D_1 - D_2)} \frac{j_x}{j},
\]

\[
d_x = \frac{1}{2} B_{xx} - \frac{1}{2D_1} B_t - \frac{1}{D_1} BB_x,
\]

\[
r_x = \frac{1}{2} B_{xx} - \frac{1}{2D_2} B_t - \frac{1}{D_2} BB_x.
\]  

(2.88)

It is noted that the classical group (2.34) is recovered when \( k \) and \( j \) approach zero. If \( \lim_{k \to 0} \left( \frac{k_x}{k} \right) \) or \( \lim_{j \to 0} \left( \frac{j_x}{j} \right) \) exists, then \( B(x, t) \) is reduced to a constant, that is,

\[
B(x, t) = p_1;
\]  

(2.89)

where \( p_1 \) denotes an arbitrary constant. From (2.88), (2.88) and (2.89), it follows that

\[
d(x, t) = d(t), \quad r(x, t) = r(t).
\]  

(2.90)

We observe that the results (2.88), (2.89) and (2.90) trivially satisfy equations (2.85) and (2.86). By (2.89) and (2.90), equations (2.82) and (2.84) simplify, giving

\[
d'(t) = 0, \quad (r - d)k_2 = 0, \quad e_t = D_1 e_{xx} - k_1 e + k_2 m,
\]

\[
(d - r)k_1 = 0, \quad r'(t) = 0, \quad m_t = D_2 m_{xx} - k_2 m + k_1 e.
\]  

(2.91)

The equations (2.91) and (2.94) imply that as \( k_1, k_2 > 0 \), we have

\[
r(t) = d(t).
\]  

(2.92)
The equations (2.91₃), (2.91₅) and (2.92) yield

\[ d(t) = r(t) = p_2; \]  \hspace{1cm} (2.93)

where \( p_2 \) denotes an arbitrary constant.

The results (2.89) and (2.93) identically satisfy (2.82), (2.84), (2.85), (2.86) and (2.88) except for (2.82₃) and (2.84₃) which yield (2.91₃) and (2.91₆). Hence from (2.89) and (2.93) with \( p_1 = \frac{d_1}{d_2}, \ p_2 = \frac{d_3}{d_2}, \ e(x, t) = \frac{\sigma(x, t)}{d_2} \) and \( m(x, t) = \frac{\beta(x, t)}{d_2} \), the group is deduced as

\[ A(x, t, u, v) = p_2u + e(x, t), \]
\[ B(x, t, u, v) = p_1, \]
\[ C(x, t, u, v) = p_2v + m(x, t); \] \hspace{1cm} (2.94)

which by definition of \( A, B \) and \( C \) in (2.61) enables recovery of the classical group (2.34). Substitution of the group (2.94) into equations (2.64) demonstrates that (2.94) satisfies equations (2.64) identically except for (2.64₁) and (2.64₁₀) which result in (2.91₃) and (2.91₆). The equations (2.91₃) and (2.91₆) are equivalent to (2.35) with \( e(x, t) \) and \( m(x, t) \) replacing \( \sigma(x, t) \) and \( \beta(x, t) \) respectively. Consequently, the functions \( e(x, t) \) and \( m(x, t) \) are solutions to the coupled system (1.1). Similarity solutions for (1.1) corresponding to the non-classical group (2.87) will now be derived. From equations (2.62) and the group (2.87), the following equations arise namely,

\[ u_t + Bu_x = d(x, t)u + k(x, t)v + e(x, t), \]
\[ v_t + Bv_x = r(x, t)v + j(x, t)u + m(x, t). \] \hspace{1cm} (2.95)

From (2.95₁),

\[ v = \frac{1}{k} [u_t + Bu_x - d(x, t)u - e(x, t)]. \] \hspace{1cm} (2.96)

From the result (2.96), it is implied that (2.95₂) yields
\[
\frac{1}{k} u_{tt} + 2 \frac{B}{k} u_{tx} + \frac{B^2}{k} u_{xx} + \left[ \left( \frac{B}{k} \right)_t + \frac{B}{k} \frac{B}{x} - \frac{(d + r)}{k} B \right] u_x + \left[ \left( \frac{B}{k} \right)_t + \frac{B}{k} \frac{B}{x} - \frac{(d + r)}{k} \right] u_t \\
+ \left[ \frac{d r}{k} - \frac{d}{k} t - B \frac{d}{k} x \right] u + \left[ \frac{r e}{k} - m - \frac{e}{k} t - B \frac{e}{k} x \right] = 0 .
\] (2.97)

From (2.95)2,
\[ u = \frac{1}{j} \left[ v_t + B v_x - r(x, t)v - m(x, t) \right] . \] (2.98)

Substituting the result (2.98) into (2.95)1 leads to
\[
\frac{1}{j} v_{tt} + 2 \frac{B}{j} v_{tx} + \frac{B^2}{j} v_{xx} + \left[ \left( \frac{B}{j} \right)_t + \frac{B}{j} \frac{B}{x} - \frac{(r + d)}{j} B \right] v_x + \left[ \left( \frac{B}{j} \right)_t + \frac{B}{j} \frac{B}{x} - \frac{(r + d)}{j} \right] v_t \\
+ \left[ \frac{r d}{j} - k - \frac{r}{j} t - B \frac{r}{j} x \right] v + \left[ \frac{d m}{j} - e - \frac{m}{j} t - B \frac{m}{j} x \right] = 0 .
\] (2.99)

Some similarity solutions for (1.1) corresponding to the non-classical group (2.87) will now be derived.

**Example:**

By setting the conditions \( j(x, t) = j, k(x, t) = k, d(x, t) = \frac{D_1}{D_2} k \) and \( r(x, t) = \frac{D_2}{D_1} j \)

where \( j \) and \( k \) denote constant terms, the equations (2.85) and (2.86) are trivially satisfied, while (2.88)2 simplifies to give
\[ B(x, t) = 0 . \] (2.100)

In addition to the conditions listed, the result (2.100) causes equations (2.88)3 and (2.88)4 to be trivially satisfied while reducing (2.82)1, (2.82)2, (2.84)1 and (2.84)2 to yield
\[ k = \frac{D_2 k j}{D_1 k_1 + (D_2 - D_1) j} . \] (2.101)

The result (2.100) reduces (2.82)3 and (2.84)3 to give
\[ e_t = D_1 e_{xx} - k_1 e + k_2 m + \left( \frac{D_1}{D_2} - 1 \right) m k, \]
\[ m_t = D_2 m_{xx} - k_2 m + k_1 e + \left( \frac{D_2}{D_1} - 1 \right) e j. \] (2.102)

Hence, based on the conditions of this example as well as the attendant results (2.100), (2.101) and (2.102), the non-classical group (2.87) for the coupled system of equations (1.1) simplifies to yield

\[ A(x, t, u, v) = \frac{D_1}{D_2} k u + k v + e(x, t), \]
\[ B(x, t, u, v) = 0, \] (2.103)
\[ C(x, t, u, v) = \frac{D_2}{D_1} j v + j u + m(x, t); \]

where \( k \) and \( j \) are constants related by (2.101) and (2.102).

Substituting the result (2.103) as well as the earlier conditions \( d(x, t) = \frac{D_1}{D_2} k \) and \( r(x, t) = \frac{D_2}{D_1} j \) into (2.97) leads to

\[ u_{tt} - \left( \frac{D_1}{D_2} k + \frac{D_2}{D_1} j \right) u_t = k m(x, t) + e_t - \frac{D_2}{D_1} j e(x, t). \] (2.104)

The equation (2.104) may be rewritten in the form

\[ u_{tt} - \alpha_1 u_t = \beta_1(x, t); \] (2.105)

where \( \alpha_1 = \frac{D_1}{D_2} k + \frac{D_2}{D_1} j \) and \( \beta_1(x, t) = k m(x, t) + e_t - \frac{D_2}{D_1} j e(x, t) \).

The corresponding homogeneous equation for (2.105) is

\[ D_t(D_t - \alpha_1) u = 0. \] (2.106)

By the integrating factor method, the general solution of (2.106) is
\[ u_h = f(x) + \exp[\alpha_1 t]g(x) \]  
(2.107)

where \( f \) and \( g \) are arbitrary functions. Rewriting (2.105) as

\[ D_t(D_t - \alpha_1)u = \beta_1(x, t) \]  
(2.108)

a particular solution of (2.108) is given by

\[ u_p = \frac{\beta_1(x, t)}{D_t(D_t - \alpha_1)} \]  
(2.109)

By partial fractions and the shift theorem, the result (2.109) may be rewritten as

\[ u_p = \frac{1}{\alpha_1} \exp[\alpha_1 t] \int \exp[-\alpha_1 s] \beta_1(x, s) ds - \frac{1}{\alpha_1} \int \beta_1(x, s) ds \]  
(2.110)

By (2.107) and (2.110), the general solution of (2.105) is given by

\[ u(x, t) = f(x) + \exp[\alpha_1 t]g(x) + \frac{1}{\alpha_1} \exp[\alpha_1 t] \int \exp[-\alpha_1 s] \beta_1(x, s) ds \]

\[ - \frac{1}{\alpha_1} \int \beta_1(x, s) ds \]  
(2.111)

Substituting the solution (2.111) into equation (2.105) confirms the validity of (2.111). By (2.111) and the given definition of \( \beta_1(x, t) \), the general solution to equation (2.104) is described by

\[ u(x, t) = f(x) + \exp[\alpha_1 t]g(x) - \frac{1}{\alpha_1} \int [km(x, s) + e_s - \frac{D_2}{D_1} je(x, s)] ds \]

\[ + \frac{1}{\alpha_1} \exp[\alpha_1 t] \int \{ \exp[-\alpha_1 s][km(x, s) + e_s - \frac{D_2}{D_1} je(x, s)] \} ds \]  
(2.112)

Substituting the result (2.103) as well as the earlier conditions \( d(x, t) = \frac{D_1}{D_2} k \) and \( r(x, t) = \frac{D_2}{D_1} j \) into (2.99) leads to
\[ v_{tt} - \left( \frac{D_2}{D_1} j + \frac{D_1}{D_2} k \right) v_t = j e(x, t) + m_t - \frac{D_1}{D_2} k m(x, t). \] (2.113)

The equation (2.113) may be rewritten as

\[ v_{tt} - \alpha_1 v_t = \beta_2(x, t); \] (2.114)

where \( \alpha_1 \) was defined earlier and \( \beta_2(x, t) = j e(x, t) + m_t - \frac{D_1}{D_2} k m(x, t). \)

In a derivation similar to that for (2.104), the general solution of (2.114) is

\[
v(x, t) = h(x) + \exp[\alpha_1 t] p(x) + \frac{1}{\alpha_1} \int_0^t \exp[-\alpha_1 s] \beta_2(x, s) ds - \frac{1}{\alpha_1} \int_0^t \beta_2(x, s) ds; \]

(2.115)

where \( h \) and \( p \) denote arbitrary functions. Substituting the solution (2.115) into (2.114) confirms the validity of (2.115). By (2.115), the general solution to (2.113) is described by

\[
v(x, t) = h(x) + \exp[\alpha_1 t] p(x) - \frac{1}{\alpha_1} \int_0^t [j e(x, s) + m_s - \frac{D_1}{D_2} k m(x, s)] ds + \frac{1}{\alpha_1} \exp[\alpha_1 t] \int_0^t \{\exp[-\alpha_1 s] [j e(x, s) + m_s - \frac{D_1}{D_2} k m(x, s)]\} ds. \]

(2.116)

Then together with the given definition of \( \alpha_1 \), the result (2.1032) and the earlier condition \( d = \frac{D_1}{D_2} k \), the solutions (2.112) and (2.116) are substituted into the equations (2.95), ultimately yielding after a great deal of derivation the following simplified equations namely,

\[ \frac{1}{D_1} [D_2 j g(x) - D_1 k p(x)] \exp[\alpha_1 t] - \frac{k}{D_2} [D_1 f(x) + D_2 h(x)] = 0, \]

(2.117)

\[ \frac{1}{D_2} [D_1 k p(x) - D_2 j g(x)] \exp[\alpha_1 t] - \frac{j}{D_1} [D_2 h(x) + D_1 f(x)] = 0; \]
implying
\[ D_1 f(x) = -D_2 h(x), \quad D_1 k p(x) = D_2 j g(x); \tag{2.118} \]

where \( k \) and \( j \) remain unrestricted constants.

Summarising briefly, from (2.112), (2.116) and the given definition of \( \alpha_1 \), the similarity solutions to the coupled system (1.1) corresponding to the group (2.103) are given by

\[
\begin{align*}
    u(x, t) & = f(x) + \exp[\alpha_1 t] g(x) - \frac{1}{\alpha_1} \int_{0}^{t} \{ k m(x, s) - \frac{D_2}{D_1} j e(x, s) \} ds \\
    & \quad + \frac{k}{\alpha_1} \exp[\alpha_1 t] \int_{0}^{t} \{ \exp[\alpha_1 s] \{ m(x, s) + \frac{D_1}{D_2} e(x, s) \} \} ds,
\end{align*}
\tag{2.119}
\]

\[
\begin{align*}
    v(x, t) & = h(x) + \exp[\alpha_1 t] p(x) - \frac{1}{\alpha_1} \int_{0}^{t} \{ j e(x, s) - \frac{D_1}{D_2} k m(x, s) \} ds \\
    & \quad + \frac{j}{\alpha_1} \exp[\alpha_1 t] \int_{0}^{t} \{ \exp[\alpha_1 s] \{ e(x, s) + \frac{D_2}{D_1} m(x, s) \} \} ds;
\end{align*}
\]

where \( \alpha_1 \) was previously defined and the solutions (2.119) are subject to the conditions (2.101), (2.102) and (2.118). Particular solutions for \( f(x), g(x), h(x) \) and \( p(x) \) can be found by substitution of (2.119) (with special solutions for \( e(x, s) \) and \( m(x, s) \)) into the coupled system (1.1).

An interesting question now arises as to whether or not the solutions (2.119) can be derived from the classical procedure. Given that non-classical solutions are non-existent for the heat equation (Arrigo, Goard and Broadbridge [7]) and that solutions for the system (1.1) can be written in terms of solutions of the heat equation (Aifantis and Hill [5]), it may indeed be that the solutions (2.119) can be derived from the classical procedure.
CHAPTER 3: SOLUTIONS TO THE UNCOUPLED SYSTEM OF REACTION-DIFFUSION EQUATIONS.

3.1: Uncoupling (1.1) By Equating To Zero The Determinant Of Its Equivalent Matrix Form.

The system (1.1) is reconsidered upon rewriting it in matrix form

\[
\begin{pmatrix}
\frac{\partial}{\partial t} - D_1 \frac{\partial^2}{\partial x^2} + k_1 & -k_2 \\
-k_1 & \frac{\partial}{\partial t} - D_2 \frac{\partial^2}{\partial x^2} + k_2
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

(3.1)

Uncoupling the system (1.1) may be accomplished by setting to zero the determinant of the system (3.1). The reader is referred to the account of this method of uncoupling equations given by Constanda [11]. Both \( u \) and \( v \) as well as their sum satisfy the following equation

\[
D_1D_2 \frac{\partial^4 v}{\partial x^4} - (D_1 + D_2) \frac{\partial^3 v}{\partial t \partial x^2} - (D_1k_2 + D_2k_1) \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial t^2} + (k_1 + k_2) \frac{\partial v}{\partial t} = 0.
\]

(3.2)

In this chapter, one-parameter transformation groups preserving the invariance of (3.2) will be determined in order to derive similarity solutions to (3.2), where \( u, v \) and their sum satisfy (3.2). Thus, we consider a general transformation of form

\[
\begin{align*}
x_1 &= x_1(x, t, y, \varepsilon) = x + \varepsilon v(x, t, y) + O(\varepsilon^2), \\
t_1 &= t_1(x, t, y, \varepsilon) = t + \varepsilon \tau(x, t, y) + O(\varepsilon^2), \\
y_1 &= y_1(x, t, y, \varepsilon) = y + \varepsilon \pi(x, t, y) + O(\varepsilon^2);
\end{align*}
\]

(3.3)

where \( y \) represents \( u, v \) or their sum. If the transformation (3.3) leaves the equation (3.2) invariant and if \( y = \gamma(x, t) \), then from \( y_1 = \gamma(x_1, t_1) \), equating terms of order \( \varepsilon \) yields
\[ v(x, t, y) \frac{\partial y}{\partial x} + \tau(x, t, y) \frac{\partial y}{\partial t} = \pi(x, t, y). \quad (3.4) \]

For known functions \( v(x, t, y) \), \( \tau(x, t, y) \) and \( \pi(x, t, y) \), the similarity variable \( y \) is obtained from the solution of (3.4) which corresponds to the functional form of the similarity solution of (3.2). We employ two procedures, the classical and the non-classical, for the purpose of ascertaining the groups keeping (3.2) invariant.

### 3.2: The Classical Procedure

From results in Appendix II of this thesis and using (3.2) to eliminate \( \frac{\partial^4 y}{\partial x^4} \), it is evident that

\[
\begin{align*}
D_1 D_2 \frac{\partial^4 y_1}{\partial x^4} - (D_1 + D_2) \frac{\partial^3 y_1}{\partial t \partial x^2} - (D_1 k_2 + D_2 k_1) \frac{\partial^2 y_1}{\partial x^2 \partial t} + (k_1 + k_2) \frac{\partial y_1}{\partial t} &= \\
D_1 D_2 \frac{\partial^4 y}{\partial x^4} - (D_1 + D_2) \frac{\partial^3 y}{\partial t \partial x^2} - (D_1 k_2 + D_2 k_1) \frac{\partial^2 y}{\partial x^2 \partial t} + (k_1 + k_2) \frac{\partial y}{\partial t} &= \\
+ \varepsilon \{D_1 D_2 \pi_{xxxx} - (D_1 + D_2) \pi_{txx} - (D_1 k_2 + D_2 k_1) \pi_{xx} + \pi_{tt} + (k_1 + k_2) \pi_t \}
+ [- D_1 D_2 \tau_{xxxx} - (D_1 + D_2) \tau_{txx}(\pi_{xy} - \tau_{tx}) + (D_1 k_2 + D_2 k_1) \tau_{xx} + 2 \pi_{tx} - \tau_{tt} \\
+ (k_1 + k_2)(4 \nu_x - \tau_x)] \frac{\partial y}{\partial t} + [D_1 D_2 (4 \pi_{xxx} - v_{xxx}) - (D_1 + D_2)(2 \pi_{tx} - v_{tx}) \\
- (D_1 k_2 + D_2 k_1)(2 \pi_{xy} - v_{xx}) - v_{tt} - (k_1 + k_2) \nu_x \frac{\partial y}{\partial x} + (D_1 + D_2) \tau_x \frac{\partial^2 y}{\partial x \partial t} \frac{\partial^2 y}{\partial t^2} \\
+ [(D_1 + D_2) \tau_{xy} + \pi_{yy} - 2 \tau_{xy}](\frac{\partial y}{\partial t})^2 + [- 4 D_1 D_2 \tau_{xxyy} + 2(D_1 k_2 + D_2 k_1) \tau_{xy} \\
- (D_1 + D_2)(2 \pi_{xy} - v_{xx}) - 2 \nu_x + 4(k_1 + k_2) \nu_y] \frac{\partial y}{\partial x} \frac{\partial y}{\partial t} - \tau_{yy} \frac{\partial y}{\partial t}^3 \\
+ [D_1 D_2 (6 \pi_{xy} - 4 v_{xxx}) - (D_1 + D_2)(\pi_{xy} - 2 \nu_{tx}) - 2(D_1 k_2 + D_2 k_1) \nu_x] \frac{\partial^2 y}{\partial x^2} \\
+ [- 4 D_1 D_2 \tau_{xxx} - (D_1 + D_2)(2 \pi_{xy} - 2 \tau_{tx} - v_{xx}) + 2(D_1 k_2 + D_2 k_1) \tau_x - 2 \nu_x] \frac{\partial^2 y}{\partial x \partial t} \\
+ 2(D_1 + D_2) \tau_x \frac{\partial^3 y}{\partial x \partial t^2} + [D_1 D_2 (6 \pi_{xxyy} - 4 v_{xxyy}) - (D_1 + D_2) \tau_{xy} - 2 \nu_{xy} \\
- (D_1 k_2 + D_2 k_1)(\pi_{yy} - 2 \nu_{xy})] \frac{\partial y}{\partial x}^2 + [(D_1 + D_2) \tau_{xx} + 4 \nu_x - 2 \tau_x] \frac{\partial^2 y}{\partial t^2} - 2 \tau_y \frac{\partial y}{\partial t} \frac{\partial^2 y}{\partial t^2} \\
+ [D_1 D_2 (12 \pi_{xxy} - 18 v_{xxy}) + 3(D_1 + D_2) \nu_{ty} - 2(D_1 k_2 + D_2 k_1) \nu_y] \frac{\partial y}{\partial x} \frac{\partial^2 y}{\partial x \partial t} \frac{\partial^2 y}{\partial t^2} \\
+ [D_1 D_2 (12 \pi_{xxy} - 18 v_{xxy}) + 3(D_1 + D_2) \nu_{ty} - 2(D_1 k_2 + D_2 k_1) \nu_y] \frac{\partial y}{\partial x} \frac{\partial^2 y}{\partial x \partial t} \frac{\partial^2 y}{\partial t^2}.\]
\]
\[ + [-6D_1D_2 \tau_{xx} - (D_1 + D_2)(\pi_{yy} - 2v_{xy} - \tau_{ty})] \frac{\partial^2 \tau_y}{\partial t \partial x^2} + [2(D_1 + D_2)\tau_{xy} - v_{yy}] \frac{\partial^2 \tau_y}{\partial x^2 \partial t} \]
\[ + [-12D_1D_2 \tau_{xy} - 2(D_1 + D_2)(\pi_{yy} - 2v_{xy} - \tau_{ty}) + 2(D_1k_2 + D_2k_1)\tau_{yy}] \frac{\partial^2 \tau_y}{\partial x^2 \partial \tau_{xy}} \]
\[ + [D_1D_2(4\pi_{xxy} - 6v_{xxy}) + (D_1 + D_2)v_{tyy} + (D_1k_2 + D_2k_1)v_{yy}](\frac{\partial \tau_y}{\partial x})^3 - 4D_1D_2 \tau_y \frac{\partial^2 \tau_y}{\partial x \partial \tau_{xy}} \]
\[ + [-6D_1D_2 \tau_{xy} - (D_1 + D_2)(\pi_{yy} - 2v_{xy} - \tau_{ty}) + (D_1k_2 + D_2k_1)\tau_{yy}] \frac{\partial^2 \tau_y}{\partial x \partial \tau_{xy}} \]
\[ + [4(D_1 + D_2)\tau_{xy} - 2v_{y}] \frac{\partial \tau_y}{\partial t} \frac{\partial^2 \tau_y}{\partial x^2} + [2(D_1 + D_2)\tau_{xy} + 4v_{y}] \frac{\partial \tau_y}{\partial x} \frac{\partial^2 \tau_y}{\partial t \partial x} + 2(D_1 + D_2)\tau_{y} \frac{\partial^2 \tau_y}{\partial x \partial \tau_{xy}} \]
\[ + [-6D_1D_2 \tau_{xx} + (D_1 + D_2)(\tau_{y} - 2v_{xy})] \frac{\partial^2 \tau_y}{\partial t \partial x^2} + [-12D_1D_2 \tau_{xy} - 2(D_1 + D_2)v_{y}] \frac{\partial^2 \tau_y}{\partial x \partial \tau_{xy}} \]
\[ + [-12D_1D_2 \tau_{xy} + 3(D_1 + D_2)v_{y}] \frac{\partial \tau_y}{\partial x} \frac{\partial^2 \tau_y}{\partial t \partial x} + 2(D_1 + D_2)\tau_{y} \frac{\partial \tau_y}{\partial x} \frac{\partial^2 \tau_y}{\partial t \partial \tau_{xy}} - 10D_1D_2 \tau_y \frac{\partial^2 \tau_y}{\partial x^2 \partial \tau_{xy}} \]
\[ + (D_1 + D_2)\tau_{y} \frac{\partial^2 \tau_y}{\partial t \partial x} + [-12D_1D_2 \tau_{xy} + 3(D_1 + D_2)v_{yy}] \frac{\partial \tau_y}{\partial x} \frac{\partial^2 \tau_y}{\partial \tau_{xy} \partial \tau_{xy}} - 4D_1D_2 \tau_x \frac{\partial \tau_y}{\partial \tau_{xy} \partial \tau_{xy}} \]
\[ + (D_1 + D_2)\tau_{yy} \frac{\partial \tau_y}{\partial t} \frac{\partial^2 \tau_y}{\partial x^2} + 4(D_1 + D_2)\tau_{xy} \frac{\partial \tau_y}{\partial x} \frac{\partial^2 \tau_y}{\partial t \partial \tau_{xy}} + (D_1 + D_2)\tau_{yy} \frac{\partial \tau_y}{\partial x} \frac{\partial^2 \tau_y}{\partial t \partial \tau_{xy}} \]
\[ + [D_1D_2(4\pi_{xxy} - 6v_{xxy}) + (D_1 + D_2)v_{y}] \frac{\partial^2 \tau_y}{\partial x \partial \tau_{xy}} + [-4D_1D_2 \tau_{xy} + (D_1 + D_2)v_{y}] \frac{\partial^2 \tau_y}{\partial x \partial \tau_{xy}} \]
\[ + [-12D_1D_2 \tau_{xy} + 3(D_1 + D_2)v_{yy}] \frac{\partial \tau_y}{\partial x} \frac{\partial^2 \tau_y}{\partial \tau_{xy} \partial \tau_{xy}} + (D_1 + D_2)\tau_{yy} \frac{\partial \tau_y}{\partial x} \frac{\partial^2 \tau_y}{\partial \tau_{xy} \partial \tau_{xy}} \]
\[ + [-4D_1D_2 \tau_{xxyy} + (D_1 + D_2)v_{y}] \frac{\partial \tau_y}{\partial x} \frac{\partial^2 \tau_y}{\partial \tau_{xy} \partial \tau_{xy} \partial \tau_{xy}} + D_1D_2(4\pi_{yy} - 16v_{xy}) \frac{\partial^2 \tau_y}{\partial x \partial \tau_{xy}} \]
\[ + D_1D_2(3\pi_{yy} - 12v_{xy})(\frac{\partial^2 \tau_y}{\partial x^2})^2 - 6D_1D_2 \tau_y \frac{\partial^2 \tau_y}{\partial x^2 \partial \tau_{xy}} - 6D_1D_2 \tau_y \frac{\partial^2 \tau_y}{\partial x^2 \partial \tau_{xy}} \]
\[ + D_1D_2(6\pi_{yy} - 24v_{xy})(\frac{\partial^2 \tau_y}{\partial x^2})^2 - 15D_1D_2 \tau_{yy} \frac{\tau_y}{\partial x} \frac{\partial^2 \tau_y}{\partial \tau_{xy} \partial \tau_{xy} \partial \tau_{xy}} - 10D_1D_2 \tau_{yy} \frac{\partial^2 \tau_y}{\partial x^2 \partial \tau_{xy} \partial \tau_{xy} \partial \tau_{xy}} \]
\[ - 4D_1D_2 \tau_{yy} \frac{\partial \tau_y}{\partial x} \frac{\partial^2 \tau_y}{\partial \tau_{xy} \partial \tau_{xy}} - 3D_1D_2 \tau_{yy} \frac{\partial \tau_y}{\partial x} \frac{\partial^2 \tau_y}{\partial \tau_{xy} \partial \tau_{xy}} - 12D_1D_2 \tau_{yy} \frac{\partial \tau_y}{\partial x} \frac{\partial \tau_y}{\partial \tau_{xy} \partial \tau_{xy}} - D_1D_2 \tau_{yy} \frac{\partial \tau_y}{\partial \tau_{xy} \partial \tau_{xy} \partial \tau_{xy}} \]
\[ - 6D_1D_2 \tau_{yy} \frac{\partial \tau_y}{\partial x} \frac{\partial \tau_y}{\partial \tau_{xy} \partial \tau_{xy} \partial \tau_{xy}} + D_1D_2(\pi_{xxyy} - 4v_{xxyy}) \frac{\partial \tau_y}{\partial x} \frac{\partial \tau_y}{\partial \tau_{xy} \partial \tau_{xy}} - 10D_1D_2 \tau_{yy} \frac{\partial \tau_y}{\partial x} \frac{\partial \tau_y}{\partial \tau_{xy} \partial \tau_{xy} \partial \tau_{xy}} \]
\[ - 6D_1D_2 \tau_{yy} \frac{\partial \tau_y}{\partial x} \frac{\partial \tau_y}{\partial \tau_{xy} \partial \tau_{xy} \partial \tau_{xy}} - 4D_1D_2 \tau_{yy} \frac{\tau_y}{\partial x} \frac{\partial \tau_y}{\partial \tau_{xy} \partial \tau_{xy} \partial \tau_{xy}} + D_1D_2 \tau_{yy} \frac{\partial \tau_y}{\partial x} \frac{\partial \tau_y}{\partial \tau_{xy} \partial \tau_{xy} \partial \tau_{xy}} \]
\[ + O(\varepsilon^2) \]
\[(3.5)\]
where the subscripts denote partial differentiation with \( x, t \) and \( y \) as independent variables (for example, \( \frac{\partial v}{\partial x} = v_x + v_y \frac{\partial v}{\partial x} \)) and partial derivatives in the form \( \frac{\partial v}{\partial x}, \frac{\partial v}{\partial t} \) are used when \( y \) is considered to be dependent on \( x \) and \( t \). From (3.5), it is observed that the invariance of (3.2) is preserved under the transformation (3.3) provided the functions \( v(x, t, y), \tau(x, t, y) \) and \( \pi(x, t, y) \) are such that the equation

