The Urban Residential Economic Model: Theoretical and Empirical Developments.

by

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PLEASE NOTE