\[
[D_1D_2\pi_{xxxx} - (D_1 + D_2)\pi_{txx} - (D_1k_2 + D_2k_1)\pi_{xx} + \tau_{tt} + (k_1 + k_2)\tau_{t}]
+ [-D_1D_2\tau_{xxxx} - (D_1 + D_2)(\pi_{xyy} - \pi_{txx}) + (D_1k_2 + D_2k_1)\tau_{xx} + 2\tau_{ty} - \tau_{tt}]
+ (k_1 + k_2)(4v_x - \tau_{t}) \frac{\partial y}{\partial t} + [D_1D_2(4\pi_{xyy} - v_{xxxx}) - (D_1 + D_2)2\pi_{txy} - v_{txx}]
- (D_1k_2 + D_2k_1)(2\pi_{xy} - v_{xx}) - \nu_{tt} - (k_1 + k_2)\nu_t] \frac{\partial y}{\partial x} + (D_1 + D_2)\tau_y \frac{\partial^2 y}{\partial x^2} \frac{\partial^2 y}{\partial t^2}
+ [(D_1 + D_2)\tau_{xyy} + \pi_{yy} - 2\tau_{ty}] \frac{\partial y}{\partial t}^2 + [-4D_1D_2\tau_{xxyy} + 2(D_1k_2 + D_2k_1)\tau_{xy}]
- (D_1 + D_2)(2\pi_{xyy} - v_{xy} - 2\tau_{txy}) - 2\nu_{ty} + 4(k_1 + k_2)\nu_y] \frac{\partial y}{\partial t} \frac{\partial y}{\partial x} - \tau_{yy} \frac{\partial y}{\partial t} \frac{\partial y}{\partial x} \frac{\partial y}{\partial t} \frac{\partial y}{\partial x}
+ [D_1D_2(6\pi_{xyy} - 4v_{xxxx}) - (D_1 + D_2)(\pi_{tyy} - 2\nu_{t}) - 2(D_1k_2 + D_2k_1)\nu_x] \frac{\partial^2 y}{\partial x^2}
+ [-4D_1D_2\tau_{xxx} - (D_1 + D_2)(2\pi_{xy} - 2\tau_{tx} - v_{xx}) + 2(D_1k_2 + D_2k_1)\pi_x - 2\nu_t] \frac{\partial^2 y}{\partial x^2} \frac{\partial y}{\partial t}
+ 2(D_1 + D_2)\tau_x \frac{\partial^3 y}{\partial x^2} \frac{\partial y}{\partial t} + [D_1D_2(6\pi_{xyyy} - 4v_{xxxx}) - (D_1 + D_2)(\pi_{yyyy} - 2\nu_{tt})].
\]
\begin{align*}
&+ \left[-6D_1D_2\tau_{xxyy} - (D_1 + D_2)(\tau_{yyyy} - 2\nu_{xxyy} - \tau_{yy}) + (D_1k_2 + D_2k_1)\tau_{yy}\right] \frac{\partial^2\nu}{\partial t \partial x^2} \\
&+ \left[4(D_1 + D_2)\tau_{xy} - 2\nu_y\right] \frac{\partial^2\nu}{\partial t \partial x^2} \frac{\partial^2\nu}{\partial x \partial t} + \left[2(D_1 + D_2)\tau_{xy} + 4\nu_y\right] \frac{\partial^2\nu}{\partial x^2} \frac{\partial^3\nu}{\partial x \partial t^2} + 2(D_1 + D_2)\tau_y \frac{\partial^2\nu}{\partial t \partial x^2} \\
&+ \left[-6D_1D_2\tau_{xx} + (D_1 + D_2)(\tau_t - 2\nu_x)\right] \frac{\partial^3\nu}{\partial t \partial x^2} \frac{\partial^3\nu}{\partial x \partial t} + \left[-12D_1D_2\tau_{xy} - 2(D_1 + D_2)\nu_y\right] \frac{\partial^3\nu}{\partial x^2} \frac{\partial^3\nu}{\partial x \partial t} \\
&+ \left[-12D_1D_2\tau_{xy} + 3(D_1 + D_2)\nu_y\right] \frac{\partial^2\nu}{\partial x \partial t^2} + 2(D_1 + D_2)\tau_y \frac{\partial^2\nu}{\partial x \partial t^2} - 10D_1D_2\nu_y \frac{\partial^2\nu}{\partial x^2} \frac{\partial^3\nu}{\partial t \partial x^3} \\
&+ (D_1 + D_2)\tau_y \frac{\partial^2\nu}{\partial t \partial x^2} + \left[-12D_1D_2\tau_{xxyy} + 3(D_1 + D_2)\nu_y\right] \frac{\partial^2\nu}{\partial t \partial x^2} \left(\frac{\partial^2\nu}{\partial t \partial x} \right)^2 - 4D_1D_2\tau_{xx} \frac{\partial^4\nu}{\partial t \partial x^3} \\
&+ (D_1 + D_2)\tau_{yy} \frac{\partial^2\nu}{\partial t \partial x^2} + 4(D_1 + D_2)\tau_{yy} \frac{\partial^2\nu}{\partial t \partial x \partial t} + (D_1 + D_2)\tau_{yy} \frac{\partial^2\nu}{\partial t \partial x} \\
&+ \left[D_1D_2(4\nu_{yy} - 6\nu_{xx}) + (D_1 + D_2)\nu_y\right] \frac{\partial^3\nu}{\partial t \partial x^3} + \left[-4D_1D_2\tau_{xy} + (D_1 + D_2)\nu_y\right] \frac{\partial^3\nu}{\partial t \partial x^3} \\
&+ \left[-12D_1D_2\tau_{xxyy} + 3(D_1 + D_2)\nu_y\right] \frac{\partial^2\nu}{\partial t \partial x^2} \frac{\partial^2\nu}{\partial t \partial x} + (D_1 + D_2)\tau_{yyyy} \frac{\partial^2\nu}{\partial t \partial x} \left(\frac{\partial^2\nu}{\partial t \partial x} \right)^2 \\
&+ \left[-4D_1D_2\tau_{xxyy} + (D_1 + D_2)\nu_y\right] \frac{\partial^3\nu}{\partial t \partial x^3} + D_1D_2(4\nu_{yy} - 16\nu_{xy}) \frac{\partial^3\nu}{\partial t \partial x^3} \\
&+ D_1D_2(3\nu_{xx} - 12\nu_{xy})(\frac{\partial^2\nu}{\partial t \partial x})^2 - 6D_1D_2\tau_{xy} \frac{\partial^2\nu}{\partial t \partial x^2} - 12D_1D_2\tau_{xy} \frac{\partial^2\nu}{\partial t \partial x \partial t} - 4D_1D_2\tau_{xy} \frac{\partial^2\nu}{\partial t \partial x^2} \\
&+ D_1D_2(6\nu_{yyyy} - 24\nu_{xxyy}) \frac{\partial^2\nu}{\partial t \partial x^2} - 15D_1D_2\nu_{yy} \frac{\partial^2\nu}{\partial t \partial x^2} \frac{\partial^2\nu}{\partial t \partial x} - 10D_1D_2\nu_{yy} \frac{\partial^3\nu}{\partial t \partial x^3} \frac{\partial^2\nu}{\partial t \partial x^2} \\
&- 4D_1D_2\tau_{yy} \frac{\partial^2\nu}{\partial t \partial x^2} \frac{\partial^3\nu}{\partial t \partial x} - 3D_1D_2\tau_{yy} \frac{\partial^2\nu}{\partial t \partial x^2} - 12D_1D_2\tau_{yy} \frac{\partial^2\nu}{\partial t \partial x \partial t} - D_1D_2\nu_{yyyy} \left(\frac{\partial^2\nu}{\partial t \partial x} \right)^5 \\
&- 6D_1D_2\tau_{yy} \frac{\partial^3\nu}{\partial t \partial x \partial t^2} \left(\frac{\partial^3\nu}{\partial t \partial x} \right)^2 + D_1D_2(\nu_{yyyy} - 4\nu_{xxyy}) \left(\frac{\partial^4\nu}{\partial t \partial x^4} - 10D_1D_2\nu_{yyyy} \frac{\partial^2\nu}{\partial t \partial x^2} \left(\frac{\partial^3\nu}{\partial t \partial x} \right)^3 \\
&- 4D_1D_2\tau_{yy} \frac{\partial^2\nu}{\partial t \partial x^2} \frac{\partial^3\nu}{\partial t \partial x} - D_1D_2\tau_{yyyy} \frac{\partial^3\nu}{\partial t \partial x} \left(\frac{\partial^4\nu}{\partial t \partial x^4} \right) = 0 ;
\end{align*}

is satisfied identically. The constants $D_i, k_i > 0 \; \forall \; i \in \{1, 2\}$ and are to remain unrestricted as far as possible.

Upon equating to zero the coefficients of $\frac{\partial^2\nu}{\partial t \partial x^2}$ and $\frac{\partial^3\nu}{\partial t \partial x^2}$ in (3.6), it is readily deduced that

$$\tau(x, t, y) = \tau(t).$$

(3.7)
The result (3.7) implies that the coefficient of $\frac{\partial^2 y}{\partial x \partial t^2}$ in (3.6) yields $\nu_y = 0$ and therefore,

$$v(x, t, y) = v(x, t).$$  \hspace{1cm} (3.8)

By (3.7) and (3.8), the coefficient of $\frac{\partial^3 y}{\partial t^2 \partial x^2}$ in (3.6) reduces to yield $2\nu_x = \tau'(t)$ and so,

$$v(x, t) = \frac{1}{2} \tau'(t)x + \rho^*(t);$$  \hspace{1cm} (3.9)

where $\rho^*$ denotes an arbitrary function.

In view of (3.7), it follows from the coefficient of $(\frac{\partial y}{\partial t})^2$ in (3.6) that $\pi_{yy} = 0$, implying

$$\pi(x, t, y) = \alpha^*(x, t)y + \beta^*(x, t);$$  \hspace{1cm} (3.10)

where $\alpha^*$ and $\beta^*$ represent further arbitrary functions.

Via the results (3.7), (3.9) and (3.10), the coefficient of $\frac{\partial^2 y}{\partial x \partial t}$ in (3.6) simplifies to give $(D_1 + D_2)\pi_{xy} = -\nu_t$ and therefore,

$$\pi_{xy} = -\frac{1}{(D_1 + D_2)} \left[ \frac{1}{2} \tau''(t)x + \rho^{**}(t) \right].$$  \hspace{1cm} (3.11)

By (3.9), the coefficient of $\frac{\partial^3 y}{\partial x^3}$ in (3.6) gives rise to $4D_1D_2\pi_{xy} = -(D_1 + D_2)v_t$ or

$$\pi_{xy} = -\frac{(D_1 + D_2)}{4D_1D_2} \left[ \frac{1}{2} \tau''(t)x + \rho^{**}(t) \right].$$  \hspace{1cm} (3.12)

Since we require $D_1$ and $D_2$ to remain unrestricted, reconciling (3.11) and (3.12) requires $\tau''(t) = 0$ and $\rho^{**}(t) = 0$, implying that

$$\tau(t) = c_1t + c_2, \quad \rho^*(t) = c_3;$$  \hspace{1cm} (3.13)

where $c_1$, $c_2$, and $c_3$ are all unrestricted constants. It should be noted that the expressions (3.11) and (3.12) can also be reconciled with $D_1 = D_2$, in which case,
the two diffusion paths will be indistinguishable. From results (3.13), inspection of (3.9), (3.11) and (3.12) yields

$$\nu(x, t) = \frac{1}{2} c_1 x + c_3, \quad \pi_{xy} = 0. \quad (3.14)$$

From (3.10) and (3.142), it immediately follows that $\alpha^*(x, t) = \alpha^*(t)$ and hence,

$$\pi(x, t, y) = \alpha^*(t) y + \beta^*(x, t). \quad (3.15)$$

By (3.131), (3.141) and (3.15), it is immediately obvious from the coefficient of $\frac{\partial y}{\partial t}$ in (3.6) that

$$\alpha^*(t) = -\frac{1}{2} (k_1 + k_2) c_1 t + c_4; \quad (3.16)$$

where $c_4$ denotes a fourth arbitrary constant. From (3.15) and (3.16), we find that

$$\pi(x, t, y) = [-\frac{1}{2} (k_1 + k_2) c_1 t + c_4] y + \beta^*(x, t). \quad (3.17)$$

By the results (3.141) and (3.17), the coefficient of $\frac{\partial^2 y}{\partial x^2}$ in (3.6) implies

$$(D_1 - D_2)(k_1 - k_2)c_1 = 0, \quad (3.18)$$

compelling $c_1 = 0$ since $D_1, D_2, k_1$ and $k_2$ are required to remain unrestricted as far as possible. Since $c_1 = 0$, the results (3.131), (3.141) and (3.17) all simplify to give

$$\tau(t) = c_2, \quad \nu(x, t) = c_3, \quad \pi(x, t, y) = c_4 y + \beta^*(x, t). \quad (3.19)$$

From the results (3.19), the term in (3.6) not involving derivatives of $y$ simplifies to give

$$D_1 D_2 \beta_{xxxx}^* - (D_1 + D_2) \beta_{txx}^* - (D_1 k_2 + D_2 k_1) \beta_{xx}^* + \beta_{tt}^* + (k_1 + k_2) \beta_t^* = 0. \quad (3.20)$$

From the results (3.19), it is clear that the remaining coefficients of derivatives of $y$ in equation (3.6) reveal no additional information.
Upon assigning \( d_1 = c_3 \), \( d_2 = c_2 \) and \( d_3 = c_4 \), it may be concluded from the results obtained thus far that the classical group derived for the equation (3.2) under the transformation (3.3) is given by

\[
\begin{align*}
\nu(x, t, y) &= d_1, \\
\tau(x, t, y) &= d_2, \\
\pi(x, t, y) &= d_3 y + \beta^*(x, t);
\end{align*}
\]  

(3.21)

such that

\[
D_1D_2 \frac{\partial^4 \beta^*}{\partial x^4} - (D_1 + D_2) \frac{\partial^3 \beta^*}{\partial t \partial x^2} - (D_1k_2 + D_2k_1) \frac{\partial^2 \beta^*}{\partial x^2} + \frac{\partial^2 \beta^*}{\partial t^2} + (k_1 + k_2) \frac{\partial \beta^*}{\partial t} = 0.
\]  

(3.22)

It should be noted that (3.21) and (3.22) are the central results in this chapter. It is observed that the equation (3.22) is equivalent to (3.2) with \( \beta^* \) replacing \( y \). The function \( \beta^*(x, t) \) is a solution to (3.2) and thus is also a solution to the uncoupled version of system (1.1). Each solution to (3.22) generates a classical group for (3.2) hence possibly enabling an infinite variety of similarity solutions to the equation (3.2) to be recovered. We shall now derive some similarity solutions to (3.2) associated with the classical group (3.21). Supposing \( d_1 = d_2 = d_3 = 1 \), the classical group (3.21) yields

\[
\begin{align*}
\nu(x, t, y) &= 1, \\
\tau(x, t, y) &= 1, \\
\pi(x, t, y) &= \beta^*(x, t) + y.
\end{align*}
\]  

(3.23)

From equation (3.4) and the group (3.23), we obtain the equation

\[
\frac{\partial \nu}{\partial x} + \frac{\partial \nu}{\partial t} = \beta^*(x, t) + y.
\]  

(3.24)

A similarity solution to (3.24) may be recovered by Lagrange's method (see [8]). Commencing the use of this technique, (3.24) may be rewritten as

\[
P(x, t, y) \frac{\partial \nu}{\partial x} + Q(x, t, y) \frac{\partial \nu}{\partial t} = R(x, t, y);
\]  

(3.25)

where \( P(x, t, y) = Q(x, t, y) = 1 \) and \( R(x, t, y) = \beta^*(x, t) + y \).
The subsidiary equations associated with (3.25) are then given by

\[
\frac{dx}{l} = \frac{dt}{1} = \frac{dy}{\beta^*(x, t) + y} .
\]  

(3.26)

As \( \frac{dx}{l} = \frac{dt}{1} \) in (3.26), integration gives

\[
x = t + e_1 ;
\]

(3.27)

where \( e_1 \) is an arbitrary constant. Hence, the first integral of (3.26) is

\[
r(x, t, y) = x - t = e_1 .
\]

(3.28)

From (3.28), it is evident that \( \frac{dt}{1} = \frac{dy}{\beta^*(x, t) + y} \) is equivalent to

\[
\frac{dy}{dt} - y = \beta^*(t + e_1 , t) .
\]

(3.29)

By the integrating factor method, (3.29) may be written as

\[
\frac{d}{dt} [e^t y] = e^t \beta^*(t + e_1 , t) ;
\]

(3.30)

integrating which gives

\[
y = e^t \int_c^{e_1} e^{-\omega} \beta^*(\omega + e_1 , \omega) d\omega ;
\]

(3.31)

where \( c \) is an arbitrary constant. Hence from (3.31),

\[
y = e^t [K^*(t, e_1) + e_2] ;
\]

(3.32)

where \( e_2 \) represents a further arbitrary constant and \( K^*(t, e_1) \) is defined as

\[
K^*(t, e_1) = \int_c^{e_1} e^{-\omega} \beta^*(\omega + e_1 , \omega) d\omega - e_2 .
\]

(3.33)

From (3.28) and (3.32), the second integral of (3.26) is given by

\[
s(x, t, y) = ye^{-t} - K^*(t, x - t) = e_2 .
\]

(3.34)
As the Jacobians \( \frac{\partial(r,s)}{\partial(x,t)} \) and \( \frac{\partial(r,s)}{\partial(x,y)} \) are all non-zero when \( K_t^* \neq -ye^{-t} \), hence in any region of space in which \( K_t^* \neq -ye^{-t} \), the general solution of (3.24) is given by \( F(r,s) = 0 \), or equivalently by (3.28) and (3.34),

\[
F(x-t, ye^{-t} - K^*(t, x-t)) = 0; \quad (3.35)
\]

where \( F \) is arbitrary. The result (3.35) may be alternatively expressed as

\[
y(x, t) = e^t K^*(t, x-t) + e^t f(x-t); \quad (3.36)
\]

where \( K^*(t, x-t) = \int_c e^{-\omega} \beta^*(\omega + x-t, \omega)d\omega \), (the right-hand side of this expression for \( K^*(t, x-t) \) includes \( e_2 \)) and \( f \) is arbitrary.

Substituting the general solution (3.36) back into (3.24) confirms its validity. To summarise, in any region of space in which \( (\int_c e^{-\omega} \beta^*(\omega + x-t, \omega)d\omega)_t \neq -ye^{-t} \), the general solution to (3.24) is given by

\[
y(x, t) = e^t f(x-t) + e^t \int_c e^{-\omega} \beta^*(\omega + x-t, \omega)d\omega; \quad (3.37)
\]

where the function \( f(x-t) \) may be determined by substituting it into (3.2) to obtain the defining equation for \( f(x-t) \). The result (3.37) is identical to that obtained in Chapter 2 under the same conditions, with \( y \) replacing \( u \) and \( v \) while \( \beta^* \) replaces \( \sigma \) and \( \beta \) in (2.52). Subject to the conditions \( d_1 = d_2 = d_3 = 1 \), a similarity solution for (3.2) using particular solutions for \( \beta^*(x, t) \) will now be derived.

**Example:**

One of the simplest solutions of (3.2) is considered here, namely

\[
\beta^*(x, t) = \frac{1}{(k_1 + k_2)} \exp[-(k_1 + k_2)t]. \quad (3.38)
\]

By making use of (3.38), the group (3.23) yields
\[ v(x, t, y) = 1, \]
\[ \tau(x, t, y) = 1, \]
\[ \pi(x, t, y) = \frac{1}{(k_1 + k_2)} \exp[-(k_1 + k_2)t] + y. \]  

(3.39)

The group (3.39) yields a solution of the form (3.37) namely,

\[ y(x, t) = e^t f(x - t) - \frac{1}{(k_1 + k_2)(k_1 + k_2 + 1)} \exp[-(k_1 + k_2)t] + d_4 e^t; \]  

(3.40)

where \( f \) is arbitrary and \( d_4 \) denotes an arbitrary constant. In order to determine \( f \), we let \( w = x - t \) before substituting the solution (3.40) into (3.2) to yield

\[ D_1 D_2 f'''(w) + (D_1 + D_2) f''(w) + \{1 - (D_1 + D_2) - (D_1 k_2 + D_2 k_1)\} f''(w) \\
- (k_1 + k_2 + 2)f'(w) + (k_1 + k_2 + 1)f(w) + (k_1 + k_2 + 1)d_4 = 0. \]  

(3.41)

A special solution of (3.41) may be obtained by setting \( f(w) = ke^w \) where \( k \) is an arbitrary constant. From (3.41) therefore, we find that

\[ [D_1 D_2 - (D_1 k_2 + D_2 k_1)]ke^w + (k_1 + k_2 + 1)d_4 = 0; \]  

(3.42)

which requires

\[ [D_1 D_2 - (D_1 k_2 + D_2 k_1)]k = 0, \quad (k_1 + k_2 + 1)d_4 = 0. \]  

(3.43)

It is implied from (3.43) that either \( D_1 D_2 = D_1 k_2 + D_2 k_1 \) or \( k = 0 \). In order for a non-trivial solution for \( f(w) \) to result, we select \( D_1 D_2 = D_1 k_2 + D_2 k_1 \). As \( k_1 \) and \( k_2 \) are strictly positive constants, (3.42) compels \( d_4 = 0 \). Hence, substituting \( f(w) = ke^w \) and \( d_4 = 0 \) into (3.40) yields the solution

\[ y(x, t) = ke^x - \frac{1}{(k_1 + k_2)(k_1 + k_2 + 1)} \exp[-(k_1 + k_2)t]; \]  

(3.44)

subject to the condition \( D_1 D_2 = D_1 k_2 + D_2 k_1 \).

Substituting the solution (3.44) together with the condition \( D_1 D_2 = D_1 k_2 + D_2 k_1 \) into equation (3.2) confirms the validity of (3.44) subject to this stated condition. As expected, the result (3.44) subject to the stated condition is a direct reflection of the results in Chapter 2.
3.3: The Non-Classical Procedure.

For the one-parameter group of transformations described in (3.3), the following terms \( A(x, t, y) \) and \( B(x, t, y) \) are introduced and defined by

\[
A(x, t, y) = \frac{\pi(x, t, y)}{\tau(x, t, y)}, \quad B(x, t, y) = \frac{\nu(x, t, y)}{\tau(x, t, y)},
\]

so that (3.4) becomes

\[
\frac{\partial y}{\partial t} = A - B \frac{\partial y}{\partial x}.
\]

Upon differentiating (3.46) partially with respect to \( x \), it is deduced that

\[
\frac{\partial^2 y}{\partial x \partial t} = A_x + (A_x - B_x) \frac{\partial y}{\partial x} - B \frac{\partial^2 y}{\partial x^2} - B_y \left( \frac{\partial y}{\partial x} \right)^2.
\]

By differentiating (3.46) partially with respect to \( t \) and using (3.46) and (3.47) to eliminate \( \frac{\partial y}{\partial t} \) and \( \frac{\partial^2 y}{\partial x \partial t} \) respectively, we obtain

\[
\frac{\partial^2 y}{\partial t^2} = A_t + AA_y - BA_x + (BB_x - 2BA_y - B_t - AB_y) \frac{\partial y}{\partial x} + B^2 \frac{\partial^2 y}{\partial x^2} + 2BB_y \left( \frac{\partial y}{\partial x} \right)^2.
\]

Differentiating (3.47) partially with respect to \( x \) yields

\[
\frac{\partial^3 y}{\partial t \partial x^2} = A_{xx} + (2A_{xy} - B_{xx}) \frac{\partial y}{\partial x} + (A_y - 2B_x) \frac{\partial^2 y}{\partial x^2} - B \frac{\partial^3 y}{\partial x^3} - 3B_y \frac{\partial y}{\partial x} \frac{\partial^2 y}{\partial x^2}
\]

\[
+ (A_{yy} - 2B_{xy}) \left( \frac{\partial y}{\partial x} \right)^2 - B_{yy} \left( \frac{\partial y}{\partial x} \right)^3.
\]

By differentiating (3.48) partially with respect to \( x \), we find that

\[
\frac{\partial^3 y}{\partial x \partial t^2} = A_{tx} + AA_{xy} + A_x A_y - BA_{xx} - A_x B_x + (3BB_x - 2BA_y - B_t - AB_y) \frac{\partial^2 y}{\partial x^2}
\]

\[
+ 6BB_y \frac{\partial y}{\partial x} \frac{\partial^2 y}{\partial x^2} + (BB_{xx} + B_x^2 - 3BA_{xy} - 2B_x A_y - B_{tx} - AB_{xy} - 2A_x B_y
\]

\[
+ A_{ty} + AA_{yy} + A_y^2) \frac{\partial y}{\partial x} + B^2 \frac{\partial^3 y}{\partial x^3} + (3BB_{xy} + 3B_x B_y - 2BA_{yy} - 3A_y B_y
\]

\[
- B_{ty} - AB_{yy}) \left( \frac{\partial y}{\partial x} \right)^2 + 2(BB_{yy} + B_y^2) \left( \frac{\partial y}{\partial x} \right)^3.
\]
Upon differentiating (3.49) partially with respect to x and using (3.2) to eliminate \( \frac{\partial^4 y}{\partial x^4} \), we deduce that

\[
\frac{\partial^4 y}{\partial x^3} = A_{xxx} - \frac{B}{D_1D_2} \left[ (D_1 + D_2)A_{xx} - A_t - (k_1 + k_2)A - AA_y + BA_x \right] \\
+ [A_y - 3B_x + \frac{(D_1 + D_2)}{D_1D_2} B^2] \frac{\partial^3 y}{\partial x^3} - 4B_y \frac{\partial^2 y}{\partial x^2} - 3B_y \frac{\partial^2 y}{\partial x^2}^2 - B_{yyy} \frac{\partial y}{\partial x}^4 \\
+ [3A_{xyy} - B_{xxx} - \frac{B}{D_1D_2} \{(D_1 + D_2)(2A_{xy} - B_{xx}) - BB_x + 2BA_y + B_t} \\
+ AB_y + (k_1 + k_2)B] \frac{\partial y}{\partial x} + [3A_{xy} - 3B_{xx} - \frac{B}{D_1D_2} \{(D_1 + D_2)(A_y - 2B_x) \\
+ D_1k_2 + D_2k_1 - B^2 \frac{\partial^2 y}{\partial x^2} + [3 \frac{(D_1 + D_2)}{D_1D_2} BB_y - 9B_{xy} + 3A_{yy}] \frac{\partial y}{\partial x} \frac{\partial^2 y}{\partial x}^2 \\
+ [- \frac{(D_1 + D_2)}{D_1D_2} B(A_{yy} - 2B_{xy}) + \frac{2}{D_1D_2} B^2B_y + 3A_{xyy} - 3B_{xyy}] \frac{\partial y}{\partial x}^2 \\
+ \frac{(D_1 + D_2)}{D_1D_2} BB_{yy} - 3B_{xyy} + A_{yy}] \frac{\partial y}{\partial x}^3 - 6B_{yy} \frac{\partial^2 y}{\partial x^2} \frac{\partial y}{\partial x}^2 . \quad (3.51)
\]

As before, the constants \( D_i, k_i > 0 \) \( \forall \ i \in \{1, 2\} \) and are to remain unrestricted as far as possible. In order for non-trivial transformations to result, it is required that \( \tau(x, t, y) \neq 0 \). Then (3.46) to (3.51) are substituted into (3.6). Subsequently, equating to zero the coefficients of \( \frac{\partial y}{\partial x} \), \( \frac{\partial^2 y}{\partial x^2} \), \( \frac{\partial^3 y}{\partial x^3} \), \( \frac{\partial^4 y}{\partial x^4} \), \( \frac{\partial^5 y}{\partial x^5} \), and the remaining terms yields the following simplified equations, namely
\[
D_1D_2A_{xxxx} + (D_1 + D_2)(A_xB_{xx} - 2A_xB_x - 2A_xA_{xy} - A_{txx} - AA_{xy}) + (D_1k_2 + D_2k_1)A_{xx} + (k_1 + k_2)(A_t + 4AB_x) + A_{tt} + 2AA_{ty} + A^2A_{yy} - 2A_xB_t + 4(A_t + AA_y - BA_x)B_x - 2AA_xB_y = 0,
\]
\[
D_1D_2(4A_{axxy} - B_{xxx}) + (k_1 + k_2)(4AB_y - 4BB_x - B_t) + (D_1k_2 + D_2k_1)(B_{xx} - 2A_{xy}) + (D_1 + D_2)(BA_{xy} + B_xB_{xx} + B_{txx} - 2A_{xy}B_x - 2A_{xx}B_y - 2A_{txy} - 2A_xA_{yy} + AB_{xx}) - 4A_xB_y + 4A_xB_{xy} + A_yB_{xx} - 2A_{xxy}) - 2BA_{ty} - B_{tt} - 2ABA_{yy} - 2AB_{ty} - A^2B_{yy} - 2A_xB_t + 4(BB_x - 8BA_y - 2B_t - 2AB_y)B_x + (4A_t + 2A_{ty} - 2BA_x)B_y = 0,
\]
\[
D_1D_2(6A_{axxy} - 4B_{xxx}) + (D_1k_2 + D_2k_1)(2B_{xy} - A_{yy}) - 4(k_1 + k_2)BB_y + (D_1 + D_2)(2B_{txy} - BB_{xx} + 2BA_{xy} + B_{xx}B_y - A_{tyy} - 2B_yA_{xy} + 3A_xB_{yy} + 2AB_{xxy} + 4A_xB_{xy} - 2A_yA_{yy} - AA_{yyy}) + B^2A_{yy} + 2BB_{ty} + 2ABB_{yy} + 10BB_x - 2B_t - 2AB_y - 6BA_y)B_y = 0,
\]
\[
D_1D_2(6A_{axxy} - 4B_{xx}) - 2(D_1k_2 + D_2k_1)B_x + 2BB_t + 4B^2B_x + 2ABB_y + (D_1 + D_2)[B_{tx} - 3A_xB_{xy} - A_{ty} - AA_{yy} + 2AB_{xy} - BB_{xx} + 2BA_{xy} + (4B - 2A_y)B_x] = 0,
\]
\[
D_1D_2(12A_{axy} - 18B_{xy}) + (D_1 + D_2)(3AB_{yy} + 3B_{ty} - 6BB_{xy} + 3BA_{yyy} + 7B_xB_y + A_yB_y) - 2(D_1k_2 + D_2k_1)B_y + 2B^2B_y = 0,
\]
\[
D_1D_2(A_{yy} - 4B_{xy}) - (D_1 + D_2)BB_y = 0 ,
\]
\[
D_1D_2(2A_{xyy} - 8B_{xxy}) + (D_1 + D_2)(B_y^2 - 2BB_{yy}) = 0 ,
\]
\[
D_1D_2(4A_{xy} - 6B_{xx}) + (D_1 + D_2)(B_t + 2BB_x + AB_y) = 0 ,
\]
\[
D_1D_2(4A_{xxy} - 6B_{xxx}) + (D_1 + D_2)[AB_{xyy} + BA_{yy} - 2BB_{xxy} + B_{tty} + (3A_y - B_x)B_{yy}] = 0 ,
\]
\[
D_1D_2(4A_{yy} - 16B_{xxy}) + (D_1 + D_2)BB_y = 0 ,
\]
\[
D_1D_2(A_{xxy} - 4B_{xyy}) - (D_1 + D_2)(BB_{xyy} + B_xB_{yy}) = 0 ,
\]
\[
B_{xyy} = 0 ,
\]
\[
B_{yy} = 0 ,
\]
\[
B_y = 0 ,
\]
\[
B_{yyyy} = 0 .
\]

The system of equations (3.52) determines the non-classical group of (3.2).

From (3.52) , it is evident that

\[
B(x , t , y) = B(x , t) .
\]

By (3.53) , inspection of (3.52) reveals that \( A_{yy} = 0 \) , implying

\[
A(x , t , y) = a(x , t)y + b(x , t) ;
\]
where a and b are arbitrary functions. The results (3.53) and (3.54) reduce the equations (3.52) to yield the following simplified system, namely

\[
D_1D_2(a_{xxxx}y + b_{xxxx}) - (D_1k_2 + D_2k_1)(a_{xx}y + b_{xx}) + (k_1 + k_2)[a_t y + b_t + 4(ay + b)B_x] \\
+ (D_1 + D_2)[(a_{xy}y + b_{xy})B_{xx} - 2(a_{xy}y + b_{xy})B_x - 2(a_{xy}y + b_{xy})a_x - (a_{xxx}y + b_{xxx}) - (ay + b)a_{xx}] \\
+ a_{tt} y + b_{tt} + 2(ay + b)a_t - 2(a_{xy}y + b_{xy})B_t + 4[a_t y + b_t + a(ay + b) - (a_{xy}y + b_{xy})]B_x = 0 ,
\]

\[
D_1D_2(4a_{xxx} - B_{xxx}) - (k_1 + k_2)(4BB_x + B_t) + (D_1k_2 + D_2k_1)(B_{xx} - 2a_x) \\
+ (4BB_x - 8aB - 2B_t)B_x + (D_1 + D_2)(Ba_{xx} + B_xB_{xx} + B_{xxx} - 2a_xB_x - 2a_t - 2a_{xx} + ab_{xx}) \\
- 2Ba_t - B_{tt} - 2aB_t = 0 ,
\]

\[
D_1D_2(6a_{xx} - 4B_{xx}) + (D_1 + D_2)[2B_{tx} - a_t - BB_{xx} + 2B_a + (4B_x - 2a)B_x] \\
- 2(D_1k_2 + D_2k_1)B_x + 2BB_t + 4B^2B_x = 0 ,
\]

\[
D_1D_2(4a_x - 6B_{xx}) + (D_1 + D_2)(B_t + 2BB_x) = 0 .
\]

In conjunction with the results (3.53) and (3.54), solution of the system of equations (3.55) gives rise to the non-classical group of (3.2) under the transformation (3.3). It is clear that the classical group (3.21) is contained within the system (3.55) and as this system is a complicated one to solve, the following assumption is made, thus enabling us to proceed.

By making the assumption that \( B(x, t) = B(t) \), equation (3.554) immediately gives

\[
a(x, t) = \theta(t) - \frac{1}{4} \frac{(D_1 + D_2)}{D_1D_2} B'(t)x ,
\]

where \( \theta \) denotes an arbitrary function. From (3.54) and (3.56), it follows that

\[
A(x, t, y) = \theta(t)y - \frac{1}{4} \frac{(D_1 + D_2)}{D_1D_2} B'(t)xy + b(x, t) .
\]

Upon making use of (3.56) and the assumption \( B(x, t) = B(t) \), it is observed from (3.553) that

\[
(D_1 + D_2)[- \theta'(t) + \frac{1}{4} \frac{(D_1 + D_2)}{D_1D_2} B''(t)x - \frac{1}{2} \frac{(D_1 + D_2)}{D_1D_2} BB'(t)] + 2BB'(t) = 0 ;
\]

\[
(3.58)
\]
and therefore,

\[ \theta(t) = \frac{1}{4} \frac{(D_1 + D_2)}{D_1 D_2} B'(t)x - \frac{(D_1 - D_2)^2}{4D_1 D_2 (D_1 + D_2)} B^2(t) + m_1 ; \]  

(3.59)

where \( m_1 \) is an arbitrary constant. However, as the right-hand side of (3.59) must be solely a function of \( t \), it is required that

\[ B(t) = m_2 ; \]  

(3.60)

where \( m_2 \) is an unrestricted constant. It follows immediately from (3.56), (3.59) and (3.60) that

\[ a(x, t) = \theta(t) = m_1 - \frac{(D_1 - D_2)^2}{4D_1 D_2 (D_1 + D_2)} m_2^2 . \]  

(3.61)

The results (3.60) and (3.61) satisfy (3.55) identically while simplifying (3.55) to give

\[ D_1 D_2 b_{xxx} - (D_1 + D_2) b_{txx} - (D_1 k_2 + D_2 k_1) b_{xx} + b_{tt} + (k_1 + k_2) b_t = 0 . \]  

(3.62)

We may conclude from (3.57) and the other results obtained in this section that under the assumption that \( B(x, t) = B(t) \), the non-classical group for (3.2) associated with the transformation (3.3) is given by

\[ A(x, t, y) = [m_1 - \frac{(D_1 - D_2)^2}{4D_1 D_2 (D_1 + D_2)} m_2^2] y + b(x, t) , \]  

(3.63)

\[ B(x, t, y) = m_2 ; \]

such that

\[ D_1 D_2 \frac{\partial^4 b}{\partial x^4} - (D_1 + D_2) \frac{\partial^3 b}{\partial t \partial x^2} - (D_1 k_2 + D_2 k_1) \frac{\partial^2 b}{\partial x^2} + \frac{\partial^2 b}{\partial t^2} + (k_1 + k_2) \frac{\partial b}{\partial t} = 0 . \]  

(3.64)

It is observed that (3.64) is identical to (3.22) and (3.2) with \( b \) replacing \( \beta^* \) and \( y \) respectively. Consequently, \( b(x, t) \) is a solution to (3.2) and therefore to the uncoupled version of system (1.1). Each solution to (3.64) gives rise to a non-classical group for (3.2) thereby possibly enabling an unending variety of similarity
solutions of (3.2) to be derived. It is interesting to note that making the simplification $B(x, t) = B(t)$ yields the classical group (3.21). This fact is evident from inspection of (3.21), (3.45) and (3.63). Similarity solutions corresponding to the group (3.63) are then identical to those associated with the group (3.21). Similarity solutions associated with the group (3.21) were obtained in section (3.2) and proven valid for the equation (3.2) thus implying their validity for the uncoupled version of system (1.1). The possibility exists that the non-classical procedure gives rise to no new similarity solutions which are not recoverable from the classical method. It is hoped that this last conjecture stimulates further research in this field.
CHAPTER 4: SOLUTIONS TO THE COUPLED SYSTEM OF REACTION-DIFFUSION EQUATIONS GOVERNING THE BURNING MODEL.

4.1: Introduction.

In this chapter, one-parameter transformation groups preserving the invariance of the coupled system (1.11) will be determined in order to obtain similarity solutions of (1.11). We therefore consider a general transformation of the form

\[
\begin{align*}
    x_1 &= x_1(x, t, C, T, \varepsilon) = x + \varepsilon \xi(x, t, C, T) + O(\varepsilon^2), \\
    t_1 &= t_1(x, t, C, T, \varepsilon) = t + \varepsilon \eta(x, t, C, T) + O(\varepsilon^2), \\
    C_1 &= C_1(x, t, C, T, \varepsilon) = C + \varepsilon \zeta(x, t, C, T) + O(\varepsilon^2), \\
    T_1 &= T_1(x, t, C, T, \varepsilon) = T + \varepsilon \chi(x, t, C, T) + O(\varepsilon^2).
\end{align*}
\]

(4.1)

If the invariance of the system (1.11) is preserved by the transformation (4.1) and if \(C = \phi(x, t), T = \psi(x, t)\); then from \(C_1 = \phi(x_1, t_1), T_1 = \psi(x_1, t_1)\), equating terms of order \(\varepsilon\) gives

\[
\begin{align*}
    \xi(x, t, C, T) \frac{\partial C}{\partial x} + \eta(x, t, C, T) \frac{\partial C}{\partial t} &= \zeta(x, t, C, T), \\
    \xi(x, t, C, T) \frac{\partial T}{\partial x} + \eta(x, t, C, T) \frac{\partial T}{\partial t} &= \chi(x, t, C, T).
\end{align*}
\]

(4.2)

The similarity variables \(C\) and \(T\) are derived from the solutions of (4.2) which correspond to the functional forms of the similarity solutions of (1.11). The groups keeping (1.11) invariant will be determined by means of the classical and non-classical procedures.
4.2: The Classical Procedure.

From results in Appendix III of this thesis and eliminating $\frac{\partial^2 C}{\partial x^2}$ and $\frac{\partial^2 T}{\partial x^2}$ by making use of (1.11), it becomes evident that

\[
\frac{\partial C}{\partial t} - \sigma \frac{\partial^2 C}{\partial x^2} - \mu + C_1 \exp[-1/T_1] = \frac{\partial C}{\partial t} - \sigma \frac{\partial^2 C}{\partial x^2} - \mu + C e^{-1/T}
\]

\[
+ \varepsilon \{ \zeta_t - \sigma \zeta_{xx} + (\zeta_C - 2 \zeta_x) \mu + \frac{\beta \sigma}{\alpha} (T - \theta_a) \zeta_t + \left[ \zeta + \left( \frac{C}{\alpha} \zeta_t + 2 \zeta_x - \zeta_C \right) C + \frac{C}{T^2} e^{-1/T} \right] \frac{\partial C}{\partial t} + \left( \frac{\sigma}{\alpha} - 1 \right) \eta_t \frac{\partial C}{\partial t} \}
\]

\[
+ \left( \frac{\sigma}{\alpha} - 1 \right) \zeta_T \frac{\partial C}{\partial \alpha} + \left[ \sigma \zeta_{xx} - 2 \sigma \zeta_x - \xi_t + \frac{\beta \sigma}{\alpha} (T - \theta_a) \zeta_T - 3 \mu \zeta_C + \left( 3 \xi_C - \sigma \xi_T \right) C e^{-1/T} \right] \frac{\partial C}{\partial \alpha}
\]

\[
+ 2(\xi_T + \sigma \eta_{xt}) \frac{\partial C}{\partial \alpha} + 2(\xi_T C e^{-1/T} - \mu \xi_T - \sigma \xi_T) \frac{\partial T}{\partial \alpha} + (1 - \frac{\sigma}{\alpha}) \frac{\partial \xi_T}{\partial \alpha} + 2 \sigma \eta_C \frac{\partial C}{\partial \alpha}
\]

\[
+ \sigma \eta_C \frac{\partial C}{\partial \alpha} + \sigma (2 \xi_T - \xi_C) \frac{\partial C}{\partial \alpha} + 2 \sigma \eta_T \frac{\partial C}{\partial \alpha} + 2 \sigma \eta_C \frac{\partial C}{\partial \alpha}
\]

\[
+ \sigma \eta_T \frac{\partial C}{\partial \alpha} + \sigma \eta_C \frac{\partial C}{\partial \alpha} + \sigma \eta_T \frac{\partial C}{\partial \alpha} + 2 \sigma \xi_C \frac{\partial C}{\partial \alpha} + 2 \sigma \xi_T \frac{\partial C}{\partial \alpha} + 2 \sigma \xi_C \frac{\partial C}{\partial \alpha}
\]

\[
+ \frac{\partial T_1}{\partial \alpha} - \alpha \frac{\partial^2 T_1}{\partial x^2} + \beta (T_1 - \theta_a) - C_1 \exp[-1/T_1] = \frac{\partial T}{\partial t} - \alpha \frac{\partial^2 T}{\partial x^2} + \beta (T - \theta_a) - C e^{-1/T}
\]

\[
+ \varepsilon \{ \chi_t - \alpha \chi_{xx} + \beta \chi + \frac{\alpha \mu}{\sigma} \chi_c + \beta (T - \theta_a) (2 \xi_x - \chi_T) + \left[ \chi_T - 2 \xi_x - \frac{\alpha}{\sigma} \chi_c \right] C - \chi - \chi C e^{-1/T} \}
\]

\[
+ (1 - \frac{\alpha}{\sigma}) \chi_c \frac{\partial C}{\partial t} + \left[ 2 \xi_x - \eta_t + \alpha \eta_\alpha - \frac{\alpha \mu}{\sigma} \eta_c + \beta (T - \theta_a) \eta_t + \left( \frac{\alpha}{\sigma} \eta_c + \frac{\alpha}{\sigma} \eta_T \right) C e^{-1/T} \right] \frac{\partial T}{\partial t}
\]

\[
+ \left( \frac{\alpha}{\sigma} - 1 \right) \eta_c \frac{\partial C}{\partial \alpha} + \left[ \sigma \zeta_{xx} - 2 \sigma \zeta_x - \xi_t - \frac{\alpha \mu}{\sigma} \zeta_C + 3 \beta (T - \theta_a) \zeta_T + \left( \frac{\alpha}{\sigma} \xi_C - 3 \xi_T \right) C e^{-1/T} \right] \frac{\partial T}{\partial \alpha}
\]

\[
+ \left( \frac{\alpha}{\sigma} - 1 \right) \zeta_C \frac{\partial C}{\partial \alpha} + \left[ \alpha \zeta_{xx} - \alpha \zeta_x - \xi_t - \frac{\alpha \mu}{\sigma} \zeta_C + 3 \beta (T - \theta_a) \zeta_T + \left( \frac{\alpha}{\sigma} \xi_C - 3 \xi_T \right) C e^{-1/T} \right] \frac{\partial T}{\partial \alpha}
\]

\[
- \alpha \chi_{xt} \frac{\partial C}{\partial \alpha} + 2(\xi_{xt} - \chi_{xt}) \frac{\partial C}{\partial \alpha} + 2 \alpha \eta_x \frac{\partial C}{\partial \alpha} + 2 \alpha \eta_{xt} \frac{\partial C}{\partial \alpha} + 2 \alpha (\xi_{xt} - \chi_{xt}) \frac{\partial C}{\partial \alpha}
\]

\[
+ \alpha (2 \xi_{xt} - \chi_{xt}) \frac{\partial C}{\partial \alpha} + 2 \alpha \eta_T \frac{\partial C}{\partial \alpha} + 2 \alpha \chi_{xx} \frac{\partial C}{\partial \alpha} + 2 \alpha \zeta_{xx} \frac{\partial C}{\partial \alpha} + 2 \alpha \zeta_{xt} \frac{\partial C}{\partial \alpha} + 2 \alpha \zeta_x \frac{\partial C}{\partial \alpha}
\]

\[
+ \alpha \eta_{ct} \frac{\partial C}{\partial \alpha} + 2 \alpha \eta_{xt} \frac{\partial C}{\partial \alpha} + \alpha \xi_{xt} \frac{\partial C}{\partial \alpha} + \alpha \eta_{tt} \frac{\partial C}{\partial \alpha} + \alpha \eta_{tt} \frac{\partial C}{\partial \alpha} + \alpha \eta_{tt} \frac{\partial C}{\partial \alpha} + O(e^2),
\]

(4.3)
where the subscripts denote partial differentiation with \( x \), \( t \), \( C \) and \( T \) as independent variables (for example, \( \frac{\partial \xi}{\partial x} = \xi_x + \xi_c \frac{\partial C}{\partial x} + \xi_T \frac{\partial T}{\partial x} \)); and partial derivatives in the form \( \frac{\partial C}{\partial x} \cdot \frac{\partial T}{\partial t} \) are used when \( C \) and \( T \) are considered to be dependent on \( x \) and \( t \). From (4.3), it is observed that the system (1.11) remains invariant under the transformation (4.1) provided the functions \( \xi(x, t, C, T), \eta(x, t, C, T), \zeta(x, t, C, T) \) and \( \chi(x, t, C, T) \) are such that the equations

\[
\xi_t - \sigma \xi_{xx} + \left( \frac{\xi_c - 2 \xi_x}{\alpha} \right) \mu - \frac{\beta \sigma}{\alpha} (T - \theta_a) \xi_T + \left[ \xi + \left( \frac{\sigma}{\alpha} \zeta_T + 2 \xi_x - \xi_c \right) C + \chi \frac{C}{T^2} \right] e^{-1/T}
+ \left[ 2 \xi_x - \eta_t + \sigma \eta_{xx} - \mu \eta_c + (\eta_c - \sigma \eta_T) C e^{-1/T} + \frac{\beta \sigma}{\alpha} (T - \theta_a) \eta_T \right] \frac{\partial C}{\partial t} + (\sigma - 1) \eta_T \frac{\partial C}{\partial t} \frac{\partial T}{\partial t}
+ \left( \frac{\sigma}{\alpha} - 1 \right) \xi_T \frac{\partial C}{\partial x} + \left[ \sigma \xi_{xx} - 2 \sigma \xi_{xc} - \xi_t + \frac{\beta \sigma}{\alpha} (T - \theta_a) \xi_T - 3 \mu \xi_c + (3 \xi_c - \sigma \xi_T) C e^{-1/T} \right] \frac{\partial C}{\partial x}
+ 2 (\xi_T + \sigma \xi_{xt}) \frac{\partial C}{\partial t} \frac{\partial T}{\partial x} + 2 (\xi_c + \sigma \xi_{xc}) \frac{\partial C}{\partial t} \frac{\partial T}{\partial x} + 2 (\xi_T - \xi_c) \frac{\partial T}{\partial x} + (1 - \frac{\sigma}{\alpha}) \xi_T \frac{\partial T}{\partial x} + 2 \eta_c \xi_T \frac{\partial C}{\partial x}
+ 2 (\xi_T + \sigma \xi_{xt}) \frac{\partial C}{\partial t} \frac{\partial T}{\partial x} + 2 (\xi_c + \sigma \xi_{xc}) \frac{\partial C}{\partial t} \frac{\partial T}{\partial x} + 2 \eta_c \frac{\partial C}{\partial t} \frac{\partial T}{\partial x} + 2 \sigma \xi_{ct} \frac{\partial C}{\partial x} \frac{\partial T}{\partial x} \frac{\partial C}{\partial x} \frac{\partial T}{\partial x} + 2 \sigma \xi_{cc} \frac{\partial C}{\partial x} \frac{\partial T}{\partial x} - \sigma \xi_{ct} \frac{\partial C}{\partial x} \frac{\partial T}{\partial x} \frac{\partial C}{\partial x} \frac{\partial T}{\partial x} = 0,
\]

(4.4)

\[
\chi_t - \alpha \chi_{xx} + \beta \chi + \frac{\alpha \mu}{\sigma} \chi_c + \beta (T - \theta_a) (2 \xi_x - \chi_t) + [(\chi_T - 2 \xi_x - \frac{\alpha}{\sigma} \chi_c) C - \chi \frac{C}{T^2}] e^{-1/T}
+ (1 - \frac{\alpha}{\sigma}) \chi_c \frac{\partial C}{\partial t} + \left[ 2 \xi_x - \eta_t + \alpha \eta_{xx} - \frac{\alpha \mu}{\sigma} \eta_c + \beta (T - \theta_a) \eta_T + \left( \frac{\alpha}{\sigma} \eta_c - \eta_T \right) C e^{-1/T} \right] \frac{\partial T}{\partial t}
+ \left( \frac{\alpha}{\sigma} - 1 \right) \chi_T \frac{\partial T}{\partial x} + \left[ \alpha \chi_{xx} - 2 \alpha \chi_{xt} - \chi_t - \frac{\alpha \mu}{\sigma} \chi_c + 3 \beta (T - \theta_a) \chi_T + \left( \frac{\alpha}{\sigma} \chi_c - 3 \chi_T \right) C e^{-1/T} \right] \frac{\partial T}{\partial x}
+ \left( \frac{\alpha}{\sigma} - 1 \right) \chi_c \frac{\partial T}{\partial x} + 2 \alpha \eta_c \frac{\partial T}{\partial x} \frac{\partial C}{\partial t} + 2 (\alpha \eta_{xt} + \chi_T) \frac{\partial T}{\partial x} + 2 \left[ \beta (T - \theta_a) \chi_c - \chi_C e^{-1/T} \right]
- \alpha \chi_{xc} \frac{\partial C}{\partial x} + 2 (\xi_c + \alpha \eta_{xc}) \frac{\partial C}{\partial t} \frac{\partial T}{\partial x} + 2 \alpha \eta_c \frac{\partial C}{\partial t} \frac{\partial T}{\partial x} + 2 \alpha \chi_{cc} \frac{\partial C}{\partial x} \frac{\partial T}{\partial x} + 2 \alpha \chi_{ct} \frac{\partial C}{\partial x} \frac{\partial T}{\partial x} + 2 \alpha \chi_{cc} \frac{\partial C}{\partial x} \frac{\partial T}{\partial x} + 2 \alpha \chi_{ct} \frac{\partial C}{\partial x} \frac{\partial T}{\partial x}
+ \alpha (2 \xi_{xt} - \chi_{tt}) \frac{\partial T}{\partial x} + 2 \alpha \eta_{xt} \frac{\partial T}{\partial x} \frac{\partial C}{\partial t} + \alpha \chi_{cc} \frac{\partial C}{\partial x} \frac{\partial T}{\partial x} + \alpha \chi_{ct} \frac{\partial C}{\partial x} \frac{\partial T}{\partial x} + \alpha \chi_{ct} \frac{\partial C}{\partial x} \frac{\partial T}{\partial x} + \alpha \eta_{tt} \frac{\partial T}{\partial x} \frac{\partial C}{\partial x} \frac{\partial T}{\partial x} = 0;
\]
are satisfied identically. The constants $\sigma$, $\mu$, $\alpha$, $\beta$ and $\theta_0$ are all assumed to be non-zero. Furthermore, all these constants are to remain unrestricted as far as possible.

Upon equating to zero the coefficients of $\frac{\partial^2 C}{\partial x \partial t}$, $\frac{\partial C}{\partial x} \frac{\partial C}{\partial t}$ and $\frac{\partial T}{\partial x} \frac{\partial^2 C}{\partial x \partial t}$ in (4.41) as well as the coefficients of $\frac{\partial^2 T}{\partial x \partial t}$, $\frac{\partial C}{\partial x} \frac{\partial^2 T}{\partial x \partial t}$ and $\frac{\partial T}{\partial x} \frac{\partial^2 T}{\partial x \partial t}$ in (4.42), it is clear that

$$\eta(x, t, C, T) = \eta(t).$$  \hspace{1cm} (4.5)

By the result (4.5), the coefficients of $\frac{\partial C}{\partial t}$ and $\frac{\partial C}{\partial x}$ in (4.41) as well as the coefficients of $\frac{\partial T}{\partial t}$ and $\frac{\partial C}{\partial x}$ in (4.42) imply that

$$\xi(x, t, C, T) = \xi(x, t).$$  \hspace{1cm} (4.6)

From (4.5) and (4.6), the coefficients of $\frac{\partial C}{\partial t}$ and $\frac{\partial T}{\partial t}$ in (4.41) and (4.42) respectively simplify to yield $2\xi_x = \eta'(t)$ and consequently,

$$\xi(x, t) = \frac{1}{2} \eta'(t)x + \rho(t);$$  \hspace{1cm} (4.7)

where $\rho$ is an arbitrary function of $t$ alone.

As the constants $\sigma$, $\mu$, $\alpha$, $\beta$ and $\theta_0$ are required to remain unrestricted as far as possible, the coefficients of $\frac{\partial T}{\partial t}$ and $\frac{\partial C}{\partial t}$ in (4.41) and (4.42) respectively compel

$$\zeta(x, t, C, T) = \zeta(x, t, C), \chi(x, t, C, T) = \chi(x, t, T).$$  \hspace{1cm} (4.8)

By (4.7) and (4.8), the coefficients of $(\frac{\partial C}{\partial x})^2$ and $(\frac{\partial T}{\partial x})^2$ in (4.41) and (4.42) respectively give rise to

$$\zeta(x, t, C) = a(x, t)C + b(x, t),$$

$$\chi(x, t, T) = c(x, t)T + d(x, t);$$  \hspace{1cm} (4.9)

where $a$, $b$, $c$ and $d$ are arbitrary functions of $x$ and $t$. 
By making use of (4.7) and (4.9), the coefficients of \( \frac{\partial C}{\partial x} \) and \( \frac{\partial T}{\partial x} \) in (4.4) and (4.4) respectively reveal that

\[
a(x, t) = -\frac{1}{8\alpha} \eta''(t)x^2 - \frac{1}{2\alpha} \rho'(t)x + e(t), \tag{4.10}
\]

\[
c(x, t) = -\frac{1}{8\alpha} \eta''(t)x^2 - \frac{1}{2\alpha} \rho'(t)x + j(t); \]

where \( e \) and \( j \) denote arbitrary functions solely of \( t \). By (4.9) and (4.10), we find that

\[
\zeta(x, t, C) = \left\{ -\frac{1}{8\alpha} \eta''(t)x^2 - \frac{1}{2\alpha} \rho'(t)x + e(t) \right\} C + b(x, t), \tag{4.11}
\]

\[
\chi(x, t, T) = \left\{ -\frac{1}{8\alpha} \eta''(t)x^2 - \frac{1}{2\alpha} \rho'(t)x + j(t) \right\} T + d(x, t).
\]

By the results (4.7) and (4.11), the terms in (4.4) and (4.4) not involving derivatives of \( C \) or \( T \) with respect to either \( x \) or \( t \) simplify to give

\[
\left[ -\frac{1}{8\alpha} \eta''(t)x^2 - \frac{1}{2\alpha} \rho''(t)x + e'(t) + \frac{1}{4} \eta''(t) + \eta(t)e^{-1/T} \right] C + d(x, t) \frac{C}{T^2} e^{-1/T}
\]

\[
+ \left[ -\frac{1}{8\alpha} \eta''(t)x^2 - \frac{1}{2\alpha} \rho'(t)x + j(t) \right] \frac{C}{T} e^{-1/T} + \left[ -\frac{1}{8\alpha} \eta''(t)x^2 - \frac{1}{2\alpha} \rho'(t)x + e(t) - \eta(t) \right] \mu
\]

\[
+ b_t - \sigma b_{xx} + be^{-1/T} = 0, \tag{4.12}
\]

\[
\left[ -\frac{1}{8\alpha} \eta''(t)x^2 - \frac{1}{2\alpha} \rho''(t)x + j'(t) + \frac{1}{4} \eta''(t) + \beta \eta(t) \right] T + d_t - \alpha d_{xx} + \beta d
\]

\[
+ \left[ -\frac{1}{8\alpha} \eta''(t)x^2 - \frac{1}{2\alpha} \rho'(t)x + j(t) - \eta(t) \right] \beta \theta_a - b(x, t) e^{-1/T} - d(x, t) \frac{C}{T^2} e^{-1/T}
\]

\[
+ \left[ \frac{1}{8} \left( \frac{1}{\alpha} - \frac{1}{\sigma} \right) \eta''(t)x^2 + \frac{1}{2} \left( \frac{1}{\alpha} - \frac{1}{\sigma} \right) \rho'(t)x + j(t) - e(t) - \eta(t) \right] C e^{-1/T}
\]

\[
+ \left[ \frac{1}{8\alpha} \eta''(t)x^2 + \frac{1}{2\alpha} \rho'(t)x - j(t) \right] \frac{C}{T} e^{-1/T} = 0.
\]
Owing to the results (4.5), (4.7) and (4.11), the remaining coefficients in (4.4.1) and (4.4.2) afford no additional information. From equations (4.12), it is required that

\[- \frac{1}{8\alpha} \eta''(t)x^2 - \frac{1}{2\alpha} \rho''(t)x + e'(t) + \frac{1}{4} \eta''(t) + \eta'(t)e^{-1/T} = 0,\]
\[- \frac{1}{8\alpha} \eta''(t)x^2 - \frac{1}{2\alpha} \rho''(t)x + \frac{1}{4} \eta''(t) + \beta \eta'(t) = 0,\]
\[\frac{1}{8\alpha} \eta''(t)x^2 + \frac{1}{2\alpha} \rho'(t)x + \frac{1}{4} \eta''(t) + \eta'(t)e^{-1/T} = 0,\]
\[\frac{1}{8\alpha} \eta''(t)x^2 + \frac{1}{2\alpha} \rho'(t)x - \frac{1}{4} \eta''(t)e^{-1/T} = 0,\]
\[d(x, t)e^{-1/T} = 0,\]
\[\left[- \frac{1}{8\alpha} \eta''(t)x^2 - \frac{1}{2\alpha} \rho'(t)x + e(t) - \eta'(t)\right]u + b_t - \sigma b_{xx} + be^{-1/T} = 0,\]
\[\left[- \frac{1}{8\alpha} \eta''(t)x^2 - \frac{1}{2\alpha} \rho'(t)x + j(t) - \eta'(t)\right]\beta \theta_a + d_t - \alpha d_{xx} + \beta d - be^{-1/T} = 0.\]

From (4.13.5), it follows that

\[d(x, t) = 0.\]  \hspace{1cm} (4.14)

It is implied from the coefficients of $x^2$, $x$ and $x^0$ in (4.13.4) that

\[\eta(t) = c_1 t + c_2, \quad \rho(t) = c_3, \quad j(t) = 0;\]  \hspace{1cm} (4.15)

where $c_1$, $c_2$ and $c_3$ all denote arbitrary constants. The results (4.15) compel the equation (4.13.2) to simplify, giving $c_1 = 0$. As $c_1 = 0$, it follows from (4.15.1) that

\[\eta(t) = c_2.\]  \hspace{1cm} (4.16)

From (4.14), (4.15.2), (4.15.3) and (4.16), the equation (4.13.7) compels

\[b(x, t) = 0.\]  \hspace{1cm} (4.17)

Substituting the results (4.15.2), (4.16) and (4.17) into (4.13.6), we find that

\[e(t) = 0.\]  \hspace{1cm} (4.18)

In view of (4.15.2), (4.15.3), (4.16) and (4.18), the equations (4.13.1) and (4.13.3) are satisfied identically. Upon assigning $a_1 = c_3$ and $a_2 = c_2$, it may be concluded from the results obtained so far that the classical group derived for the coupled system (1.11) under the transformation (4.1) is given by
\[ \xi(x, t, C, T) = a_1, \]
\[ \eta(x, t, C, T) = a_2, \]
\[ \zeta(x, t, C, T) = 0, \]
\[ \chi(x, t, C, T) = 0. \]  
(4.19)

From equations (4.2) and the group (4.19), the following equations are obtained, namely
\[ \frac{\partial C}{\partial x} + \frac{a_2}{a_1} \frac{\partial C}{\partial t} = 0, \quad \frac{\partial T}{\partial x} + \frac{a_2}{a_1} \frac{\partial T}{\partial t} = 0. \]  
(4.20)

The general solutions of (4.20) correspond to the functional forms of the similarity solutions \( C(x, t) \) and \( T(x, t) \) of the coupled system (1.11). By means of the method contained in [12], the general solutions of (4.20) are given by
\[ C(x, t) = f(a_1 t - a_2 x), \quad T(x, t) = g(a_1 t - a_2 x); \]  
(4.21)

where \( f \) and \( g \) are arbitrary functions of \( a_1 t - a_2 x \), while \( a_1 \) and \( a_2 \) denote arbitrary constants.

Upon letting \( w = a_1 t - a_2 x \) and substituting the solutions (4.21) into (1.11), the following coupled system of equations is obtained, namely
\[ \sigma a_2^2 f''(w) - a_1 f'(w) - \exp[-\frac{1}{g(w)}]f'(w) + \mu = 0, \]  
(4.22)
\[ \alpha a_2^2 g''(w) - a_1 g'(w) - \beta g(w) + \exp[-\frac{1}{g(w)}]g'(w) + \beta \theta_a = 0. \]

The general solutions \( f(w) \) and \( g(w) \) of equations (4.22) correspond to the general similarity solutions \( C(x, t) \) and \( T(x, t) \) of (1.11) under the transformation (4.1).

It is noted that the classical group (4.19) with $a_1 = 0$ and $a_2 = -1$ gives rise to the equations governing the steady state solutions. By (4.21) and (4.22), these governing equations are therefore given by

$$\sigma C''(x) - \exp\left[-\frac{1}{T(x)}\right]C(x) + \mu = 0,$$

$$\alpha T''(x) - \beta T(x) + \exp\left[-\frac{1}{T(x)}\right]C(x) + \beta \theta_n = 0.$$  \hspace{1cm} (4.23)

Equations (4.23) correspond to the one-dimensional steady equations obtained by Forbes [15] and which govern the form of the concentration and temperature profiles $C(x)$ and $T(x)$, respectively. The boundary conditions for equations (4.23) found appropriate by Forbes [15] are the Neumann conditions listed in (1.6). The concentration and temperature profiles are easily obtained via the Mathematica NDSolve command for moderate concentration amplitudes $A_C$. The concentration amplitude is defined by Forbes [15] as

$$A_C = C(0) - C(1/m);$$  \hspace{1cm} (4.24)

which, at least for the linearised solution presented by Forbes [15], represents the difference between the maximum and the minimum points in the concentration profile, occurring at the points $x = 0$ and $x = 1/m$ for the $m$-th eigensolution of equations (4.23).

According to Forbes [15], a moderate concentration amplitude is $A_C = 0.015$ and the greatest value for $A_C$ yielding numerical solutions is $A_C = 0.031533185$. The Mathematica NDSolve command experienced some difficulty with the latter value for $A_C$, namely $A_C = 0.031533185$. Several temperature and concentration profiles for different concentration amplitudes $A_C$, relating to equations (4.23) are displayed below.
Figure 4(a). Temperature profile for $\theta_a = 0.18$, $\mu = 0.5$, $\beta = 1$, $\alpha = 0$, $\sigma = 0.0195$, $C(0) = 2.184$, $T(0) = 0.66$.

Figure 4(b). Concentration profile for $\theta_a = 0.18$, $\mu = 0.5$, $\beta = 1$, $\alpha = 0$, $\sigma = 0.0195$, $C(0) = 2.184$, $T(0) = 0.66$. 
Figure 4(c). Temperature profile for $\theta_a = 0.18$, $\mu = 0.5$, $\beta = 1$, $\alpha = 0.001$, $\sigma = 0.25$, $C(0) = 2.22$, $T(0) = 0.68$.

Figure 4(d). Concentration profile for $\theta_a = 0.18$, $\mu = 0.5$, $\beta = 1$, $\alpha = 0.001$, $\sigma = 0.25$, $C(0) = 2.22$, $T(0) = 0.68$. 
Figure 4(e). Temperature profile for $\theta_a = 0.18$, $\mu = 0.5$, $\beta = 1$, $\alpha = 0.001$, $\sigma = 0.25$, $C(0) = 2.1922$, $T(0) = 0.68$.

Figure 4(f). Temperature profile for $\theta_a = 0.18$, $\mu = 0.5$, $\beta = 1$, $\alpha = 0.001$, $\sigma = 0.25$, $C(0) = 2.1912$, $T(0) = 0.68$.

4.4: The Linearised Solution Of Forbes.

Forbes [15] considered a linearised solution in which he assumed $C(x, t)$ and $T(x, t)$ to take the form of the following perturbation expansions.
\[ C(x, t) = C_e + \varepsilon^* C_1(x, t) + O(\varepsilon^{*2}) , \]
\[ T(x, t) = T_e + \varepsilon^* T_1(x, t) + O(\varepsilon^{*2}) ; \]

where \( \varepsilon^* \) is some small parameter related to pattern amplitude. We note that the parameter \( \varepsilon^* \) is distinct from the small parameter \( \varepsilon \) featured in the general transformations involved in similarity methods. Thus, from (4.25) and the one-dimensional case (1.11) of the equations (1.5) governing the second stage of the burning process, Forbes [15] obtained

\[ \frac{\partial C_1}{\partial t} = \sigma \frac{\partial^2 C_1}{\partial x^2} - C_1 \exp[-1/T_e] - \frac{\mu}{T_e^2} T_1 , \]
\[ \frac{\partial T_1}{\partial t} = \alpha \frac{\partial^2 T_1}{\partial x^2} + C_1 \exp[-1/T_e] + \left( \frac{\mu}{T_e^2} - \beta \right) T_1 ; \]

where the equilibrium concentration and temperature are found to be

\[ C_e = \mu \exp \left[ \frac{1}{\theta_a + \mu / \beta} \right] , \quad T_e = \theta_a + \frac{\mu}{\beta} . \]

It should be mentioned that equations (4.26) have been thoroughly investigated. For instance, solutions were obtained for a limiting case of the equations (4.26) as \( \beta \to 0 \) (Aifantis and Hill [5] and Hill and Aifantis [18]). These equations have been studied from the perspective of one-parameter transformation groups in Chapter 2 of this thesis. The reader is referred to equations (1.1) of this thesis.

A more general system was investigated by Lee and Hill [22]. The reader's attention is drawn to equations (1.16) as well as Chapter 6 of this thesis. For the general linear system of coupled diffusion equations with cross-effects, of which equations (1.16) form a one-dimensional case, Lee and Hill [22] considered a solution procedure requiring a number of constraints to be placed on the constants of the system. The coefficients of the system were discovered to obey the properties listed in (1.9) and (1.10).

Comparison of (1.16) and (1.11) implies that for the linear version (4.26) of the one-dimensional case (1.11) for the reaction-diffusion system of equations (1.5) (ruling the second stage of the burning model) obtained by Forbes [15], it is
required that $D_1 = \sigma$, $D_2 = \alpha$, $E_1 = E_2 = 0$, $A_1 = B_2 = \exp[-1/T_e]$, $B_1 = -\frac{\mu}{T_e^2}$ and $A_2 = \beta - \frac{\mu}{T_e^2}$. With these requirements, equations (1.9) and (1.10) yield

\[(\exp[-1/T_e] + \frac{\mu}{T_e^2} - \beta)(\sigma - \alpha) > 0,\]  
\[(\exp[-1/T_e])^2 - 2(\beta + \frac{\mu}{T_e^2})\exp[-1/T_e] + (\frac{\mu}{T_e^2} - \beta)^2 > 0.\] (4.28)

It is observed that Forbes [15] used values of $\sigma$ greater than those of $\alpha$. From (4.28) therefore, we have $\exp[-1/T_e] + \frac{\mu}{T_e^2} - \beta > 0$ and hence,

$$\beta - \frac{\mu}{T_e^2} < \exp[-1/T_e].$$  
(4.29)

From (4.28), it is evident that

$$\exp[-1/T_e] < \beta + \frac{\mu}{T_e^2} - 2\frac{\sqrt{\beta\mu}}{T_e}$$  
or

$$\exp[-1/T_e] > \beta + \frac{\mu}{T_e^2} + 2\frac{\sqrt{\beta\mu}}{T_e}.$$  
(4.30)

From (4.29) and (4.30), it is implied that $\beta - \frac{\mu}{T_e^2} < \beta + \frac{\mu}{T_e^2} - 2\sqrt{\beta\mu}/T_e$ and so it follows that

$$T_e < \sqrt{\frac{\mu}{\beta}}.$$  
(4.31)

In view of (4.27), the inequality (4.31) yields (1.7) which is precisely the condition Forbes [15] required in his linear analysis. In the same paper, Forbes noted that the condition (1.7) gives rise to the condition (1.8) due to the positivity of the ambient temperature $\theta_a$. Forbes [15] was subsequently able to deduce from the condition (1.8) that pattern formation can take place only if the ambient temperature $\theta_a$ is not too high and if Newtonian cooling of the burning surface occurs at a sufficiently high rate where $\beta$ is the coefficient of Newtonian
cooling through the substrate surface. Further details of the more general linear system of coupled diffusion equations with cross-effects are contained in Lee and Hill [22] together with solutions for the system. For example, equations (4.26) with the boundary conditions

$$\frac{\partial C_1}{\partial x} = \frac{\partial T_1}{\partial x} = 0 \quad \text{for } x = 0, 1; \quad (4.32)$$

will have the following solutions which may be read directly from those in Lee and Hill [22]:

$$C_1(x, t) = e^{\sigma t} \int e^{-\mu \xi^*} [K_1(t, \xi^*)h_1(x, \xi^*) + K_2(t, \xi^*)h_2(x, \xi^*)]d\xi^* \alpha t$$

$$+ \exp[-t \exp(-1/T_e)]h_1(x, \alpha t), \quad (4.33)$$

$$T_1(x, t) = e^{\sigma t} \int e^{-\mu \xi^*} [K_3(t, \xi^*)h_2(x, \xi^*) + K_4(t, \xi^*)h_1(x, \xi^*)]d\xi^* \alpha t$$

$$+ \exp[-(\beta - \mu/T_e^2)t]h_2(x, \alpha t);$$

where

$$\omega = \frac{\{\alpha \exp[-1/T_e] - (\beta - \mu/T_e^2)\sigma\}}{(\sigma - \alpha)},$$

$$\nu = \frac{\{\exp[-1/T_e] + \mu/T_e^2 - \beta\}}{(\sigma - \alpha)},$$

$$K_1(t, \xi^*) = -\frac{\sqrt{\mu \exp[-1/T_e]}}{(\sigma - \alpha)T_e} \sqrt{\frac{\xi^* - \alpha t}{\xi^* - \alpha t}} J_1(\eta^*),$$

$$K_2(t, \xi^*) = -\frac{\mu}{(\sigma - \alpha)T_e^2} J_0(\eta^*), \quad (4.34)$$

$$K_3(t, \xi^*) = -\frac{\sqrt{\mu \exp[-1/T_e]}}{(\sigma - \alpha)T_e} \sqrt{\frac{\alpha t - \xi^*}{\xi^* - \alpha t}} J_1(\eta^*),$$

$$K_4(t, \xi^*) = \frac{\exp[-1/T_e]}{(\sigma - \alpha)} J_0(\eta^*),$$

$$\eta^* = 2\frac{\sqrt{\mu \exp[-1/T_e]}}{(\sigma - \alpha)T_e} \sqrt{(\alpha t - \xi^*)(\xi^* - \alpha t)};$$
where $J_0$ and $J_1$ are Bessel functions of order zero and one respectively while the functions $h_\nu(x,t)$ (where $\nu = 1, 2$) satisfy the relation

$$\frac{\partial h_\nu}{\partial t} = \frac{\partial^2 h_\nu}{\partial x^2} ; \tag{4.35}$$

subject to

$$\frac{\partial h_\nu}{\partial x} = 0 \quad \text{for} \quad x = 0, 1 . \tag{4.36}$$

Solutions to (4.35) under the condition (4.36) may be found in Crank [34]. For instance, if initially we have $h_\nu(x,0) = f_\nu(x)$ (where $\nu = 1, 2$), the solutions of (4.35) subject to the condition (4.36) are given by

$$h_\nu(x,t) = \int_0^1 f_\nu(y)dy + 2 \sum_{n=1}^{\infty} \exp[-\pi^2n^2t] \cos(n\pi x) \int_0^1 f_\nu(y) \cos(n\pi y)dy . \tag{4.37}$$

Some of these solutions may be generated by results similar to those presented in Chapter 2.

4.5: Stability Of Nonlinear Steady Patterns.

In his investigations of the stability of steady state patterns of chemical concentration and temperature, Forbes [15] considered a small time-dependent perturbation to the steady solution, in the form

$$C(x,t) = C_S(x) + \varepsilon^* C_1(x,t) + O(\varepsilon^*^2) ,$$

$$T(x,t) = T_S(x) + \varepsilon^* T_1(x,t) + O(\varepsilon^*^2) ; \tag{4.38}$$

where $C_S(x)$ and $T_S(x)$ denote the steady solutions of the system while $\varepsilon^*$ is a small parameter. By substituting (4.38) into equations (1.11), Forbes [15] obtained
\[
\frac{\partial C_1}{\partial t} = \sigma \frac{\partial^2 C_1}{\partial x^2} - \frac{T_1}{T_s^2} C_s \exp[-1/T_s] - C_1 \exp[-1/T_s],
\]
\[
\frac{\partial T_1}{\partial t} = \alpha \frac{\partial^2 T_1}{\partial x^2} + \left( \frac{C_s}{T_s^2} \exp[-1/T_s] - \beta \right) T_1 + C_1 \exp[-1/T_s];
\]

which is of the form

\[
\frac{\partial}{\partial t} \{C_1(x, t)\} = \sigma \frac{\partial^2}{\partial x^2} \{C_1(x, t)\} - f(x)C_1(x, t) - g(x)T_1(x, t),
\]
\[
\frac{\partial}{\partial t} \{T_1(x, t)\} = \alpha \frac{\partial^2}{\partial x^2} \{T_1(x, t)\} + f(x)C_1(x, t) + \{g(x) - \beta\} T_1(x, t);
\]

where \(f\) and \(g\) are functions of \(x\). Equations (4.40) are a generalisation of the equations (1.1) studied in Chapter 2 of this thesis. Further research would be required to determine if the group method sheds any light on equations (4.40).

### 4.6: The Sal'nikov Model.

The model of a burning process proposed and developed by Sal'nikov in 1949 did not take into consideration the effects of chemical diffusion and thermal conductivity. This model corresponds to equations (1.5) with \(\sigma = \alpha = 0\). Hence with \(\sigma = \alpha = 0\) in our one-dimensional case (1.11) of the system (1.5), the classical group derived for the coupled system (1.11) under the transformation (4.1) is given by

\[
\xi(x, t, C, T) = \xi(x),
\]
\[
\eta(x, t, C, T) = \eta(x),
\]
\[
\zeta(x, t, C, T) = 0,
\]
\[
\chi(x, t, C, T) = 0.
\]
From the equations (4.2), the group (4.41) and via the method contained in [12], it was revealed that by letting \( \xi(x) = a_1 \) and \( \eta(x) = a_2 \) in (4.41), the functional forms of the similarity solutions \( C(x, t) \) and \( T(x, t) \) for the coupled system (1.11) with \( \sigma = \alpha = 0 \) are given by the results (4.21) where the coupled system (4.22) reduces to yield

\[
-a_1 f'(w) - \exp[-\frac{1}{g(w)}]ff(w) + \mu = 0,
\]

\[
-a_1 g'(w) - \beta g(w) + \exp[-\frac{1}{g(w)}]ff(w) + \beta \theta_a = 0;
\]

with \( w = a_1 t - a_2 x \). The steady state solution for the Sal'nikov model is obtained by considering the group (4.41) with \( \xi(x) = a_1 \) and \( \eta(x) = a_2 \), characterised by \( a_1 = 0 \) and \( a_2 = 1 \). Consequently from (4.42), for the steady state solution we obtain

\[
T(x) = \theta_a + \frac{\mu}{\beta}, \quad C(x) = \frac{\mu}{\beta} \exp\left[\frac{1}{(\theta_a + \mu/\beta)}\right];
\]

so that the temperature and concentration profiles are the constants \( T_e \) and \( C_e \) respectively, the equilibrium temperature and concentration obtained in Forbes' linearised solution discussed in Section (4.4) of Chapter 4 in this thesis.

### 4.7: The Non-Classical Procedure.

For the one-parameter group of transformations described in (4.1), the terms \( A(x, t, C, T) \), \( B(x, t, C, T) \) and \( G(x, t, C, T) \) are introduced and defined by

\[
A(x, t, C, T) = \frac{\xi(x, t, C, T)}{\eta(x, t, C, T)},
\]

\[
B(x, t, C, T) = \frac{\xi(x, t, C, T)}{\eta(x, t, C, T)},
\]

\[
G(x, t, C, T) = \frac{\chi(x, t, C, T)}{\eta(x, t, C, T)}.
\]
so that (4.2) becomes

\[
\frac{\partial C}{\partial t} = A - B \frac{\partial C}{\partial x}, \tag{4.45}
\]

\[
\frac{\partial T}{\partial t} = G - B \frac{\partial T}{\partial x}.
\]

By differentiating (4.45) partially with respect to x and making use of (1.11) and (4.45), we may deduce that

\[
\frac{\partial^2 C}{\partial x \partial t} = A_x - \frac{1}{\sigma} AB + \frac{\mu}{\sigma} B - \frac{1}{\sigma} B C e^{1/T} + (A_c + \frac{1}{\sigma} B^2 - B_x) \frac{\partial C}{\partial x} + A_T \frac{\partial T}{\partial x} - B_c (\frac{\partial C}{\partial x})^2 \\
- B_T \frac{\partial C}{\partial x} \frac{\partial T}{\partial x} \tag{4.46}
\]

\[
\frac{\partial^2 T}{\partial x \partial t} = G_x - \frac{1}{\alpha} GB - \frac{\beta}{\alpha} (T - \theta_d)B + \frac{1}{\alpha} B C e^{1/T} + (G_T + \frac{1}{\alpha} B^2 - B_x) \frac{\partial T}{\partial x} + G_c \frac{\partial C}{\partial x} \\
- B_T (\frac{\partial T}{\partial x})^2 - B_c \frac{\partial C}{\partial x} \frac{\partial T}{\partial x}.
\]

The constants \(\sigma, \mu, \alpha, \beta\) and \(\theta_d\) are all assumed to be non-zero as in the classical procedure. All these constants should remain unrestricted unless otherwise forced. Since only the identity transformation results from \(\eta(x, t, C, T) = 0\), we consider only the case \(\eta(x, t, C, T) \neq 0\) in order to obtain non-trivial transformations. The results (4.45) and (4.46) are then substituted into equations (4.4). Then equating to zero the coefficients of \((\frac{\partial C}{\partial x})^3, (\frac{\partial T}{\partial x})^3, \frac{\partial C}{\partial x}, \frac{\partial C}{\partial x}, \frac{\partial C}{\partial x}, (\frac{\partial T}{\partial x})^2, \frac{\partial T}{\partial x} (\frac{\partial C}{\partial x})^2, (\frac{\partial T}{\partial x})^2, \frac{\partial C}{\partial x}, (\frac{\partial C}{\partial x})^2\) and the remaining terms yields the equations
\[ B_{CC} = 0, \]
\[ B_{CT} = 0, \]
\[ B_{TT} = 0, \]
\[ A_{TT} = 0, \]
\[ 2\sigma A_{CT} = 2\sigma B_{XT} - \left( \frac{\sigma}{\alpha} + 1 \right)BB_T, \]
\[ \sigma A_{CC} = 2\sigma B_{XC} - 2BB_C, \]
\[ 2\sigma A_{XT} = \left( \frac{\sigma}{\alpha} - 1 \right)BA_T + 2(A + Ce^{-1/T} - \mu)B_T, \]
\[ 2\sigma A_{XC} = \sigma B_{XX} - 2BB_X - B_t + [2A + 3(Ce^{-1/T} - \mu)]B_C \]
\[ + \{ \frac{\sigma}{\alpha} [\beta(T - \theta_a) - Ce^{-1/T}] + \left( \frac{\sigma}{\alpha} - 1 \right)G \} B_T, \]
\[ A_t = \sigma A_{XX} + (Ce^{-1/T} - \mu)A_C + \left\{ \left( \frac{\sigma}{\alpha} - 1 \right)G + \frac{\sigma}{\alpha} [\beta(T - \theta_a) - Ce^{-1/T}] \right\} A_T \]
\[ + 2(\mu - Ce^{-1/T} - A)B_X - Ae^{-1/T} - G \frac{C}{T^2} e^{-1/T}, \]
\[ G_{CC} = 0, \]
\[ 2\alpha G_{CT} = 2\alpha B_{XC} - \left( \frac{\alpha}{\sigma} + 1 \right)BB_C, \]
\[ \alpha G_{TT} = 2\alpha B_{XT} - 2BB_T, \]
\[ 2\alpha G_{XC} = \left( \frac{\alpha}{\sigma} - 1 \right)BG_C + 2[G - Ce^{-1/T} + \beta(T - \theta_a)]B_C, \]
\[ 2\alpha G_{XT} = \alpha B_{XX} - 2BB_X - B_t + [2G + 3[\beta(T - \theta_a) - Ce^{-1/T}]]B_T \]
\[ + \{ \frac{\alpha}{\sigma} (Ce^{-1/T} - \mu) + \left( \frac{\alpha}{\sigma} - 1 \right)A \} B_C, \]
\[ G_t = \alpha G_{XX} + [\beta(T - \theta_a) - Ce^{-1/T}]G_T + \left\{ \left( \frac{\alpha}{\sigma} - 1 \right)A + \frac{\alpha}{\sigma} (Ce^{-1/T} - \mu) \right\} G_C \]
\[ + 2[Ce^{-1/T} - \beta(T - \theta_a) - G]B_X + Ae^{-1/T} + G \frac{C}{T^2} e^{-1/T} - \beta G. \]

The system of equations (4.47) determines the non-classical group of (1.11).

From (4.47) and (4.47), it is clear that \( B_T = a^*(x, t) \) and therefore,

\[ B(x, t, C, T) = a^*(x, t)T + \gamma(x, t, C); \quad (4.48) \]

where \( a^* \) is an arbitrary function of \( x \) and \( t \) while \( \gamma \) denotes an unrestricted function of \( x, t \) and \( C \).

By (4.48), examination of (4.47) reveals that \( \gamma_{CC} = 0 \) and so,

\[ \gamma(x, t, C) = b^*(x, t)C + c^*(x, t); \quad (4.49) \]

where \( b^* \) and \( c^* \) represent further arbitrary functions of \( x \) and \( t \).
The results (4.48) and (4.49) then yield
\[ B(x, t, C, T) = a^*(x, t)T + b^*(x, t)C + c^*(x, t). \] (4.50)

By making use of (4.50), it follows from (4.47\(_5\)) that
\[ A_T = -\frac{1}{4} \left( \frac{1}{\alpha} + \frac{1}{\sigma} \right) a^* b^* C^2 + \left[ a_x^* - \frac{1}{2} \left( \frac{1}{\alpha} + \frac{1}{\sigma} \right) (a^{*2} T + a^* c^*) \right] C + d^*(x, t, T); \] (4.51)
where \( d^* \) is an arbitrary function of \( x, t \) and \( T \).

From (4.47\(_4\)) and (4.51), it is clear that \( -\frac{1}{2} \left( \frac{1}{\alpha} + \frac{1}{\sigma} \right) a^*^2 C + d^* T = 0 \) which compels
\[ a^*(x, t) = 0, \quad d^*(x, t, T) = d^*(x, t). \] (4.52)

The results (4.52) have the effect of simplifying (4.50) and (4.51) to yield
\[ B(x, t, C, T) = b^*(x, t)C + c^*(x, t), \] (4.53)
\[ A(x, t, C, T) = d^*(x, t)T + \kappa(x, t, C); \]
where \( \kappa \) denotes an arbitrary function of \( x, t \) and \( C \). By the results (4.53), the equation (4.47\(_6\)) gives
\[ \kappa(x, t, C) = k^*(x, t) + e^*(x, t)C + (b^* - \frac{1}{\sigma} b^* c^*) C^2 - \frac{1}{3\sigma} b^*^2 C^3; \] (4.54)

where \( e^* \) and \( k^* \) are further arbitrary functions of \( x \) and \( t \). Hence from (4.53\(_2\)) and (4.54), it follows immediately that
\[ A(x, t, C, T) = k^*(x, t) + e^*(x, t)C + d^*(x, t)T + (b^* - \frac{1}{\sigma} b^* c^*) C^2 - \frac{1}{3\sigma} b^*^2 C^3. \] (4.55)

In view of (4.53\(_1\)) and (4.55), it is implied from (4.47\(_7\)) that
\[ d^*_x + \frac{1}{2} \left( \frac{1}{\alpha} - \frac{1}{\sigma} \right) (b^* C + c^*) d^* = 0. \] (4.56)

Upon substituting (4.53\(_1\)) and (4.55) into (4.47\(_8\)), it is deduced that
\[-\frac{2}{3\sigma} b^* C^3 + (4b^*_x - \frac{2}{\sigma} b^* c^*) b^* C^2 + [2b^* e^* + 3(b^* e^{-1/T} - \sigma b^*_x) - b^*_t + 2(b^* e^*)] C
+ 2 b^* d^* T + [(2k^* - 3\mu)b^* + \sigma c^*_x - 2c^* c^*_x - c^*_t - 2\sigma e^*_x] = 0. \tag{4.57}\]

The coefficient of $C^3$ in (4.57) requires $b^*(x, t) = 0$ and upon setting $e^*(x, t) = B(x, t)$, the results obtained from (4.53), (4.55), (4.56) and (4.57) are deduced as

\[
B(x, t, C, T) = B(x, t),
\]

\[
A(x, t, C, T) = k^*(x, t) + e^*(x, t)C + d^*(x, t)T, \tag{4.58}
\]

\[
d^*_x + \frac{1}{2} \left( \frac{1}{\sigma} - \frac{1}{\alpha} \right) B(x, t) d^* = 0,
\]

\[
e^*_x = \frac{1}{2} B_{xx} - \frac{1}{\sigma} B_{x} - \frac{1}{2\sigma} B_t.
\]

From (4.58), the equations (4.47) imply that $G_T = m(x, t)$ and so,

\[
G(x, t, C, T) = m(x, t)T + \varphi(x, t, C); \tag{4.59}
\]

where $m$ denotes an arbitrary function of $x$ and $t$ while $\varphi$ is an arbitrary function of $x$, $t$ and $C$. By (4.47) and (4.59), it follows that $\varphi_{CC} = 0$ and hence,

\[
\varphi(x, t, C) = n(x, t)C + p(x, t); \tag{4.60}
\]

where $n$ and $p$ are arbitrary functions of $x$ and $t$. It follows from (4.59) and (4.60) that

\[
G(x, t, C, T) = p(x, t) + m(x, t)T + n(x, t)C. \tag{4.61}
\]

The results (4.58) and (4.61) together with equation (4.47) imply that

\[
n_x + \frac{1}{2} \left( \frac{1}{\alpha} - \frac{1}{\sigma} \right) B(x, t) n = 0. \tag{4.62}
\]

By (4.58) and (4.61), the equation (4.47) is simplified to yield

\[
m_x = \frac{1}{2} B_{xx} - \frac{1}{\alpha} B_x - \frac{1}{2\alpha} B_t. \tag{4.63}
\]

Reconciling (4.58) and (4.63) requires
\[
B_x = \frac{2}{(\sigma - \alpha)} (\sigma e^* - \alpha m) + r(t); \tag{4.64}
\]

where \(r\) represents an arbitrary function solely of \(t\).

Substituting (4.58), (4.58) and (4.61) into the equation (4.47) leads to

\[
\begin{align*}
\sigma k^*_{xx} - k^*_t - \mu e^* + \left(\frac{\sigma}{\alpha} - 1\right)p d^* + 2(\mu - k^*)B_x - \frac{\sigma}{\alpha} \beta \theta d^* - k^* e^{-1/T} - \frac{m \sigma}{T} e^{-1/T} \\
+ \left[\sigma e^*_{xx} - e^*_t + \left(\frac{\sigma}{\alpha} - 1\right)n d^* - 2e^*B_x - \left(\frac{\sigma}{\alpha} d^* + 2B_x\right)e^{-1/T}\right]C - \frac{C}{T^2} e^{-1/T} \\
+ \left[\sigma d^*_{xx} - d^*_t - d^* e^{-1/T} + \left(\frac{\sigma}{\alpha} - 1\right)m d^* + \frac{\sigma}{\alpha} \beta d^* - 2d^*B_x\right]T - \frac{C^2}{T^2} e^{-1/T} = 0.
\end{align*}
\]

(4.65)

The coefficients of \(\frac{C}{T}, \frac{C}{T^2}\) and \(\frac{C^2}{T^2}\) in (4.65) require \(m(x, t) = p(x, t) = n(x, t) = 0\) and so from (4.61), we obtain

\[
G(x, t, C, T) = 0. \tag{4.66}
\]

Equating to zero the coefficients of \(T\) and \(C\) in (4.65) compels

\[
d^*(x, t) = 0, \quad B(x, t) = B(t). \tag{4.67}
\]

The coefficient of \(C^0T^0\) in (4.65) yields

\[
k^*(x, t) = 0. \tag{4.68}
\]

It follows from (4.61), (4.66), (4.67) and (4.68) that (4.65) gives \(e^*(x, t)\mu = 0\) and so,

\[
e^*(x, t) = 0. \tag{4.69}
\]

As \(d^*(x, t) = n(x, t) = 0\), the equations (4.58) and (4.62) are satisfied identically.

By (4.58), (4.63), (4.67), (4.69) and as \(m(x, t) = 0\), it is evident that

\[
B(t) = d_1; \tag{4.70}
\]

where \(d_1\) denotes an arbitrary constant.

It follows from (4.58), (4.67), (4.68) and (4.69) that

\[
A(x, t, C, T) = 0. \tag{4.71}
\]
We see that the results \((4.66), (4.70)\) and \((4.71)\) satisfy equation \((4.47_{15})\) identically. The results derived in this section enable us to deduce that the non-classical group for the coupled system \((1.11)\) under the transformation \((4.1)\) is given by

\[
A(x, t, C, T) = 0, \quad B(x, t, C, T) = d_1, \quad G(x, t, C, T) = 0; \quad (4.72)
\]

which satisfies equations \((4.47)\) identically. From this section, we observed that the results obtained bear a marked resemblance to those derived in Chapter 2 of this thesis, lending credence to the statement made in Section \((4.5)\) of this chapter to the effect that equations \((4.40)\) are a generalisation of equations \((1.1)\) studied in Chapter 2. By letting \(d_1 = \frac{a_1}{a_2}\), the non-classical group \((4.72)\) in conjunction with the definitions of \(A, B\) and \(G\) given in \((4.44)\) enables the classical group \((4.19)\) to be recovered. From equations \((4.45)\) and the group \((4.72)\), the following equations are obtained, namely

\[
\frac{\partial C}{\partial t} + d_1 \frac{\partial C}{\partial x} = 0, \quad \frac{\partial T}{\partial t} + d_1 \frac{\partial T}{\partial x} = 0. \quad (4.73)
\]

The general solutions of \((4.73)\) correspond to the functional forms of the similarity solutions \(C(x, t)\) and \(T(x, t)\) of the coupled system \((1.11)\). Via the method contained in [12], the general solutions of \((4.73)\) are given by

\[
C(x, t) = f^*(x - d_1 t), \quad T(x, t) = g^*(x - d_1 t); \quad (4.74)
\]

where \(f^*\) and \(g^*\) denote arbitrary functions of \(x - d_1 t\).

Upon letting \(w^* = x - d_1 t\) and substituting the solutions \((4.74)\) into \((1.11)\), the following coupled system of equations is obtained, namely

\[
\alpha f^*(w^*) + d_1 f^*(w^*) - \exp[- \frac{1}{g^*(w^*)}] f^*(w^*) + \mu = 0, \quad (4.75)
\]

\[
\alpha g^*(w^*) + d_1 g^*(w^*) - \beta g^*(w^*) + \exp[- \frac{1}{g^*(w^*)}] f^*(w^*) + \beta \theta = 0;
\]

which is identical to \((4.22)\) with \(a_1 = -d_1, a_2 = -1\) and \(f = f^*\).
The general solutions $f^*(w^*)$ and $g^*(w^*)$ of equations (4.75) correspond to the general similarity solutions $C(x, t)$ and $T(x, t)$ of (1.11) under the transformation (4.1). The results derived in this section seem to indicate that there are no solutions obtainable from the non-classical method for the coupled system (1.11) which cannot be recovered via the classical approach.
CHAPTER 5: SOLUTIONS TO A COUPLED SYSTEM OF SEMI-LINEAR EQUATIONS.

5.1: Introduction.

In this chapter, similarity solutions for the coupled system of equations (1.13) will be derived by means of one-parameter transformation groups. By the introduction of characteristic coordinates \( p = x - a_1 t \) and \( q = x - a_2 t \), it follows that the coupled system of equations (1.13) may be rewritten as

\[
\frac{\partial u}{\partial q} = \frac{\lambda_1}{(a_1 - a_2)} u v, \quad \quad \frac{\partial v}{\partial p} = \frac{\lambda_2}{(a_2 - a_1)} u v. \tag{5.1}
\]

One-parameter transformation groups maintaining the invariance of the coupled system (5.1) will now be deduced in order to derive similarity solutions of the coupled system of equations (5.1). A general transformation of the following form is considered, namely

\[
p_1 = p_1(p, q, u, v, \varepsilon) = p + \varepsilon \zeta(p, q, u, v) + O(\varepsilon^2), \\
q_1 = q_1(p, q, u, v, \varepsilon) = q + \varepsilon \eta(p, q, u, v) + O(\varepsilon^2), \\
u_1 = u_1(p, q, u, v, \varepsilon) = u + \varepsilon \zeta(p, q, u, v) + O(\varepsilon^2), \\
v_1 = v_1(p, q, u, v, \varepsilon) = v + \varepsilon \chi(p, q, u, v) + O(\varepsilon^2). \tag{5.2}
\]

If the invariance of the coupled system of equations (5.1) is retained by the transformation (5.2) and if \( u = \phi(p, q) \), \( v = \psi(p, q) \); then from \( u_1 = \phi(p_1, q_1) \) and \( v_1 = \psi(p_1, q_1) \), equating terms of order \( \varepsilon \) gives

\[
\xi(p, q, u, v) \frac{\partial u}{\partial p} + \eta(p, q, u, v) \frac{\partial u}{\partial q} = \zeta(p, q, u, v), \\
\xi(p, q, u, v) \frac{\partial v}{\partial p} + \eta(p, q, u, v) \frac{\partial v}{\partial q} = \chi(p, q, u, v). \tag{5.3}
\]
The similarity variables \( u \) and \( v \) are obtained directly from the solutions of (5.3) which correspond to the functional forms of the similarity solutions of the coupled system of equations (5.1). The group leaving the coupled system (5.1) invariant is ascertained by the following approach to the coupled system of equations (5.1).

5.2: The Classical Approach To Equations (5.1).

From results in Appendix IV of this thesis and upon eliminating \( \frac{\partial u}{\partial q} \) and \( \frac{\partial v}{\partial p} \) by using (5.1), it is clear that

\[
\begin{align*}
\frac{\partial u_1}{\partial q_1} - \frac{\lambda_1}{(a_1 - a_2)} u_1 v_1 &= \frac{\partial u}{\partial q} - \frac{\lambda_1}{(a_1 - a_2)} uv \\
+ \epsilon \left( \zeta_q + \frac{\lambda_1}{(a_1 - a_2)} (\zeta_u - \eta_q) uv - (\zeta u + \zeta v) \right) &= \frac{\lambda_1^2}{(a_1 - a_2)^2} \eta_u (uv)^2 - \xi_v \frac{\partial u}{\partial p} \\
+ \left[ \zeta_v - \frac{\lambda_1}{(a_1 - a_2)} \eta_v uv \right] \frac{\partial v}{\partial q} - \left[ \xi_q + \frac{\lambda_1}{(a_1 - a_2)} \xi_u uv \right] \frac{\partial v}{\partial p} \right) + O(\epsilon^2),
\end{align*}
\]

(5.4)

where the subscripts denote partial differentiation with \( p \), \( q \), \( u \) and \( v \) as independent variables (for example, \( \frac{\partial \xi}{\partial p} = \xi_p + \xi_u \frac{\partial u}{\partial p} + \xi_v \frac{\partial v}{\partial p} \)); and partial derivatives in the form \( \frac{\partial \xi}{\partial p}, \frac{\partial \xi}{\partial q} \) are used when \( u \) and \( v \) are considered to be dependent on \( p \) and \( q \).

It is observed from equations (5.4) that the invariance of the coupled system of equations (5.1) is preserved under the transformation (5.2) provided the functions \( \xi(p, q, u, v), \eta(p, q, u, v), \zeta(p, q, u, v) \) and \( \chi(p, q, u, v) \) are such that the following equations

\[ \frac{\partial \xi}{\partial q} = \frac{\partial u}{\partial q} \]
\[ \xi_q + \frac{\lambda_1}{(a_1 - a_2)} [(\xi_u - \eta_q)uv - (\chi u + \xi v)] - \frac{\lambda_1^2}{(a_1 - a_2)^2} \eta_u (uv)^2 - \xi_v \frac{\partial u}{\partial p} \frac{\partial v}{\partial q} + \left[ \xi_v - \frac{\lambda_1}{(a_1 - a_2)} \eta_v uv \right] \frac{\partial v}{\partial q} - \left[ \xi_q + \frac{\lambda_1}{(a_1 - a_2)} \xi_u uv \right] \frac{\partial u}{\partial p} = 0, \]

\[ \chi_p + \frac{\lambda_2}{(a_2 - a_1)} [(\chi_v - \xi_p)uv - (\chi u + \xi v)] - \frac{\lambda_2^2}{(a_2 - a_1)^2} \xi_v (uv)^2 - \eta_u \frac{\partial u}{\partial p} \frac{\partial v}{\partial q} + \left[ \chi_u - \frac{\lambda_2}{(a_2 - a_1)} \xi_u uv \right] \frac{\partial u}{\partial p} - \left[ \eta_p + \frac{\lambda_2}{(a_2 - a_1)} \eta_v uv \right] \frac{\partial v}{\partial q} = 0; \]

are satisfied identically. It is assumed that the constants \( a_1, a_2, \lambda_1 \) and \( \lambda_2 \) are all non-zero. In this study of non-linearly interacting waves, only the case of unequal wave speeds \( a_1 \neq a_2 \) is interesting and will be considered. It is noted that if \( a_1 = a_2 \), the procedure for solving the system in question reduces to solving a very simple differential equation and so the case \( a_1 = a_2 \) will not be considered here.

From the coefficients of \( \frac{\partial u}{\partial p} \frac{\partial v}{\partial q} \) in equations (5.51) and (5.52) respectively, it follows that

\[ \xi(p, q, u, v) = \xi(p, q, u), \quad \eta(p, q, u, v) = \eta(p, q, v). \]  

(5.6)

Via the result (5.61), equating to zero the terms not involving \( v \) in the coefficient of \( \frac{\partial u}{\partial p} \) in (5.51) implies that \( \xi_q = 0 \) and so,

\[ \xi(p, q, u) = \xi(p, u). \]  

(5.7)

By use of the result (5.62), equating to zero the terms not involving \( u \) in the coefficient of \( \frac{\partial v}{\partial q} \) in (5.52) reveals that \( \eta_p = 0 \) and thus,

\[ \eta(p, q, v) = \eta(q, v). \]  

(5.8)

Upon using the results (5.62) and (5.8), equating to zero the term involving \( p \) in the coefficient of \( \frac{\partial v}{\partial q} \) in (5.51) yields \( \zeta_v = 0 \) and so,

\[ \zeta(p, q, u, v) = \zeta(p, q, u). \]  

(5.9)
By (5.61) and (5.7), equating to zero the term involving \( q \) in the coefficient of \( \frac{\partial u}{\partial p} \) in (5.52) gives \( \chi_u = 0 \) and so,

\[
\chi(p, q, u, v) = \chi(p, q, v). \tag{5.10}
\]

By the results (5.6), (5.7), (5.8), (5.9) and (5.10), the terms in (5.51) and (5.52) not involving derivatives of \( u \) and \( v \) with respect to either \( p \) or \( q \) give rise to the equations

\[
\zeta_q + \frac{\lambda_1}{(a_1 - a_2)} [(\zeta_u - \eta_q)uv - (\chi_u + \zeta_v)] = 0, \tag{5.11}
\]

\[
\chi_p + \frac{\lambda_2}{(a_2 - a_1)} [(\chi_v - \xi_p)uv - (\chi_u + \zeta_v)] = 0.
\]

In view of the results (5.6), (5.7), (5.8), (5.9) and (5.10), equating to zero the terms in (5.111) not involving \( v \) and the terms in (5.112) not involving \( u \) respectively yields \( \zeta_q = 0 \) and \( \chi_p = 0 \) and so from (5.9) and (5.10), it is clear that

\[
\zeta(p, q, u) = \zeta(p, u), \quad \chi(p, q, v) = \chi(q, v). \tag{5.12}
\]

From (5.12), the equations (5.11) simplify to give

\[
(\zeta_u - \eta_q)uv = \chi_u + \zeta_v, \tag{5.13}
\]

\[
(\chi_v - \xi_p)uv = \chi_u + \zeta_v.
\]

Reconciling (5.131) and (5.132) requires \( \zeta_u - \eta_q = \chi_v - \xi_p \) or \( \zeta_u + \xi_p = \chi_v + \eta_q \) and from the results (5.6), (5.7), (5.8), (5.9), (5.10) and (5.12), it is evident that

\[
\zeta_u = - \xi_p + b_1, \quad \chi_v = - \eta_q + b_1; \tag{5.14}
\]

where \( b_1 \) is an arbitrary constant.

By (5.61) and (5.7), the coefficient of \( \frac{\partial u}{\partial p} \) in (5.51) yields \( (\zeta_u)uv = 0 \) and dividing through by \( uv \) gives

\[
\zeta(p, u) = \zeta(p). \tag{5.15}
\]
By (5.62) and (5.8), the coefficient of \( \frac{\partial \nu}{\partial q} \) in (5.52) gives \((\eta_v)uv = 0\) and dividing through by \(uv\) gives
\[
\eta(q, v) = \eta(q).
\]
(5.16)

From (5.9), (5.10), (5.12), (5.14), (5.15) and (5.16), it follows that
\[
\zeta(p, u) = [-\xi'(p) + b_1]u + b_2(p),
\]
\[
\chi(q, v) = [-\eta'(q) + b_1]v + b_3(q);
\]
(5.17)

where \(b_2\) and \(b_3\) are arbitrary functions of \(p\) and \(q\) respectively.

Substituting the results (5.15), (5.16) and (5.17) into equations (5.5) yields the relation
\[
b_1uv + b_3(q)u + b_2(p)v = 0.
\]
(5.18)

The coefficients of \(uv\), \(u\) and \(v\) in (5.18) result in
\[
b_1 = 0, \quad b_2(p) = 0, \quad b_3(q) = 0.
\]
(5.19)

From (5.17) and (5.19), it is clear that
\[
\zeta(p, u) = -\xi'(p)u, \quad \chi(q, v) = -\eta'(q)v.
\]
(5.20)

In conclusion, the group derived for the coupled system of equations (5.1) under the transformation (5.2) and subject to the constraint \(a_1 \neq a_2\), is
\[
\xi(p, q, u, v) = \xi(p),
\]
\[
\eta(p, q, u, v) = \eta(q),
\]
\[
\zeta(p, q, u, v) = -\xi'(p)u,
\]
\[
\chi(p, q, u, v) = -\eta'(q)v.
\]
(5.21)

The group (5.21) identically satisfies equations (5.51) and (5.52).

From equations (5.3) and the group (5.21), the following equations arise, namely
\[
\xi(p) \frac{\partial u}{\partial p} + \eta(q) \frac{\partial u}{\partial q} = - \xi'(p)u,
\]

(5.22)

\[
\xi(p) \frac{\partial v}{\partial p} + \eta(q) \frac{\partial v}{\partial q} = - \eta'(q)v.
\]

The general solutions to equations (5.22) may be recovered via the method of Lagrange (see [8]) and correspond to the functional forms of the similarity solutions \( u(p, q) \) and \( v(p, q) \) for the coupled system of equations (5.1) subject to the condition \( a_1 \neq a_2 \). By this method, the subsidiary equations associated with equations (5.22) are

\[
\frac{dp}{\xi(p)} = \frac{dq}{\eta(q)} = - \frac{du}{\xi'(p)u}, \quad \frac{dp}{\xi(p)} = \frac{dq}{\eta(q)} = - \frac{dv}{\eta'(q)v}.
\]

(5.23)

Integration of \( \frac{dp}{\xi(p)} = \frac{dq}{\eta(q)} \) in equations (5.23) gives

\[
h(p) = k(q) + c_1;
\]

(5.24)

where \( h(p) = \int dp^{*} / \xi(p^{*}) \) and \( k(q) = \int dq^{*} / \eta(q^{*}) \) while \( c_1 \) is an arbitrary constant. The first integrals of equations (5.23\_1) and (5.23\_2) respectively are then

\[
r_1(p, q, u) = r_2(p, q, v) = h(p) + m(q) = c_1;
\]

(5.25)

where \( m(q) = -k(q) \). From (5.23), we next consider the equations

\[
\frac{\xi'(p)}{\xi(p)} dp = - \frac{du}{u}, \quad \frac{\eta'(q)}{\eta(q)} dq = - \frac{dv}{v};
\]

(5.26)

integrating both of which gives rise to the second integrals of equations (5.23\_1) and (5.23\_2) respectively, namely

\[
s_1(p, q, u) = \xi'(p)u = c_2, \quad s_2(p, q, v) = \eta'(q)v = c_3;
\]

(5.27)

where \( c_2 \) and \( c_3 \) are arbitrary constants.

The Jacobians \( \frac{\partial (r_1, s_1)}{\partial (p, q)}, \frac{\partial (r_1, s_1)}{\partial (p, u)} \) and \( \frac{\partial (r_1, s_1)}{\partial (q, u)} \) are all non-zero in any region of space in which \( \xi(p) \neq 0 \) and \( m'(q)\xi'(p)u \neq 0 \) (considering only the case \( a_1 \neq a_2 \)).
Hence in any region of space in which $\xi(p) \neq 0$ and $m'(q)\xi(p)u \neq 0$, the general solution of equation (5.22) is

$$u(p, q) = h'(p)M[h(p) + m(q)] ;$$

(5.28)

where $M$ denotes an arbitrary function.

The Jacobians $\frac{\partial (r_1, s_2)}{\partial (p, q)}$, $\frac{\partial (r_2, s_2)}{\partial (p, q)}$, and $\frac{\partial (r_2, s_2)}{\partial (q, v)}$ are all non-zero in any region of space in which $\eta(q) \neq 0$ and $h'(p)\eta(q)v \neq 0$ (considering only the case $\alpha_1 \neq \alpha_2$).

Hence in any region of space in which $\eta(q) \neq 0$ and $h'(p)\eta(q)v \neq 0$, the general solution of equation (5.22) is

$$v(p, q) = -m'(q)N[h(p) + m(q)] ;$$

(5.29)

where $N$ represents an arbitrary function. The general solutions (5.28) and (5.29) are verified upon substitution into equations (5.22).

Setting $w = h(p) + m(q)$ and substituting (5.28) and (5.29) into (5.1) gives rise to the following coupled system of equations, namely

$$\frac{(a_2 - a_1)}{\lambda_1} M'(w) = M(w)N(w), \quad \frac{(a_2 - a_1)}{\lambda_2} N'(w) = M(w)N(w).$$

(5.30)

The interested reader is referred to Gorbuzov [35], where equations such as those listed in (5.30) have been studied in detail.

A simple pair of solutions to equations (5.30) is given by

$$M(w) = \frac{(a_1 - a_2)}{\lambda_2 w}, \quad N(w) = \frac{(a_1 - a_2)}{\lambda_1 w}.$$  

(5.31)

Solutions (5.31) are verified upon substitution into equations (5.30).

From (5.28), (5.29) and (5.31) where $w = h(p) + m(q)$, the similarity solutions for the coupled system of equations (5.1) under the transformation (5.2) and associated with the group (5.21) subject to the condition $\alpha_1 \neq \alpha_2$ are

$$u(p, q) = \frac{(a_1 - a_2)}{\lambda_2} \frac{h'(p)}{[h(p) + m(q)]}, \quad v(p, q) = \frac{(a_2 - a_1)}{\lambda_1} \frac{m'(q)}{[h(p) + m(q)]}.$$  

(5.32)

Substituting the solutions (5.32) into the coupled system of equations (5.1) confirms the validity of these similarity solutions. Further solutions of (5.30) may
be obtained, but the resulting expressions for \( u(p, q) \) and \( v(p, q) \) may be shown to be equivalent to (5.32). Since \( p = x - a_1 t \) and \( q = x - a_2 t \), the solutions (5.32) may be rewritten as

\[
\begin{align*}
    u(x, t) &= \frac{(a_1 - a_2)}{\lambda_2} \frac{h'(x - a_1 t)}{[h(x - a_1 t) + m(x - a_2 t)]}, \\
    v(x, t) &= \frac{(a_2 - a_1)}{\lambda_1} \frac{m'(x - a_2 t)}{[h(x - a_1 t) + m(x - a_2 t)]}.
\end{align*}
\]

(5.33)

Solutions (5.33) form the general solution of the system of equations (1.13) and are equivalent to the solutions presented by Hasimoto [16]. By letting

\[
    m(x - a_2 t) = x - a_2 t, \quad h(x - a_1 t) = 2(x - a_1 t); \quad (5.34)
\]
a particular pair of similarity solutions may be obtained from the results (5.33), namely

\[
\begin{align*}
    u(x, t) &= \frac{2(a_1 - a_2)}{[3x - (2a_1 + a_2)t]\lambda_2}, \\
    v(x, t) &= \frac{(a_2 - a_1)}{[3x - (2a_1 + a_2)t]\lambda_1}.
\end{align*}
\]

(5.35)

Graphs (for different values of \( t \)) corresponding to the similarity solutions (5.35) with numerical values for the constants \( a_1, a_2, \lambda_1 \) and \( \lambda_2 \) are illustrated below in Figures 5(a) to 5(f). It will be observed that since solutions (5.33) are merely travelling wave solutions, Figures 5(c), 5(d), 5(e) and 5(f) are merely translates of Figures 5(a), 5(b), 5(c) and 5(d) respectively by \( 5 \frac{5}{6} \) units in the horizontal direction.
Figure 5(a). Graph of $u(x, 0)$ for $a_1 = 2$, $a_2 = -0.5$, $\lambda_1 = 1$, $\lambda_2 = 1$.

Figure 5(b). Graph of $v(x, 0)$ for $a_1 = 2$, $a_2 = -0.5$, $\lambda_1 = 1$, $\lambda_2 = 1$. 
Figure 5(c). Graph of $u(x, 5)$ for $a_1 = 2, a_2 = -0.5, \lambda_1 = 1, \lambda_2 = 1$.

Figure 5(d). Graph of $v(x, 5)$ for $a_1 = 2, a_2 = -0.5, \lambda_1 = 1, \lambda_2 = 1$. 
Figure 5(e). Graph of $u(x, 10)$ for $a_1 = 2, a_2 = -0.5, \lambda_1 = 1, \lambda_2 = 1$.

Figure 5(f). Graph of $v(x, 10)$ for $a_1 = 2, a_2 = -0.5, \lambda_1 = 1, \lambda_2 = 1$. 
An alternative derivation of the general solution to equations (1.13) constitutes the next and final section of this chapter.

5.3: An Alternative Approach.

In this section, a one-parameter transformation group maintaining the invariance of the coupled system of equations (1.13) is deduced in order to derive the general similarity solution of the coupled system (1.13). A general transformation of the following form is considered, namely

\[ x_1 = x_1(x, t, u, v, \varepsilon) = x + \varepsilon \xi(x, t, u, v) + O(\varepsilon^2), \]
\[ t_1 = t_1(x, t, u, v, \varepsilon) = t + \varepsilon \eta(x, t, u, v) + O(\varepsilon^2), \]
\[ u_1 = u_1(x, t, u, v, \varepsilon) = u + \varepsilon \zeta(x, t, u, v) + O(\varepsilon^2), \]
\[ v_1 = v_1(x, t, u, v, \varepsilon) = v + \varepsilon \chi(x, t, u, v) + O(\varepsilon^2). \] (5.36)

If the invariance of the coupled system of equations (1.13) is retained by the transformation (5.36) and if \( u = \phi(x, t) \), \( v = \psi(x, t) \); then from \( u_1 = \phi(x_1, t_1) \) and \( v_1 = \psi(x_1, t_1) \), equating terms of order \( \varepsilon \) gives

\[ \xi(x, t, u, v) \frac{\partial u}{\partial x} + \eta(x, t, u, v) \frac{\partial u}{\partial t} = \zeta(x, t, u, v), \] (5.37)
\[ \xi(x, t, u, v) \frac{\partial v}{\partial x} + \eta(x, t, u, v) \frac{\partial v}{\partial t} = \chi(x, t, u, v). \]

The similarity variables \( u \) and \( v \) are obtained directly from the solutions of (5.37) which correspond to the functional forms of the similarity solutions of the coupled system of equations (1.13). The group leaving the coupled system (1.13) invariant is ascertained as follows.

From results in Appendix IV of this thesis and upon eliminating \( \frac{\partial u}{\partial t} \) and \( \frac{\partial v}{\partial t} \) by using (1.13), it is clear that
\[ \frac{\partial u}{\partial t_1} + a_1 \frac{\partial u}{\partial x_1} - \lambda_1 u_1 v_1 = \frac{\partial u}{\partial t} + a_1 \frac{\partial u}{\partial x} - \lambda_1 uv \\
+ \varepsilon \{ (\zeta_t + a_1 \zeta_x - (\chi u + \zeta v)\lambda_1 + [(\zeta_u - \eta_t - a_1 \eta_{1x})\lambda_1 + \lambda_2 \zeta_v]uv \\
- \lambda_1 [\lambda_1 \eta_u + \lambda_2 \eta_v](uv)^2 + (a_1 - a_2)(\zeta_v - \lambda_1 \eta_{1v}) \frac{\partial u}{\partial x} + (a_1 - a_2)(a_1 \eta - \zeta_v)\lambda_1 \frac{\partial u}{\partial x} \} + O(\varepsilon^2) , \]
\]
\[ (5.38) \]
\[
\frac{\partial v}{\partial t_1} + a_2 \frac{\partial v}{\partial x_1} - \lambda_2 u_1 v_1 = \frac{\partial v}{\partial t} + a_2 \frac{\partial v}{\partial x} - \lambda_2 uv \\
+ \varepsilon \{ (\chi_t + a_2 \chi_x - (\chi u + \zeta v)\lambda_2 + [(\chi_v - \eta_t - a_2 \eta_{1x})\lambda_2 + \lambda_1 \chi_u]uv \\
- \lambda_2 [\lambda_2 \eta_v + \lambda_1 \eta_u](uv)^2 + (a_2 - a_1)(\chi_u - \lambda_2 \eta_{1u}) \frac{\partial u}{\partial x} + (a_1 - a_2)(\zeta - a_2 \eta_{1v}) \frac{\partial u}{\partial x} \} \frac{\partial v}{\partial x} + O(\varepsilon^2); \]

where the subscripts denote partial differentiation with \( x, t, u \) and \( v \) as independent variables (for example, \( \frac{\partial \zeta}{\partial x} = \zeta_x + \zeta_u \frac{\partial u}{\partial x} + \zeta_v \frac{\partial v}{\partial x} \)), and partial derivatives in the form \( \frac{\partial \zeta}{\partial x}, \frac{\partial \zeta}{\partial t} \) are used when \( u \) and \( v \) are considered to be dependent on \( x \) and \( t \).

It is observed from equations (5.38) that the invariance of the coupled system of equations (1.13) is maintained under the transformation (5.36) provided the functions \( \zeta(x, t, u, v), \eta(x, t, u, v), \chi(x, t, u, v) \) and \( \chi(x, t, u, v) \) are such that the following equations
\[
\zeta_t + a_1 \zeta_x - (\chi u + \zeta v)\lambda_1 + [(\zeta_u - \eta_t - a_1 \eta_{1x})\lambda_1 + \lambda_2 \zeta_v]uv \\
- \lambda_1 [\lambda_1 \eta_u + \lambda_2 \eta_v](uv)^2 + (a_1 - a_2)(\zeta_v - \lambda_1 \eta_{1v}) \frac{\partial u}{\partial x} + (a_1 - a_2)(a_1 \eta - \zeta_v)\lambda_1 \frac{\partial u}{\partial x} \} \frac{\partial u}{\partial x} = 0, \]
\]
\[ (5.39) \]
\[
\chi_t + a_2 \chi_x - (\chi u + \zeta v)\lambda_2 + [(\chi_v - \eta_t - a_2 \eta_{1x})\lambda_2 + \lambda_1 \chi_u]uv \\
- \lambda_2 [\lambda_2 \eta_v + \lambda_1 \eta_u](uv)^2 + (a_2 - a_1)(\chi_u - \lambda_2 \eta_{1u}) \frac{\partial u}{\partial x} + (a_1 - a_2)(\zeta - a_2 \eta_{1v}) \frac{\partial u}{\partial x} \} \frac{\partial v}{\partial x} = 0; \]
are satisfied identically. It is assumed that the constants \( a_1, a_2, \lambda_1 \) and \( \lambda_2 \) are all non-zero. In this study of non-linearly interacting waves, only the case of unequal wave speeds \( a_1 \neq a_2 \) is interesting and will be considered. If \( a_1 = a_2 \), the procedure for solving the system (1.13) reduces to solving a very simple differential equation and so the case \( a_1 = a_2 \) will not be considered here.

For the case \( a_1 \neq a_2 \), the coefficients of \( \frac{\partial u}{\partial x} \) in (5.39.1) and (5.39.2) respectively imply that \( (\xi - a_1 \eta)_x = 0 \) and \( (\xi - a_2 \eta)_u = 0 \). Therefore, we have

\[
\xi - a_1 \eta = f(x, t, u), \quad \xi - a_2 \eta = g(x, t, v);
\]

resulting in

\[
\eta(x, t, u, v) = \frac{1}{(a_2 - a_1)} [f(x, t, u) - g(x, t, v)],
\]

\[
\xi(x, t, u, v) = \frac{1}{(a_2 - a_1)} [a_2 f(x, t, u) - a_1 g(x, t, v)];
\]

where \( f \) and \( g \) denote arbitrary functions.

For the case \( a_1 \neq a_2 \) and in view of the results (5.41), the coefficients of \( \frac{\partial u}{\partial x} \) and \( \frac{\partial v}{\partial x} \) in equations (5.39.1) and (5.39.2) respectively yield

\[
f_t + a_1 f_x + \lambda_1 f_u u v = 0, \quad g_t + a_2 g_x + \lambda_2 g_v u v = 0.
\]

Reconciling (5.42.1) and (5.42.2) yields

\[
(f - g)_t + (a_1 f - a_2 g)_x + (\lambda_1 f_u - \lambda_2 g_v) u v = 0.
\]

The coefficient of \( u v \) in (5.43) gives \( \lambda_1 f_u = \lambda_2 g_v \), implying that

\[
f(x, t, u) = \frac{\lambda_2}{\lambda_1} g_v u + a(x, t),
\]

and since \( g \) is a function of \( x, t \) and \( v \), it follows that

\[
g(x, t, v) = b(x, t) v + c(x, t);
\]

where \( a, b \) and \( c \) all represent arbitrary functions of \( x \) and \( t \).

From (5.44) and (5.45), it follows that
\[ f(x, t, u) = \frac{\lambda_2}{\lambda_1} b(x, t) u + a(x, t). \quad (5.46) \]

Substituting (5.45) and (5.46) into equation (5.43) yields
\[ (a - c)_t + (a_1 a - a_2 c)_x + (b_t + a_1 b_x) \frac{\lambda_2}{\lambda_1} u - (b_t + a_2 b_x) v = 0. \quad (5.47) \]

Equating to zero the coefficients of \( u \) and \( v \) in equation (5.47) gives
\[ b_t + a_1 b_x = 0, \quad b_t + a_2 b_x = 0. \quad (5.48) \]

For the case \( a_1 \neq a_2 \), reconciling (5.48)\(_1\) and (5.48)\(_2\) yields
\[ b(x,t) = b(t). \quad (5.49) \]

From (5.48) and (5.49), it is clear that \( b'(t) = 0 \) and so,
\[ b(t) = d_1; \quad (5.50) \]

where \( d_1 \) is an arbitrary constant.

By the results (5.49) and (5.50), the equation (5.47) simplifies to give
\[ (a - c)_t + (a_1 a - a_2 c)_x = 0, \quad (5.51) \]

and the results (5.46) and (5.45) in turn yield
\[ f(x, t, u) = \frac{\lambda_2}{\lambda_1} d_1 u + a(x, t), \quad g(x, t, v) = d_1 v + c(x, t). \quad (5.52) \]

Substituting the results (5.52) into the equations (5.42) yields
\[ a_t + a_1 a_x + \lambda_2 d_1 u v = 0, \quad c_t + a_2 c_x + \lambda_2 d_1 u v = 0. \quad (5.53) \]

Equating to zero the coefficients of \( uv \) in (5.53)\(_1\) and (5.53)\(_2\) yields \( d_1 = 0 \), simplifying the results (5.52) to give
\[ f(x, t, u) = f(x, t) = a(x, t), \quad g(x, t, v) = g(x, t) = c(x, t). \quad (5.54) \]

As \( d_1 = 0 \), it follows that (5.53) yields
\[ a_t + a_1 a_x = 0, \quad c_t + a_2 c_x = 0. \quad (5.55) \]
It is noted that reconciling \((5.53_1)\) and \((5.53_2)\) gives rise to \((5.51)\). The general solutions to equations \((5.55)\) are recovered via the method contained in \([12]\) and thus, the general solutions of equations \((5.55_1)\) and \((5.55_2)\) are respectively given by
\[
a(x, t) = h(x - a_1 t), \quad c(x, t) = k(x - a_2 t); \quad (5.56)
\]
where \(h\) and \(k\) denote arbitrary functions of \(x - a_1 t\) and \(x - a_2 t\) respectively.

From \((5.54)\) and \((5.56)\), it is evident that
\[
f(x, t) = h(x - a_1 t), \quad g(x, t) = k(x - a_2 t). \quad (5.57)
\]

From \((5.41)\) and \((5.57)\), it is clear that
\[
\eta(x, t, u, v) = \eta(x, t) = \frac{1}{(a_2 - a_1)} \left[ h(x - a_1 t) - k(x - a_2 t) \right],
\]
\[
\xi(x, t, u, v) = \xi(x, t) = \frac{1}{(a_2 - a_1)} \left[ a_2 h(x - a_1 t) - a_1 k(x - a_2 t) \right]. \quad (5.58)
\]

For the case \(a_1 \neq a_2\) and using the result \((5.58_1)\), the coefficients of \(\frac{\partial v}{\partial x}\) and \(\frac{\partial u}{\partial x}\) in \((5.39_1)\) and \((5.39_2)\) respectively reveal that
\[
\zeta(x, t, u, v) = \zeta(x, t, u), \quad \chi(x, t, u, v) = \chi(x, t, v). \quad (5.59)
\]

By the expressions \((5.58_1)\) and \((5.59)\), the terms in \((5.39_1)\) and \((5.39_2)\) not involving derivatives of \(u\) or \(v\) with respect to \(x\) give rise to
\[
\zeta_t + a_1 \zeta_x - \left[ \chi(x, t, v) u + \zeta(x, t, u) v \right] \lambda_1 + \left[ \zeta_u - k'(x - a_2 t) \right] \lambda_1 u v = 0,
\]
\[
\chi_t + a_2 \chi_x - \left[ \chi(x, t, v) u + \zeta(x, t, u) v \right] \lambda_2 + \left[ \chi_v - h'(x - a_1 t) \right] \lambda_2 u v = 0. \quad (5.60)
\]

Equating to zero respectively the terms in \((5.60_1)\) not involving \(v\) and the terms in \((5.60_2)\) not involving \(u\) gives the relations
\[
\zeta_t + a_1 \zeta_x = 0, \quad \chi_t + a_2 \chi_x = 0. \quad (5.61)
\]

From \((5.61)\), the equations \((5.60)\) simplify to yield
\[ [\zeta_u - k'(x - a_2 t)]uv = \chi(x, t, v)u + \zeta(x, t, u)v, \] 
\[ [\chi_v - h'(x - a_1 t)]uv = \chi(x, t, v)u + \zeta(x, t, u)v. \] 

(5.62)

Reconciling (5.62_1) and (5.62_2) requires
\[ \zeta_u - k'(x - a_2 t) = \chi_v - h'(x - a_1 t). \] 

(5.63)

In conjunction with (5.59), it is implied from (5.63) that
\[ \zeta(x, t, u) = m(x, t)u + n(x, t), \quad \chi(x, t, v) = m^*(x, t)v + n^*(x, t); \] 

(5.64)

where \( m, m^*, n \) and \( n^* \) all denote arbitrary functions of \( x \) and \( t \).

Substituting (5.64) into equations (5.62) gives rise to
\[ n^*(x, t)u + n(x, t)v + [m^*(x, t) + k'(x - a_2 t)]uv = 0, \]
\[ n^*(x, t)u + n(x, t)v + [m(x, t) + h'(x - a_1 t)]uv = 0. \] 

(5.65)

The coefficients of \( u, v \) and \( uv \) in equations (5.65) compel
\[ n^*(x, t) = 0, \]
\[ n(x, t) = 0, \]
\[ m^*(x, t) = -k'(x - a_2 t), \]
\[ m(x, t) = -h'(x - a_1 t). \] 

(5.66)

It follows immediately from (5.64) and (5.66) that
\[ \zeta(x, t, u) = -h'(x - a_1 t)u, \] 
\[ \chi(x, t, v) = -k'(x - a_2 t)v. \] 

(5.67)
The results (5.67) satisfy equations (5.60), (5.61), (5.62) and (5.63) identically. From results (5.58), (5.59) and (5.67), the group derived for the coupled system of equations (1.13) (for the case \( a_1 \neq a_2 \)) under the transformation (5.36) is

\[
\xi(x, t, u, v) = \frac{1}{(a_2 - a_1)} \left[ a_2 F(x - a_1 t) + a_1 G(x - a_2 t) \right],
\]

\[
\eta(x, t, u, v) = \frac{1}{(a_2 - a_1)} \left[ F(x - a_1 t) + G(x - a_2 t) \right],
\]

\[
\zeta(x, t, u, v) = -F(x - a_1 t)u,
\]

\[
\chi(x, t, u, v) = G'(x - a_2 t)v;
\]

(5.68)

where \( F(x - a_1 t) = h(x - a_1 t) \) and \( G(x - a_2 t) = -k(x - a_2 t) \). The group (5.68) identically satisfies equations (5.39) and (5.39').

From equations (5.37) and the group (5.68), the following equations arise, namely

\[
[a_2 F(x - a_1 t) + a_1 G(x - a_2 t)] \frac{\partial u}{\partial x} + [F(x - a_1 t) + G(x - a_2 t)] \frac{\partial u}{\partial t} = (a_1 - a_2)F'(x - a_1 t)u,
\]

(5.69)

\[
[a_2 F(x - a_1 t) + a_1 G(x - a_2 t)] \frac{\partial v}{\partial x} + [F(x - a_1 t) + G(x - a_2 t)] \frac{\partial v}{\partial t} = (a_2 - a_1)G'(x - a_2 t)v.
\]

The general solutions to equations (5.69) may be recovered via the method of Lagrange (see [8]) and correspond to the functional forms of the similarity solutions for the coupled system of equations (1.13) (for the case \( a_1 \neq a_2 \)). By this method, the subsidiary equations associated with (5.69) are

\[
\frac{dx}{[a_2 F(x - a_1 t) + a_1 G(x - a_2 t)]} = \frac{dt}{[F(x - a_1 t) + G(x - a_2 t)]} = \frac{du}{(a_1 - a_2)F'(x - a_1 t)u},
\]

(5.70)

\[
\frac{dx}{[a_2 F(x - a_1 t) + a_1 G(x - a_2 t)]} = \frac{dt}{[F(x - a_1 t) + G(x - a_2 t)]} = \frac{dv}{(a_2 - a_1)G'(x - a_2 t)v}.
\]

From equations (5.70), it is evident that
\[
\frac{dx}{dt} = \frac{a_2 F(x - a_1 t) + a_1 G(x - a_2 t)}{[F(x - a_1 t) + G(x - a_2 t)]}.
\] (5.71)

Upon the introduction of characteristic coordinates \( p = x - a_1 t \) and \( q = x - a_2 t \), it follows that

\[
\frac{dp}{dq} = \frac{dx}{dt} \frac{1}{a_1 - a_2}.
\] (5.72)

Substituting (5.71) into (5.72) yields the separable equation

\[
\frac{dp}{dq} = \frac{F(p)}{G(q)},
\] (5.73)

whose general solution is

\[
f(p) = -g(q) + m_1.
\] (5.74)

where \( f(p) = \int \frac{dp}{F(p)} \) and \( g(q) = \int \frac{dq}{G(q)} \) while \( m_1 \) is an arbitrary constant.

From (5.74) together with the definitions of \( p \) and \( q \), the first integrals of equations (5.70) and (5.702) respectively are

\[
r_1(x, t, u) = r_2(x, t, v) = f(x - a_1 t) + g(x - a_2 t) = m_1.
\] (5.75)

From equations (5.70), we next consider

\[
\frac{dt}{[F(x - a_1 t) + G(x - a_2 t)]} = \frac{du}{(a_1 - a_2)F(x - a_1 t)u},
\] (5.76)

\[
\frac{dt}{[F(x - a_1 t) + G(x - a_2 t)]} = \frac{dv}{(a_2 - a_1)G(x - a_2 t)v}.
\]

As \( p = x - a_1 t \) and \( q = x - a_2 t \), we have \( t = \frac{q - p}{a_1 - a_2} \) which simplifies equations (5.76) to yield

\[
\frac{dq - dp}{F(p) + G(q)} = \frac{du}{F(p)u}, \quad \frac{dp - dq}{F(p) + G(q)} = \frac{dv}{G(q)v}.
\] (5.77)

From (5.73) and (5.77), it is evident that
\[
\frac{F'(p)}{F(p)} \, dp = -\frac{du}{u}, \quad \frac{G'(q)}{G(q)} \, dq = -\frac{dv}{v}. \tag{5.78}
\]

With \( p = x - a_1 t \) and \( q = x - a_2 t \), integrating equations (5.78) gives rise to the second integrals of equations (5.70_1) and (5.70_2) respectively, namely

\[
s_1(x, t, u) = F(x - a_1 t)u = m_2, \quad s_2(x, t, v) = G(x - a_2 t)v = m_3; \tag{5.79}
\]

where \( m_2 \) and \( m_3 \) both denote arbitrary constants.

Since the Jacobians \( \frac{\partial (r_1, s_1)}{\partial (x, t)} \) and \( \frac{\partial (r_1, s_1)}{\partial (x, u)} \) are all non-zero when \( F(x - a_1 t)u \neq 0 \), \( F(x - a_1 t) + G(x - a_2 t) 
eq 0 \) and \( a_2 F(x - a_1 t) + a_1 G(x - a_2 t) \neq 0 \), (considering only the case where \( a_1 \neq a_2 \),) in any region of space in which the conditions \( F(x - a_1 t)u \neq 0 \), \( F(x - a_1 t) + G(x - a_2 t) \neq 0 \), \( a_2 F(x - a_1 t) + a_1 G(x - a_2 t) = 0 \) hold, the general solution of equation (5.69_1) is

\[
u(x, t) = f'(x - a_1 t)S[f(x - a_1 t) + g(x - a_2 t)]; \tag{5.80}
\]

where \( S \) represents an arbitrary function.

As the Jacobians \( \frac{\partial (r_2, s_2)}{\partial (x, t)} \) and \( \frac{\partial (r_2, s_2)}{\partial (x, v)} \) are all non-zero when \( G(x - a_2 t)v \neq 0 \), \( F(x - a_1 t) + G(x - a_2 t) \neq 0 \) and \( a_2 F(x - a_1 t) + a_1 G(x - a_2 t) \neq 0 \), (considering only the case where \( a_1 \neq a_2 \),) in any region of space in which the conditions \( G(x - a_2 t)v \neq 0 \), \( F(x - a_1 t) + G(x - a_2 t) \neq 0 \), \( a_2 F(x - a_1 t) + a_1 G(x - a_2 t) = 0 \) hold, the general solution of equation (5.69_2) is

\[
v(x, t) = g'(x - a_2 t)T[f(x - a_1 t) + g(x - a_2 t)]; \tag{5.81}
\]

where \( T \) represents an arbitrary function.

The general solutions (5.80) and (5.81) are verified upon substitution into equations (5.69). Letting \( w = f(x - a_1 t) + g(x - a_2 t) \) and substituting (5.80) and (5.81) into equations (1.13) gives rise to the following coupled system of equations, namely

\[
\frac{(a_1 - a_2)}{\lambda_1} S'(w) = S(w)T(w), \tag{5.82}
\]

\[
\frac{(a_2 - a_1)}{\lambda_2} T'(w) = S(w)T(w).
\]
The interested reader is referred to Gorbuzov [35], where equations such as those listed in (5.82) have been studied in detail.

A simple pair of solutions to equations (5.82) is given by

\[ S(w) = \frac{(a_1 - a_2)}{\lambda_2 w}, \quad T(w) = \frac{(a_2 - a_1)}{\lambda_1 w}. \]  

(5.83)

Solutions (5.83) are verified upon substitution into equations (5.82). From (5.80), (5.81) and (5.83) where \( w = f(x - a_1 t) + g(x - a_2 t) \), it follows that the similarity solutions for the coupled system of equations (1.13) under the transformation (5.36) and associated with the group (5.68) for the case \( a_1 \neq a_2 \) are given by

\[ u(x, t) = \frac{(a_1 - a_2)}{\lambda_2} \frac{f'(x - a_1 t)}{[f(x - a_1 t) + g(x - a_2 t)]}, \]

\[ v(x, t) = \frac{(a_2 - a_1)}{\lambda_1} \frac{g'(x - a_2 t)}{[f(x - a_1 t) + g(x - a_2 t)]}. \]  

(5.84)

Substituting the solutions (5.84) into the coupled system of equations (1.13) attests to the validity of these similarity solutions. Further solutions of (5.82) may be obtained but the resulting expressions for \( u(x, t) \) and \( v(x, t) \) may be shown to be equivalent to (5.84). Solutions (5.84) and (5.33) are equivalent to each other and form the general solution of the system of equations (1.13). It should be noted that solutions (5.33) and (5.84) are equivalent to the solutions presented by Hasimoto [16].
BIBLIOGRAPHY.


**Additional References.**


APPENDIX I

FORMULAE FOR PARTIAL DERIVATIVES FOR THE CLASSICAL PROCEDURE OF CHAPTER 2.

\[
\frac{\partial u_1}{\partial t_1} = \frac{\partial u}{\partial t} + \varepsilon [\xi_t + (\zeta_u - \eta_t) \frac{\partial u}{\partial t} + \zeta_v \frac{\partial v}{\partial t} - \eta_v \left( \frac{\partial u}{\partial t} \right)^2 - \eta_v \frac{\partial u}{\partial t} \frac{\partial v}{\partial t} - \xi_u \frac{\partial u}{\partial t} - \xi_v \frac{\partial v}{\partial t}] + O(\varepsilon^2),
\]

\[
\frac{\partial v_1}{\partial t_1} = \frac{\partial v}{\partial t} + \varepsilon [\chi_t + \chi_u \frac{\partial u}{\partial t} + (\chi_v - \eta_t) \frac{\partial v}{\partial t} - \eta_u \left( \frac{\partial u}{\partial t} \right)^2 - \xi_u \frac{\partial u}{\partial t} - \xi_v \frac{\partial v}{\partial t}] + O(\varepsilon^2),
\]

\[
\frac{\partial^2 u_1}{\partial x_1^2} = \frac{\partial^2 u}{\partial x^2} + \varepsilon [\zeta_{xx} + (2\zeta_{xu} - \xi_{xx}) \frac{\partial u}{\partial x} + (\zeta_u - 2\xi_{xx}) \frac{\partial^2 u}{\partial x^2} - 2\eta_x \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial t} - \eta_{xx} \frac{\partial u}{\partial x} - 2\eta_u \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial t}]
- \eta_u \frac{\partial^2 u}{\partial x \partial t} - 2\eta_{ux} \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial t} + \eta_{uv} \frac{\partial^2 u}{\partial x \partial t} + 2\eta_{uxx} \frac{\partial^2 u}{\partial x \partial t} - 2\eta_{xv} \frac{\partial^2 u}{\partial x \partial t} - 3\xi_{xu} \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial t}
+ (\zeta_{uu} - 2\xi_{xu}) \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial t} - \eta_{uu} \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial t} - 2\eta_{uv} \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial t} - 2\eta_{uv} \frac{\partial^2 u}{\partial x \partial t} - \xi_{uu} \frac{\partial^2 u}{\partial x^2}
- \eta_{vv} \frac{\partial^2 u}{\partial x^2} - 2\xi_{uv} \frac{\partial^2 u}{\partial x^2} + \xi_{vv} \frac{\partial^2 u}{\partial x^2}] + O(\varepsilon^2),
\]

\[
\frac{\partial^2 v_1}{\partial x_1^2} = \frac{\partial^2 v}{\partial x^2} + \varepsilon [\chi_{xx} + 2\chi_{xu} \frac{\partial u}{\partial x} + \chi_u \frac{\partial^2 u}{\partial x^2} + (2\chi_{xy} - \xi_{xx}) \frac{\partial v}{\partial x} + (\chi_v - 2\xi_{xx}) \frac{\partial^2 v}{\partial x^2} - 2\eta_x \frac{\partial v}{\partial x} \frac{\partial^2 v}{\partial x \partial t}]
+ 2(\chi_{uv} - \xi_{xx}) \frac{\partial u}{\partial x} \frac{\partial^2 v}{\partial x \partial t} - \eta_{xx} \frac{\partial v}{\partial x} \frac{\partial^2 v}{\partial x \partial t} - 2\eta_{ux} \frac{\partial u}{\partial x} \frac{\partial^2 v}{\partial x \partial t} + (\chi_{uv} - 2\xi_{xx}) \frac{\partial^2 v}{\partial x \partial t} - 2\eta_x \frac{\partial^2 v}{\partial x \partial t}
- \frac{\partial v}{\partial x} \frac{\partial^2 u}{\partial x \partial t} - 2\eta_u \frac{\partial v}{\partial x} \frac{\partial^2 u}{\partial x \partial t} - 2\eta_{xu} \frac{\partial u}{\partial x} \frac{\partial^2 v}{\partial x \partial t} - 2\eta_{uv} \frac{\partial u}{\partial x} \frac{\partial^2 v}{\partial x \partial t} - \eta_{uv} \frac{\partial^2 u}{\partial x^2}
- \eta_{vv} \frac{\partial^2 u}{\partial x^2} - 2\xi_{uv} \frac{\partial^2 u}{\partial x^2} + 2\xi_{vv} \frac{\partial^2 u}{\partial x^2} - 2\eta_{uv} \frac{\partial^2 u}{\partial x^2} - \xi_{uv} \frac{\partial^2 u}{\partial x^2} + \xi_{vv} \frac{\partial^2 u}{\partial x^2}] + O(\varepsilon^2).
\]
APPENDIX II

FORMULAE FOR PARTIAL DERIVATIVES FOR THE CLASSICAL PROCEDURE OF CHAPTER 3.

\[
\frac{\partial y_1}{\partial t_1} = \frac{\partial y}{\partial t} + \varepsilon[\pi_t + (\pi_y - \tau_t) \frac{\partial y}{\partial t} - v_t \frac{\partial y}{\partial x} - \tau_y (\frac{\partial y}{\partial t})^2 - v_y \frac{\partial y}{\partial t} \frac{\partial y}{\partial x}] + O(\varepsilon^2),
\]

\[
\frac{\partial^2 y_1}{\partial t_1^2} = \frac{\partial^2 y}{\partial x^2} + \varepsilon[\pi_{xx} + (2\pi_{xy} - \nu_{xx}) \frac{\partial y}{\partial x} + (\pi_y - 2\nu_y) \frac{\partial^2 y}{\partial x^2} - \tau_{xx} \frac{\partial y}{\partial t} - 2\tau_x \frac{\partial^2 y}{\partial x \partial t} + (\pi_{yy} - 2\nu_{xy}) (\frac{\partial y}{\partial x})^2 - 3\nu_y \frac{\partial^2 y}{\partial x \partial t} - 2\tau_y \frac{\partial^2 y}{\partial x \partial t} - \tau_y \frac{\partial^2 y}{\partial t^2} - 2\tau_{xy} \frac{\partial y}{\partial x} \frac{\partial y}{\partial t} - v_y \frac{\partial y}{\partial x} \frac{\partial y}{\partial t}] + O(\varepsilon^2),
\]

\[
\frac{\partial^3 y_1}{\partial t_1^3} = \frac{\partial^3 y}{\partial x^3} + \varepsilon[\pi_{txx} + (2\pi_{txy} - \nu_{txx} - 2\tau_{tx}) \frac{\partial^2 y}{\partial x \partial t} + (2\pi_{txxy} - \nu_{txxx}) \frac{\partial y}{\partial x} + (\pi_y - 2\nu_x - \tau_t) \frac{\partial^3 y}{\partial t^3} + \tau_{tx} \frac{\partial^2 y}{\partial t \partial x}] + O(\varepsilon^2).
\]
\[ \frac{\partial^4 y}{\partial x_1^4} = \frac{\partial^4 y}{\partial x^4} + \epsilon[\pi_{xxxx} + (6\pi_{xxyy} - 4\nu_{xxyy})\frac{\partial^2 y}{\partial x^2 \partial x^2} + (4\pi_{xxyy} - \nu_{xxxx})\frac{\partial y}{\partial x} - 4\tau_{xxx} \frac{\partial^2 y}{\partial x \partial t} - \tau_{xxxxx} \frac{\partial y}{\partial t}] \\
+ (4\pi_{xy} - 6\nu_{xx})\frac{\partial^2 y}{\partial x \partial x^3} - 6\tau_{xx} \frac{\partial^3 y}{\partial x \partial x \partial t} + (\pi_{y} - 4\nu_{yx})\frac{\partial^4 y}{\partial x^4} - 4\tau_{x} \frac{\partial^4 y}{\partial t \partial x \partial x^2} - 12\tau_{xy} \frac{\partial^3 y}{\partial x \partial t \partial x^2} \\
+ (4\pi_{yy} - 16\nu_{xy})(\frac{\partial y}{\partial x})^2 + (3\pi_{yy} - 12\nu_{xyy})(\frac{\partial^2 y}{\partial x^2})^2 + (12\pi_{xyy} - 18\nu_{xyy})(\frac{\partial y}{\partial x})^3 \\
+ (6\pi_{xxyy} - 4\nu_{xxyy})(\frac{\partial^2 y}{\partial x \partial x^3}) - 5\nu_{y} \frac{\partial^2 y}{\partial x^2} \frac{\partial^3 y}{\partial x^3} - 10\nu_{y} \frac{\partial^2 y}{\partial x^2} \frac{\partial^3 y}{\partial x^3} - 4\tau_{y} \frac{\partial^4 y}{\partial t \partial x \partial x^2} \\
- 6\tau_{y} \frac{\partial^2 y}{\partial x \partial x^2} - 12\tau_{xy} \frac{\partial^2 y}{\partial x \partial x^2} - 4\tau_{y} \frac{\partial^3 y}{\partial x \partial x^2} \frac{\partial y}{\partial t} - 12\tau_{xy} \frac{\partial^2 y}{\partial x \partial x} \frac{\partial y}{\partial t} - 4\tau_{xy} \frac{\partial^3 y}{\partial x \partial x} \frac{\partial y}{\partial t} \\
- 4\tau_{xxx} \frac{\partial y}{\partial x} \frac{\partial^3 y}{\partial x \partial t} + \tau_{y} \frac{\partial y}{\partial t} \frac{\partial^4 y}{\partial x^4} - 6\tau_{xy} \frac{\partial^2 y}{\partial x \partial t} + (6\pi_{yyyy} - 24\nu_{xyy})(\frac{\partial y}{\partial x})^2 \frac{\partial^2 y}{\partial x^2} \\
- 15\nu_{yy} \frac{\partial^2 y}{\partial x^2} \frac{\partial^2 y}{\partial x \partial x^3} - 10\nu_{y} \frac{\partial^3 y}{\partial x \partial x^3} + (4\pi_{xxyy} - 6\nu_{xxxx})(\frac{\partial y}{\partial x})^3 - 12\tau_{xy} \frac{\partial^2 y}{\partial t \partial x} \frac{\partial y}{\partial x} \\
- 4\tau_{yy} \frac{\partial^2 y}{\partial t \partial x} - 3\nu_{y} \frac{\partial^2 y}{\partial x \partial x^3} - 12\tau_{xy} \frac{\partial^2 y}{\partial t \partial x} \frac{\partial y}{\partial x} \frac{\partial^2 y}{\partial x \partial x^3} \\
- 6\tau_{xxyy} \frac{\partial y}{\partial x} \frac{\partial^2 y}{\partial x^2} - 6\tau_{yy} \frac{\partial^2 y}{\partial x^2} \frac{\partial^2 y}{\partial x \partial x^3} + (\pi_{yyyy} - 4\nu_{yyyy})(\frac{\partial y}{\partial x})^4 - 10\nu_{yyyy} \frac{\partial^2 y}{\partial x^2} \frac{\partial^3 y}{\partial x^3} \\
- 4\tau_{xyyy} \frac{\partial^2 y}{\partial x \partial x^2} + 6\tau_{yy} \frac{\partial^2 y}{\partial x^2} \frac{\partial^3 y}{\partial x \partial x} - 4\tau_{yyyy} \frac{\partial^2 y}{\partial x \partial x} \frac{\partial^3 y}{\partial x^3} - 6\nu_{yyyy} \frac{\partial^3 y}{\partial x^3} \\
- \tau_{yyyy} \frac{\partial y}{\partial t} \frac{\partial^3 y}{\partial x^4} + O(\epsilon^2). \]
APPENDIX III

FORMULAE FOR PARTIAL DERIVATIVES FOR THE CLASICAL PROCEDURE OF CHAPTER 4.

\[ \frac{\partial C}{\partial t} = \frac{\partial C}{\partial t} + \varepsilon \left[ \frac{\partial C}{\partial t} + (\xi_c - \eta_t) \frac{\partial C}{\partial t} + \xi_t \frac{\partial T}{\partial t} - \eta_c \left( \frac{\partial C}{\partial t} \right)^2 - \eta_t \frac{\partial C}{\partial t} - \xi_c \frac{\partial C}{\partial t} - \xi_c \frac{\partial C}{\partial t} \right] + O(\varepsilon^2), \]

\[ \frac{\partial T}{\partial t} = \frac{\partial C}{\partial t} + \varepsilon \left[ \chi_c + \chi_t \frac{\partial C}{\partial t} + (\chi_t - \eta_t) \frac{\partial T}{\partial t} - \chi_c \frac{\partial T}{\partial t} - \eta_t \left( \frac{\partial T}{\partial t} \right)^2 - \eta_t \frac{\partial C}{\partial t} - \xi_c \frac{\partial C}{\partial t} \right] + O(\varepsilon^2), \]

\[ \frac{\partial^2 C}{\partial x^2} = \frac{\partial^2 C}{\partial x^2} + \varepsilon \left[ \frac{\partial C}{\partial x} + (2\xi_{xx} - \xi_{xx}) \frac{\partial C}{\partial x} + (\xi_c - 2\xi_x) \frac{\partial C}{\partial x} - 2\eta_x \frac{\partial^2 C}{\partial x \partial t} - \eta_{xx} \frac{\partial C}{\partial t} - 2\eta_c \frac{\partial C}{\partial t} \right] + O(\varepsilon^2), \]

\[ \frac{\partial^2 T}{\partial x^2} = \frac{\partial^2 T}{\partial x^2} + \varepsilon \left[ \frac{\partial C}{\partial x} + (2\xi_{xx} - \xi_{xx}) \frac{\partial C}{\partial x} + (\xi_c - 2\xi_x) \frac{\partial C}{\partial x} - 2\eta_x \frac{\partial^2 T}{\partial x \partial t} - \eta_{xx} \frac{\partial C}{\partial t} - 2\eta_c \frac{\partial C}{\partial t} \right] + O(\varepsilon^2). \]
APPENDIX IV

FORMULAE FOR PARTIAL DERIVATIVES FOR THE PROCEDURES OF CHAPTER 5.

\[
\frac{\partial u}{\partial q_1} = \frac{\partial u}{\partial q} + \varepsilon [\frac{\partial u}{\partial q} + (\xi_u - \eta_u) \frac{\partial u}{\partial q} + \xi_v \frac{\partial v}{\partial q} - \eta_v \frac{\partial v}{\partial q} - \xi_u \frac{\partial v}{\partial p} - \xi_v \frac{\partial u}{\partial p}] + O(\varepsilon^2),
\]

\[
\frac{\partial v_1}{\partial q_1} = \frac{\partial v}{\partial q} + \varepsilon [\chi_q] + \chi_u \frac{\partial u}{\partial q} + (\chi_v - \eta_q) \frac{\partial v}{\partial q} - \eta_u \frac{\partial v}{\partial q} - \eta_v \frac{\partial v}{\partial q} - \xi_u \frac{\partial v}{\partial p} - \xi_v \frac{\partial v}{\partial p} + O(\varepsilon^2),
\]

\[
\frac{\partial u}{\partial p_1} = \frac{\partial u}{\partial p} + \varepsilon [\frac{\partial u}{\partial p} + (\xi_u - \xi_p) \frac{\partial u}{\partial p}] - \eta_p \frac{\partial u}{\partial p} - \eta_u \frac{\partial u}{\partial p} - \eta_v \frac{\partial u}{\partial p} - \xi_u \frac{\partial v}{\partial p} + O(\varepsilon^2),
\]

\[
\frac{\partial v_1}{\partial p_1} = \frac{\partial v}{\partial p} + \varepsilon [\chi_p] + \chi_u \frac{\partial u}{\partial p} + (\chi_v - \xi_p) \frac{\partial v}{\partial p} - \xi_u \frac{\partial u}{\partial p} - \xi_v \frac{\partial v}{\partial p} - \eta_p \frac{\partial u}{\partial q} - \eta_u \frac{\partial u}{\partial q} - \eta_v \frac{\partial u}{\partial q} + O(\varepsilon^2),
\]

\[
\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial t} + \varepsilon [\frac{\partial u}{\partial t} + (\xi_u - \eta_t) \frac{\partial u}{\partial t} + \xi_v \frac{\partial v}{\partial t} - \eta_v \frac{\partial v}{\partial t} - \xi_u \frac{\partial u}{\partial x} - \xi_v \frac{\partial u}{\partial x} + O(\varepsilon^2),
\]

\[
\frac{\partial v_1}{\partial t_1} = \frac{\partial v}{\partial t} + \varepsilon [\chi_t] + \chi_u \frac{\partial u}{\partial t} + (\chi_v - \eta_t) \frac{\partial v}{\partial t} - \eta_u \frac{\partial u}{\partial t} - \eta_v \frac{\partial v}{\partial t} - \xi_u \frac{\partial v}{\partial x} - \xi_v \frac{\partial v}{\partial x} + O(\varepsilon^2),
\]

\[
\frac{\partial u}{\partial x_1} = \frac{\partial u}{\partial x} + \varepsilon [\frac{\partial u}{\partial x} + (\xi_u - \xi_x) \frac{\partial u}{\partial x}] - \eta_x \frac{\partial u}{\partial x} - \eta_u \frac{\partial u}{\partial x} - \eta_v \frac{\partial v}{\partial x} - \xi_u \frac{\partial v}{\partial x} + O(\varepsilon^2),
\]

\[
\frac{\partial v_1}{\partial x_1} = \frac{\partial v}{\partial x} + \varepsilon [\chi_x] + \chi_u \frac{\partial u}{\partial x} + (\chi_v - \xi_x) \frac{\partial v}{\partial x} - \xi_u \frac{\partial v}{\partial x} - \xi_v \frac{\partial v}{\partial x} - \eta_x \frac{\partial u}{\partial t} - \eta_u \frac{\partial u}{\partial t} - \eta_v \frac{\partial v}{\partial t} + O(\varepsilon^2).
\]
APPENDIX V

FORMULAE FOR PARTIAL DERIVATIVES FOR THE CLASSICAL PROCEDURE OF CHAPTER 6.

\[
\frac{\partial u_1}{\partial t} = \frac{\partial u}{\partial t} + \varepsilon [u \frac{\partial \zeta}{\partial t} + (\zeta - \frac{\partial \eta}{\partial t}) \frac{\partial u}{\partial t} - \frac{\partial \zeta}{\partial t} \frac{\partial u}{\partial \xi}] + O(\varepsilon^2),
\]

\[
\frac{\partial v_1}{\partial t} = \frac{\partial v}{\partial t} + \varepsilon [v \frac{\partial \chi}{\partial t} + (\chi - \frac{\partial \eta}{\partial t}) \frac{\partial v}{\partial t} - \frac{\partial \chi}{\partial t} \frac{\partial v}{\partial \xi}] + O(\varepsilon^2),
\]

\[
\frac{\partial w_1}{\partial t} = \frac{\partial w}{\partial t} + \varepsilon [w \frac{\partial \lambda}{\partial t} + (\lambda - \frac{\partial \eta}{\partial t}) \frac{\partial w}{\partial t} - \frac{\partial \lambda}{\partial t} \frac{\partial w}{\partial \xi}] + O(\varepsilon^2),
\]

\[
\frac{\partial^2 u_1}{\partial x_1^2} = \frac{\partial^2 u}{\partial x_1^2} + \varepsilon [(\zeta - 2 \frac{\partial \zeta}{\partial x_1}) \frac{\partial^2 u}{\partial x_1^2} + (2 \frac{\partial \zeta}{\partial x_1} \frac{\partial^2 \zeta}{\partial x_1^2}) \frac{\partial u}{\partial x_1} + u \frac{\partial \zeta}{\partial x_1} \frac{\partial^2 \zeta}{\partial x_1^2} \frac{\partial u}{\partial t} - 2 \frac{\partial \zeta}{\partial x_1} \frac{\partial^2 u}{\partial x_1 \partial t}] + O(\varepsilon^2),
\]

\[
\frac{\partial^2 v_1}{\partial x_1^2} = \frac{\partial^2 v}{\partial x_1^2} + \varepsilon [(\chi - 2 \frac{\partial \chi}{\partial x_1}) \frac{\partial^2 v}{\partial x_1^2} + (2 \frac{\partial \chi}{\partial x_1} \frac{\partial^2 \chi}{\partial x_1^2}) \frac{\partial v}{\partial x_1} + v \frac{\partial \chi}{\partial x_1} \frac{\partial^2 \chi}{\partial x_1^2} \frac{\partial v}{\partial t} - 2 \frac{\partial \chi}{\partial x_1} \frac{\partial^2 v}{\partial x_1 \partial t}] + O(\varepsilon^2),
\]

\[
\frac{\partial^2 w_1}{\partial x_1^2} = \frac{\partial^2 w}{\partial x_1^2} + \varepsilon [(\lambda - 2 \frac{\partial \lambda}{\partial x_1}) \frac{\partial^2 w}{\partial x_1^2} + (2 \frac{\partial \lambda}{\partial x_1} \frac{\partial^2 \lambda}{\partial x_1^2}) \frac{\partial w}{\partial x_1} + w \frac{\partial \lambda}{\partial x_1} \frac{\partial^2 \lambda}{\partial x_1^2} \frac{\partial w}{\partial t} - 2 \frac{\partial \lambda}{\partial x_1} \frac{\partial^2 w}{\partial x_1 \partial t}] + O(\varepsilon^2).
\]
CHAPTER 6: APPLICATIONS OF A SIMPLER TRANSFORMATION.

6.1: Similarity Solutions To The Triples System Representing A One-Dimensional Case Of Diffusion In The Presence Of Three Diffusion Paths.

In this section one-parameter transformation groups preserving the invariance of the triple system (1.14) will be presented along with the resulting similarity solutions for the triple system (1.14). A special transformation of the following form is considered, namely

\[ x_1 = x_1(x, t, \varepsilon) = x + \varepsilon \xi(x, t) + O(\varepsilon^2), \]
\[ t_1 = t_1(x, t, \varepsilon) = t + \varepsilon \eta(x, t) + O(\varepsilon^2), \]
\[ u_1 = u_1(x, t, \varepsilon) u = u + \varepsilon \zeta(x, t) u + O(\varepsilon^2), \]
\[ v_1 = v_1(x, t, \varepsilon) v = v + \varepsilon \chi(x, t) v + O(\varepsilon^2), \]
\[ w_1 = w_1(x, t, \varepsilon) w = w + \varepsilon \lambda(x, t) w + O(\varepsilon^2). \] (6.1)

It should be noted that solutions obtained under the transformation (6.1) take the form of linear combinations of exponential terms \( e^{(ax + bt)} \), where \( a \) and \( b \) denote arbitrary constants.

If the transformation (6.1) preserves the invariance of the triple system (1.14) and if \( u = \phi(x, t), v = \psi(x, t) \) and \( w = \omega(x, t) \); then from \( u_1 = \phi(x_1, t_1), v_1 = \psi(x_1, t_1) \) and \( w_1 = \omega(x_1, t_1) \), equating terms of order \( \varepsilon \) gives

\[ \xi(x, t) \frac{\partial u}{\partial x} + \eta(x, t) \frac{\partial u}{\partial t} = \zeta(x, t) u, \]
\[ \xi(x, t) \frac{\partial v}{\partial x} + \eta(x, t) \frac{\partial v}{\partial t} = \chi(x, t) v, \]
\[ \xi(x, t) \frac{\partial w}{\partial x} + \eta(x, t) \frac{\partial w}{\partial t} = \lambda(x, t) w. \] (6.2)
The similarity variables $u$, $v$ and $w$ are directly obtained from the solutions of equations (6.2) which correspond to the functional forms of the similarity solutions $u(x, t)$, $v(x, t)$ and $w(x, t)$ for the tripled system of equations (1.14). The groups leaving the tripled system (1.14) invariant are determined by the classical and non-classical procedures, which have been outlined in previous chapters of this thesis. Assuming that $D_i$ and $a_{ij}$ $\forall$ $i, j \in \{1, 2, 3\}$ in the system (1.14) are all positive constants, it may be deduced by the classical procedure (outlined in previous chapters of this thesis) that a classical group for the system (1.14) under the transformation (6.1) is

$$
\zeta(x, t) = c_2, \quad \eta(x, t) = c_1, \quad \zeta(x, t) = \lambda(x, t) = c_3; \quad (6.3)
$$

where $c_1$, $c_2$ and $c_3$ all represent arbitrary constants.

From the equations (6.2), the classical group (6.3) and by assigning $u = z_1$, $v = z_2$ and $w = z_3$, a method similar to that used in earlier chapters of this thesis was used to ultimately derive similarity solutions of the tripled system of equations (1.14), namely

$$
z_i(x, t) = \sum_{j=1}^{6} d_{ij} \exp[k_j c_2 t + \frac{c_3}{c_2} - k_j c_1] x, \quad \forall \ i \in \{1, 2, 3\}; \quad (6.4)
$$

where $c_1$, $d_{ij}$ and $k_j$ are arbitrary constants $\forall \ i \in \{1, 2, 3\}$ and $\forall \ j \in \{1, \ldots, 6\}$.

Following the non-classical method (outlined in previous chapters of this thesis) and using (6.1), the terms $A(x, t)$, $B(x, t)$, $C(x, t)$ and $G(x, t)$ are introduced and defined by

$$
A(x, t) = \frac{\zeta(x, t)}{\eta(x, t)}, \quad B(x, t) = \frac{\zeta(x, t)}{\eta(x, t)}, \quad C(x, t) = \frac{\chi(x, t)}{\eta(x, t)}, \quad G(x, t) = \frac{\lambda(x, t)}{\eta(x, t)}; \quad (6.5)
$$

so that equations (6.2) yield

$$
\frac{\partial u}{\partial t} = Au - B \frac{\partial u}{\partial x}, \quad \frac{\partial v}{\partial t} = Cv - B \frac{\partial v}{\partial x}, \quad \frac{\partial w}{\partial t} = Gw - B \frac{\partial w}{\partial x}. \quad (6.6)
$$

It is assumed that $D_i$ and $a_{ij}$ $\forall$ $i, j \in \{1, 2, 3\}$ in the system (1.14) are all positive constants. In order for non-trivial transformations to be obtained, only the case $\eta(x, t) \neq 0$ is considered. The non-classical group of (1.14) is determined by the following system of equations, namely
\[
\frac{\partial A}{\partial t} = D_1 \frac{\partial^2 A}{\partial x^2} - 2(A + a_{11}) \frac{\partial B}{\partial x},
\]
\[
A = C + 2 \frac{\partial B}{\partial x},
\]
\[
A = G + 2 \frac{\partial B}{\partial x},
\]
\[
\frac{\partial B}{\partial t} = D_1 \frac{\partial^2 B}{\partial x^2} - 2B \frac{\partial B}{\partial x} - 2D_1 \frac{\partial A}{\partial x},
\]
\[
\frac{\partial C}{\partial t} = D_2 \frac{\partial^2 C}{\partial x^2} - 2(C + a_{22}) \frac{\partial B}{\partial x},
\]
\[
C = A + 2 \frac{\partial B}{\partial x},
\]
\[
C = G + 2 \frac{\partial B}{\partial x},
\]
\[
\frac{\partial B}{\partial t} = D_2 \frac{\partial^2 B}{\partial x^2} - 2B \frac{\partial B}{\partial x} - 2D_2 \frac{\partial C}{\partial x},
\]
\[
\frac{\partial G}{\partial t} = D_3 \frac{\partial^2 G}{\partial x^2} - 2(G + a_{33}) \frac{\partial B}{\partial x},
\]
\[
G = A + 2 \frac{\partial B}{\partial x},
\]
\[
G = C + 2 \frac{\partial B}{\partial x},
\]
\[
\frac{\partial B}{\partial t} = D_3 \frac{\partial^2 B}{\partial x^2} - 2B \frac{\partial B}{\partial x} - 2D_3 \frac{\partial G}{\partial x}.
\]

(6.7)

In order to reconcile (6.72) and (6.73) as well as (6.76) and (6.77), it is required that

\[
A(x, t) = C(x, t) = G(x, t) = \Phi(x, t);
\]

(6.8)

where \( \Phi(x, t) \) is an unknown function of \( x \) and \( t \).

From (6.710) and (6.8), it is implied that

\[
B(x, t) = B(t).
\]

(6.9)

Reconciling (6.74) and (6.78) while making use of the results (6.8) and (6.9) yields \((D_1 - D_2)\Phi_x = 0\) and so as to avoid restricting \( D_1 \) and \( D_2 \), it is required that

\[
\Phi(x, t) = \Phi(t).
\]

(6.10)

From the results (6.8), (6.9) and (6.10), it follows that (6.74) yields \( B'(t) = 0 \) and therefore,
\[ B(t) = k_1; \]  
\[ (6.11) \]

where \( k_1 \) denotes an arbitrary constant.

It is implied from \((6.7_1),(6.8),(6.10)\) and \((6.11)\) that \( \Phi'(t) = 0 \) and thus,

\[ \Phi(t) = k_2; \]
\[ (6.12) \]

where \( k_2 \) is a further arbitrary constant.

Hence it is evident from \((6.8),(6.10)\) and \((6.12)\) that

\[ A(x, t) = C(x, t) = G(x, t) = k_2. \]
\[ (6.13) \]

By inspection, it is clear that the results \((6.11)\) and \((6.13)\) identically satisfy equations \((6.7)\) and in fact are the only non-trivial solutions of equations \((6.7)\).

Upon assigning \( k_1 = m_2 \) and \( k_2 = m_1 \), where \( m_1 \) and \( m_2 \) are unrestricted constants, it may be concluded that a non-classical group for the system \((1.14)\) subject to the transformation \((6.1)\) is

\[ A(x, t) = m_1, \]
\[ B(x, t) = m_2, \]
\[ C(x, t) = m_1, \]
\[ G(x, t) = m_1. \]
\[ (6.14) \]

From equations \((6.6)\) in conjunction with the non-classical group \((6.14)\), the following equations are obtained, namely

\[ \frac{\partial u}{\partial t} + m_2 \frac{\partial u}{\partial x} = m_1 u, \quad \frac{\partial v}{\partial t} + m_2 \frac{\partial v}{\partial x} = m_1 v, \quad \frac{\partial w}{\partial t} + m_2 \frac{\partial w}{\partial x} = m_1 w. \]
\[ (6.15) \]

Via the definitions \((6.5)\), it is evident that the classical group \((6.3)\) with the relations \( \frac{c_2}{c_1} = m_1 \) and \( \frac{c_2}{c_1} = m_2 \) yields the non-classical group \((6.14)\). Therefore, similarity solutions of the tripled system of equations \((1.14)\) associated with the group \((6.14)\) are equivalent to the similarity solutions \((6.4)\) corresponding to the group \((6.3)\), obtained by the classical procedure. Evidently, under the simple transformation \((6.1)\), there are no solutions to the system \((1.14)\) obtainable from the non-classical approach which cannot be retrieved using the classical method.
6.2: Similarity Solutions To The Coupled System Representing A One-Dimensional Case Of The General Linear System Of Coupled Diffusion Equations With Cross-Effects.

In this section, one-parameter transformation groups leaving the coupled system (1.16) invariant will be presented together with the corresponding similarity solutions for the coupled system (1.16). A special transformation of the following form is considered, namely

\[ x_1 = x_1(x, t, \varepsilon) = x + \varepsilon \xi(x, t) + O(\varepsilon^2), \]
\[ t_1 = t_1(x, t, \varepsilon) = t + \varepsilon \eta(x, t) + O(\varepsilon^2), \]
\[ u_1 = u_1(x, t, \varepsilon)u = u + \varepsilon \zeta(x, t)u + O(\varepsilon^2), \]
\[ v_1 = v_1(x, t, \varepsilon)v = v + \varepsilon \chi(x, t)v + O(\varepsilon^2). \]  

(6.16)

It should be mentioned that solutions obtained using transformation (6.16) have the form of linear combinations of exponential terms \( e^{(ax + bt)} \), where \( a \) and \( b \) denote arbitrary constants.

If the invariance of the coupled system (1.16) is maintained by the transformation (6.16) and if \( u = \phi(x, t) \), \( v = \psi(x, t) \); then from \( u_1 = \phi(x_1, t_1) \) and \( v_1 = \psi(x_1, t_1) \), equating terms of order \( \varepsilon \) gives

\[ \xi(x, t) \frac{\partial u}{\partial x} + \eta(x, t) \frac{\partial u}{\partial t} = \zeta(x, t)u, \]
\[ \xi(x, t) \frac{\partial v}{\partial x} + \eta(x, t) \frac{\partial v}{\partial t} = \chi(x, t)v. \]  

(6.17)

The similarity variables \( u \) and \( v \) are obtained from the solutions of equations (6.17) which correspond to the functional forms of the similarity solutions \( u(x, t) \) and \( v(x, t) \) for the coupled system of equations (1.16). The groups preserving the invariance of the coupled system (1.16) are obtained from the classical and non-classical procedures, described in earlier chapters of this thesis. By the assumption that \( A_i, B_i, D_i \) and \( E_i \forall i \in \{1, 2\} \) in the system (1.16) are positive constants
and that $D_1D_2 - E_1E_2 \neq 0$, it may be concluded via the classical procedure that a classical group for the system (1.16) associated with the transformation (6.16) is

\[ \xi(x, t) = n_1, \]
\[ \eta(x, t) = n_2, \]
\[ \zeta(x, t) = n_3, \]
\[ \chi(x, t) = n_3; \] (6.18)

where $n_1$, $n_2$ and $n_3$ all represent arbitrary constants.

From the equations (6.17), the classical group (6.18) and upon assigning $u = z_1$ and $v = z_2$, a method similar to that employed in previous chapters of this thesis was used to eventually derive similarity solutions of the coupled system of equations (1.16), namely

\[ z_i(x, t) = \sum_{j=1}^{4} d_{ij} \exp[k_j n_1 t + \left( \frac{n_3}{n_1} - k_j n_2 \right) x], \forall i \in \{1, 2\}; \] (6.19)

where $d_{ij}$ and $k_j$ are arbitrary constants $\forall i \in \{1, 2\}$ and $\forall j \in \{1, ..., 4\}$.

By the non-classical method (outlined in previous chapters of this thesis) and using (6.16), the terms $A(x, t)$, $B(x, t)$ and $C(x, t)$ are introduced and defined by

\[ A(x, t) = \frac{\xi(x, t)}{\eta(x, t)}, \quad B(x, t) = \frac{\zeta(x, t)}{\eta(x, t)}, \quad C(x, t) = \frac{\chi(x, t)}{\eta(x, t)}; \] (6.20)

so that equations (6.17) yield

\[ \frac{\partial u}{\partial t} = Au - B \frac{\partial u}{\partial x}, \quad \frac{\partial v}{\partial t} = Cv - B \frac{\partial v}{\partial x}. \] (6.21)

It is assumed that $A_i, B_i, D_i$ and $E_i \forall i \in \{1, 2\}$ in the system (1.16) are all positive constants and that $D_1D_2 - E_1E_2 \neq 0$. In order to obtain non-trivial transformations, only the case $\eta(x, t) \neq 0$ is considered. The non-classical group of (1.16) is determined by the following system of equations, namely
\[
\frac{\partial A}{\partial t} = D_1 \frac{\partial^2 A}{\partial x^2} - 2(A + A_1) \frac{\partial B}{\partial x} + \frac{[(A + A_1)E_2 + B_2D_1]E_1}{(D_1D_2 - E_1E_2)} (A - C),
\]
\[
\frac{\partial B}{\partial t} = D_1 \frac{\partial^2 B}{\partial x^2} - 2B \frac{\partial B}{\partial x} - 2D_1 \frac{\partial A}{\partial x} + \frac{E_1E_2}{(D_1D_2 - E_1E_2)} (A - C)B,
\]
\[
2B_1 \frac{\partial B}{\partial x} = \frac{[(C + A_2)E_1 + B_1D_2]D_1}{(D_1D_2 - E_1E_2)} (A - C) - E_1 \frac{\partial^2 C}{\partial x^2},
\]
\[
\frac{\partial^2 B}{\partial x^2} = 2 \frac{\partial C}{\partial x} + \frac{D_1}{(D_1D_2 - E_1E_2)} (A - C)B,
\]
\[
\frac{\partial C}{\partial t} = D_2 \frac{\partial^2 C}{\partial x^2} - 2(C + A_2) \frac{\partial B}{\partial x} + \frac{[(C + A_2)E_1 + B_1D_2]E_2}{(D_1D_2 - E_1E_2)} (C - A),
\]
\[
\frac{\partial B}{\partial t} = D_2 \frac{\partial^2 B}{\partial x^2} - 2B \frac{\partial B}{\partial x} - 2D_2 \frac{\partial C}{\partial x} - \frac{E_1E_2}{(D_1D_2 - E_1E_2)} (A - C)B,
\]
\[
2B_2 \frac{\partial B}{\partial x} = \frac{[(A + A_1)E_2 + B_2D_1]D_2}{(D_1D_2 - E_1E_2)} (C - A) - E_2 \frac{\partial^2 A}{\partial x^2},
\]
\[
\frac{\partial^2 B}{\partial x^2} = 2 \frac{\partial A}{\partial x} + \frac{D_2}{(D_1D_2 - E_1E_2)} (C - A)B.
\]

In order to reconcile equations (6.222) and (6.226), it is required that
\[
(D_1 - D_2) \frac{\partial^2 B}{\partial x^2} - 2(D_1 \frac{\partial A}{\partial x} - D_2 \frac{\partial C}{\partial x}) + 2 \frac{E_1E_2}{(D_1D_2 - E_1E_2)} (A - C)B = 0. \tag{6.23}
\]

Adding \(D_1\) times (6.228) to - \(D_2\) times (6.224) yields
\[
(D_1 - D_2) \frac{\partial^2 B}{\partial x^2} - 2(D_1 \frac{\partial A}{\partial x} - D_2 \frac{\partial C}{\partial x}) + 2 \frac{D_1D_2}{(D_1D_2 - E_1E_2)} (A - C)B = 0. \tag{6.24}
\]

For (6.23) and (6.24) to be reconciled, it is required that \((A - C)B = 0\) and in order to avoid trivialising \(B(x, t)\), it follows that
\[
A(x, t) = C(x, t) = \Psi(x, t); \tag{6.25}
\]

where \(\Psi(x, t)\) denotes an unknown function of \(x\) and \(t\).

In conjunction with the result (6.25), equations (6.22) simplify to yield
\[ \frac{\partial \Psi}{\partial t} = D_1 \frac{\partial^2 \Psi}{\partial x^2} - 2(A_1 + \Psi) \frac{\partial B}{\partial x}, \]
\[ \frac{\partial B}{\partial t} = D_1 \frac{\partial^2 B}{\partial x^2} - 2B \frac{\partial B}{\partial x} - 2D_1 \frac{\partial \Psi}{\partial x}, \]
\[ 2B_1 \frac{\partial B}{\partial x} = -E_1 \frac{\partial^2 \Psi}{\partial x^2}, \]
\[ \frac{\partial^2 B}{\partial x^2} = 2 \frac{\partial \Psi}{\partial x}, \] (6.26)
\[ \frac{\partial \Psi}{\partial t} = D_2 \frac{\partial^2 \Psi}{\partial x^2} - 2(A_2 + \Psi) \frac{\partial B}{\partial x}, \]
\[ \frac{\partial B}{\partial t} = D_2 \frac{\partial^2 B}{\partial x^2} - 2B \frac{\partial B}{\partial x} - 2D_2 \frac{\partial \Psi}{\partial x}, \]
\[ 2B_2 \frac{\partial B}{\partial x} = -E_2 \frac{\partial^2 \Psi}{\partial x^2}. \]

Multiplying (6.263) by \( E_2 \) and (6.267) by \( E_1 \), then reconciling the resulting equations gives rise to the relation \( (B_1 E_2 - B_2 E_1) \frac{\partial B}{\partial x} = 0 \), resulting in two possible cases, namely (I). \( B(x, t) = B(t) \) and (II). \( B_1 E_2 = B_2 E_1 \).

**Case (I):** \( B(x, t) = B(t) \).

In this case, equations (6.26) simplify to yield
\[ \frac{\partial \Psi}{\partial t} = D_1 \frac{\partial^2 \Psi}{\partial x^2}, \]
\[ B'(t) = -2D_1 \frac{\partial \Psi}{\partial x}, \]
\[ \frac{\partial \Psi}{\partial x} = 0, \] (6.27)
\[ \frac{\partial^2 \Psi}{\partial t^2} = D_2 \frac{\partial^2 \Psi}{\partial x^2}, \]
\[ B'(t) = -2D_2 \frac{\partial \Psi}{\partial x}. \]
From equation (6.27\textsubscript{3}), it follows that
\[ \Psi(x, t) = \Psi(t). \] (6.28)

By making use of (6.28), it follows from (6.27\textsubscript{1}) and (6.27\textsubscript{4}) that \( \Psi'(t) = 0 \) and so,
\[ \Psi(t) = c_1; \] (6.29)

where \( c_1 \) denotes an arbitrary constant.

Inspection of (6.27\textsubscript{2}) and (6.27\textsubscript{5}) in association with results (6.28) and (6.29) reveals \( B'(t) = 0 \), implying
\[ B(t) = c_2; \] (6.30)

where \( c_2 \) represents a further arbitrary constant.

By inspection, it is evident that the results (6.29) and (6.30) identically satisfy equations (6.26). Thus, together with the result (6.25), the results (6.28), (6.29) and (6.30) also identically satisfy equations (6.22). In conclusion, a non-classical group for the coupled system of equations (1.16) under the transformation (6.16) and in association with case (I) is given by
\[ A(x, t) = c_1, \quad B(x, t) = c_2, \quad C(x, t) = c_1. \] (6.31)

**Case (II):** \( B_1E_2 = B_2E_1 \) or equivalently, \( B_2 = \frac{B_1E_2}{E_1} \).

Equations (6.26) then simplify to yield
\[
\begin{align*}
\frac{\partial \Psi}{\partial t} &= D_1 \frac{\partial^2 \Psi}{\partial x^2} - 2(A_1 + \Psi) \frac{\partial B}{\partial x}, \\
\frac{\partial B}{\partial t} &= D_1 \frac{\partial^2 B}{\partial x^2} - 2B \frac{\partial B}{\partial x} - 2D_1 \frac{\partial \Psi}{\partial x}, \\
2B_1 \frac{\partial B}{\partial x} &= -E_1 \frac{\partial^2 \Psi}{\partial x^2}, \\
\frac{\partial^2 B}{\partial x^2} &= 2 \frac{\partial \Psi}{\partial x}, \quad (6.32) \\
\frac{\partial \Psi}{\partial t} &= D_2 \frac{\partial^2 \Psi}{\partial x^2} - 2(A_2 + \Psi) \frac{\partial B}{\partial x}, \\
\frac{\partial B}{\partial t} &= D_2 \frac{\partial^2 B}{\partial x^2} - 2B \frac{\partial B}{\partial x} - 2D_2 \frac{\partial \Psi}{\partial x}.
\end{align*}
\]

From (6.32\textsubscript{2}), (6.32\textsubscript{4}) and (6.32\textsubscript{6}), it is clear that
\[
\frac{\partial B}{\partial t} = -2B \frac{\partial B}{\partial x} .
\] (6.33)

Reconciling (6.321) and (6.325) requires \( \frac{\partial^2 \Psi}{\partial x^2} = 2 \frac{(A_1 - A_2)}{(D_1 - D_2)} \frac{\partial B}{\partial x} \) and so,

\[
\frac{\partial \Psi}{\partial x} = 2 \frac{(A_1 - A_2)}{(D_1 - D_2)} B(x, t) + a(t) ;
\] (6.34)

where \( a \) denotes an unknown function of \( t \). Equation (6.323) implies that

\[
\frac{\partial \Psi}{\partial x} = -2 \frac{B_1}{E_1} B(x, t) + b(t) ;
\] (6.35)

where \( b \) is a second unknown function of \( t \). Reconciling (6.34) and (6.35) requires

\[
2 \left( \frac{B_1}{E_1} + \frac{(A_1 - A_2)}{(D_1 - D_2)} \right) B(x, t) = (b - a)(t) ;
\] (6.36)

giving rise to two possibilities: (II)(i). \( B(x, t) = B(t) \) and (II)(ii). \((D_1 - D_2)B_1 + (A_1 - A_2)E_1 = 0 \), in which case \( b(t) = a(t) = \phi(t) \) where \( \phi \) is some unknown function of \( t \).

It is noted that case (II)(i) leads to the non-classical group as in case (I) being obtained.

Case (II)(ii): (\(D_1 - D_2\))B_1 + (A_1 - A_2)E_1 = 0 or \( B_1 = \frac{(A_2 - A_1)}{(D_1 - D_2)} E_1 \);

and \( b(t) = a(t) = \phi(t) \).

Equation (6.34) (which is also consistent with (6.35) by case (II)(ii)) then implies

\[
\frac{\partial \Psi}{\partial x} = 2 \frac{(A_1 - A_2)}{(D_1 - D_2)} B(x, t) + \phi(t) .
\] (6.37)

From (6.32), (6.33) and (6.37), the following system of equations is obtained, namely
\[ \frac{\partial \Psi}{\partial t} = \frac{2}{(D_1 - D_2)} \{ A_1 D_2 - A_2 D_1 - (D_1 - D_2) \Psi \} \frac{\partial B}{\partial x}, \]

\[ \frac{\partial B}{\partial t} = -2B \frac{\partial B}{\partial x}, \]

\[ \frac{\partial^2 \Psi}{\partial x^2} = 2 \frac{(A_1 - A_2) \partial B}{(D_1 - D_2) \partial x}, \]

\[ \frac{\partial^2 B}{\partial x^2} + 4 \frac{(A_2 - A_1)}{(D_1 - D_2)} B(x, t) - 2\phi(t) = 0. \] (6.38)

Differentiating equation (6.38) partially with respect to \( x \) gives

\[ \frac{\partial^3 B}{\partial x^3} + 4 \frac{(A_2 - A_1)}{(D_1 - D_2)} \frac{\partial B}{\partial x} = 0; \] (6.39)

which can be solved by the method of linear differential operators (see [8]) to yield the general solution

\[ B(x, t) = f(t) + \exp[-2i \sqrt{\frac{A_2 - A_1}{D_1 - D_2}} x]g(t) + \exp[2i \sqrt{\frac{A_2 - A_1}{D_1 - D_2}} x]h(t); \] (6.40)

where \( f, g \) and \( h \) all denote arbitrary functions of \( t \).

Substituting (6.40) into (6.38) yields

\[ \phi(t) = 2 \frac{(A_2 - A_1)}{(D_1 - D_2)} f(t). \] (6.41)

From (6.38) and (6.40), it follows that

\[ f'(t) + [g'(t) - 2icf(t)g(t)]e^{icx} + [h'(t) + 2icf(t)h(t)]e^{icx} \]

\[ - 2ic[g(t)]^2e^{-2icx} + 2ic[h(t)]^2e^{2icx} = 0; \] (6.42)

where \( c = 2 \sqrt{\frac{A_2 - A_1}{D_1 - D_2}} \). Equating to zero the terms in (6.42) not containing \( x \) gives \( f'(t) = 0 \) and so,

\[ f(t) = d_1; \] (6.43)
where $d_1$ is an arbitrary constant. It is assumed that $A_1 \neq A_2$ and so $c \neq 0$. Consequently, equating to zero the coefficients of $e^{-2icx}$ and $e^{2icx}$ gives

$$g(t) = 0, \quad h(t) = 0.$$  \hfill (6.44)

The results (6.43) and (6.44) automatically satisfy equation (6.42). Thus (6.40) and (6.41) yield

$$B(x, t) = d_1, \quad \phi(t) = 2 \frac{(A_2 - A_1)}{(D_1 - D_2)} d_1.$$  \hfill (6.45)

From (6.38) and (6.45), it is implied that $\frac{\partial \psi}{\partial t} = 0$ and so,

$$\Psi(x, t) = \Psi(x).$$  \hfill (6.46)

By (6.37), (6.45) and (6.46), we have $\Psi'(x) = 0$ and so,

$$\Psi(x) = d_2.$$  \hfill (6.47)

where $d_2$ is a second arbitrary constant. The results (6.45) and (6.47) satisfy equations (6.26), (6.32) and (6.38) identically. Together with (6.25), the results (6.45), (6.46) and (6.47) identically satisfy equations (6.22). By letting $d_1 = c_2$ and $d_2 = c_1$, the non-classical group (6.31) emerges. It is thus concluded that the special case $B_1E_2 = B_2E_1$ does not lead to additional symmetries, but only to the non-classical group as in case (I) being obtained.

From equations (6.21) in association with the group (6.31), the following equations are obtained, namely

$$\frac{\partial u}{\partial t} + c_2 \frac{\partial u}{\partial x} = c_1 u, \quad \frac{\partial v}{\partial t} + c_2 \frac{\partial v}{\partial x} = c_1 v.$$  \hfill (6.48)

By (6.20), it is clear that the classical group (6.18) with the relations $\frac{n_3}{n_2} = c_1$ and $\frac{n_1}{n_2} = c_2$ gives rise to the non-classical group (6.31). Consequently, similarity solutions of the coupled system of equations (1.16) associated with the group (6.31) are equivalent to the similarity solutions (6.19) corresponding to the group (6.18), derived from the classical procedure. Evidently, under the special transformation (6.16), no solutions to the coupled system (1.16) can be obtained from the non-classical approach which are not recoverable from the classical procedure.
CHAPTER 7: CONCLUSION.

For the one-dimensional case of a coupled system of diffusion equations obtained by Aifantis and Hill [5] and which was discussed in Chapter 2, a classical group was obtained in terms of two functions both of which are solutions to this one-dimensional case. It follows that any solution to this one-dimensional case generates a classical group for it, hence raising the possibility that an infinite variety of similarity solutions to this system under discussion can be generated in this manner. Similarity solutions to this system were derived for particular cases of the classical and non-classical groups obtained. It was observed that under certain conditions, the non-classical group for this system gives rise to the classical group mentioned above. It can be conjectured that the solutions for this system obtained by the non-classical method are recoverable from the classical procedure owing to the fact that non-classical similarity solutions do not exist for the heat equation (Arrigo, Goard and Broadbridge [7]) and that solutions for this system can be expressed in terms of solutions for the heat equation (Aifantis and Hill [5]). Doubtless, more remains to be discovered about this.

Uncoupling this system in Chapter 3 yielded results directly reflecting those derived from the coupled system of Chapter 2. A simplification enabled the classical group of the uncoupled system to be retrieved from the non-classical group of the uncoupled system, implying that subject to this simplification, similarity solutions arising from the non-classical method are identical to those derived via the classical procedure.

The results derived for the system investigated in Chapter 2 were reflected by those obtained in seeking to derive a non-classical group for the system representing a one-dimensional case of a reaction-diffusion system of equations obtained by Forbes [15] (in his investigations of the formation of stationary patterns of temperature and chemical concentration in a model of a burning process proposed by Sal'nikov [29]), studied in Chapter 4. This reflection lends credence to the statement made that the coupled system of equations obtained by Forbes [15] (in his investigations of stability of non-linear steady state patterns of chemical concentration and temperature) is a generalisation of the equations considered in Chapter 2 of this thesis. For the system under scrutiny in Chapter 4, a classical group consisting entirely of constants was derived and yielded similarity solutions. These solutions are a generalisation of the solutions of the one-
dimensional steady equations obtained by Forbes [15] and which govern the form of the concentration and temperature profiles in the reaction-diffusion system of equations Forbes [15] derived from the aforementioned model of a burning process. Subject to the Neumann conditions used by Forbes [15], the linear version of the system discussed in Chapter 4 was found to have solutions which may be read directly from those in Lee and Hill [22] relating to the general linear system of coupled diffusion equations with cross-effects of which the equations discussed in Section 6.2 of Chapter 6 form a one-dimensional case. Some of these solutions may be generated by results similar to those presented in Chapter 2. If $\sigma = \alpha = 0$ in the system studied in Chapter 4, a specialisation of the resulting classical group gives rise to similarity solutions corresponding to the steady state solution for the Sal'nikov model [29].

The introduction of characteristic coordinates into the classical approach to the coupled system of semi-linear equations (5.1) (which is an alternative form of the coupled system (1.13)) treated in Chapter 5 led to the derivation of a classical system (containing arbitrary functions) for this system. This classical group ultimately gave rise to the general similarity solution (in terms of arbitrary functions) for the coupled system (1.13), equivalent to the solution presented by Hasimoto [16]. In conjunction with the transformation (1.3), an alternative approach to the system (1.13) using the one-parameter group method employed in previous chapters of this thesis also enabled the general similarity solution (in terms of arbitrary functions) for the coupled system (1.13) to be retrieved.

For a one-dimensional case of the simple, discrete random walk model developed by Hill [17] for diffusion in the presence of three diffusion paths, and a one-dimensional case of the general linear system of coupled diffusion equations with cross-effects, discussed in Sections 6.1 and 6.2 respectively of Chapter 6, simple transformations of the forms (1.15) and (1.2) respectively resulted in one-parameter groups consisting solely of constants. These groups merely generated solutions in the form of linear combinations of exponential terms $e^{(ax + bt)}$ where $a$ and $b$ are arbitrary constants. It is evident that by using these simple transformations, no solutions of either system under discussion are obtainable from the non-classical approach which are not recoverable via the classical method. Thus for the coupled system of Chapter 2 under the simple transformation (1.2), there are no solutions from the non-classical approach which cannot be obtained via the classical method.
SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS
AND GROUP METHODS

TANYA L.M. CHOW

A thesis submitted in fulfilment
of the requirements for the degree of
Master of Science (Honours)

University of Western Sydney (Macarthur)
1996
I declare that no part of this thesis has been submitted for a higher degree at this or at any other institution and that the work is my own.

(Tanya Lye Ming Chow)
PLEASE NOTE

The greatest amount of care has been taken while scanning this thesis,

and the best possible result has been obtained.
ABSTRACT

This thesis is concerned with the derivation of similarity solutions for one-dimensional coupled systems of reaction-diffusion equations, a semi-linear system and a one-dimensional tripled system.

The first area of research in this thesis involves a coupled system of diffusion equations for the existence of two distinct families of diffusion paths. Constructing one-parameter transformation groups preserving the invariance of this system of equations enables similarity solutions for this coupled system to be derived via the classical and non-classical procedures outlined in Chapter 2 of this thesis. This system of equations is then uncoupled in the hope of recovering further similarity solutions for the system. Once again, one-parameter groups leaving the uncoupled system invariant are obtained, enabling similarity solutions for the system to be elicited.

A one-dimensional pattern formation in a model of burning forms the next component of this thesis. In this model, a substance undergoes a two-stage decay via an intermediate chemical to form a final product. The second stage takes place at a temperature-sensitive rate and results in the production of heat. The effects of thermal induction are considered and it is assumed that the intermediate chemical is capable of diffusing through the decomposing substrate. The governing equations thus form a reaction-diffusion system thus making it possible for spatially inhomogeneous behaviour to arise. The primary focus of this area is the determination of similarity solutions for this reaction-diffusion system by means of one-parameter transformation group methods. Consequently, similarity solutions which are a generalisation of the solutions of the one-dimensional steady equations derived by Forbes [15] are deduced.

Attention in this thesis is then directed toward a semi-linear coupled system representing a predator-prey relationship. Two approaches to solving this system are made using the classical procedure, leading to one-parameter transformation groups which are instrumental in eliciting the general similarity solution for this system. The solution obtained is equivalent to that presented by Hasimoto [16].

A tripled system of equations representing a one-dimensional case of diffusion in the presence of three diffusion paths constitutes the next theme of this thesis. In association with the classical and non-classical procedures, the derivation of one-parameter transformation groups leaving this system invariant enables similarity solutions for this system to be deduced.
The final strand of this thesis involves a one-dimensional case of the general linear system of coupled diffusion equations with cross-effects for which one-parameter transformation group methods are once more employed. The one-parameter groups constructed for this system prove instrumental in enabling the attainment of similarity solutions for this system to be accomplished.
PREFACE

In this thesis, the similarity solutions obtained for the various systems of partial differential equations under consideration form an original contribution. Where the work of other authors has been used, this has always been specifically acknowledged in the relevant sections of the text.
ACKNOWLEDGMENTS

I am indebted to my supervisor Dr. Alec Lee whose intellectual stimulation, enthusiasm, good humour and boundless patience have guided my sojourn in the realms of research for the past two and a half years.

My grateful thanks is also due to my father, Sew Wah Chow, for his unabating faith in me. Without his unfailing support, both morally and financially, the last few years would have been harrowing indeed. His positive spirit has imbued me with confidence to face life’s challenges.

I would like to express my warm appreciation of my mother, Indira Chow, for her support of me through the difficult times and for the understanding she has displayed.

Furthermore I thank my deceased sweetheart, Lindsay Welsh for his constant love, devotion and absolute confidence in me during the sixteen halcyon months we had together.

All these factors have combined to make this thesis a reality.
This thesis is dedicated with love and respect to my father, Sew Wah Chow.

"...O how vast the shores of learning,
There are still uncharted seas,
And they call to bold adventure,
Those who turn from sloth and ease..."

Excerpt from "A Student's Prayer"

Author unknown
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>CONTENTS</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>SOLUTIONS TO THE COUPLED SYSTEM OF REACTION - DIFFUSION EQUATIONS</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>2.1 Introduction</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>2.2 The Classical Procedure</td>
<td>11</td>
</tr>
<tr>
<td></td>
<td>2.3 The Non- Classical Procedure</td>
<td>24</td>
</tr>
<tr>
<td>3</td>
<td>SOLUTIONS TO THE UNCOUPLED SYSTEM OF REACTION - DIFFUSION EQUATIONS</td>
<td>37</td>
</tr>
<tr>
<td></td>
<td>3.1 Uncoupling (1.1) By Equating To Zero The Determinant Of Its Equivalent Matrix Form</td>
<td>37</td>
</tr>
<tr>
<td></td>
<td>3.2 The Classical Procedure</td>
<td>38</td>
</tr>
<tr>
<td></td>
<td>3.3 The Non- Classical Procedure</td>
<td>48</td>
</tr>
<tr>
<td>4</td>
<td>SOLUTIONS TO THE COUPLED SYSTEM OF REACTION - DIFFUSION EQUATIONS</td>
<td>54</td>
</tr>
<tr>
<td></td>
<td>GOVERNING THE BURNING MODEL</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4.1 Introduction</td>
<td>54</td>
</tr>
<tr>
<td></td>
<td>4.2 The Classical Procedure</td>
<td>55</td>
</tr>
<tr>
<td></td>
<td>4.3 The Steady State Solution Of Forbes</td>
<td>61</td>
</tr>
<tr>
<td></td>
<td>4.4 The Linearised Solution Of Forbes</td>
<td>64</td>
</tr>
</tbody>
</table>
4.5 Stability Of Nonlinear Steady Patterns
4.6 The Sal'nikov Model
4.7 The Non-Classical Procedure

CHAPTER 5: SOLUTIONS TO A COUPLED SYSTEM OF SEMI-LINEAR EQUATIONS
5.1 Introduction
5.2 The Classical Approach To Equations (5.1)
5.3 An Alternative Approach

CHAPTER 6: APPLICATIONS OF A SIMPLER TRANSFORMATION
6.1 Similarity Solutions To The Tripled System Representing A One-Dimensional Case Of Diffusion In The Presence Of Three Diffusion Paths
6.2 Similarity Solutions To The Coupled System Representing A One-Dimensional Case Of The General Linear System Of Coupled Diffusion Equations With Cross-Effects

CHAPTER 7: CONCLUSION

BIBLIOGRAPHY

APPENDICES
I Formulae For Partial Derivatives For The Classical Procedure Of Chapter 2
II Formulae For Partial Derivatives For The Classical Procedure Of Chapter 3
III  Formulae For Partial Derivatives For The Classical Procedure Of Chapter 4  120

IV  Formulae For Partial Derivatives For The Procedures Of Chapter 5  121

V  Formulae For Partial Derivatives For The Classical Procedure Of Chapter 6  122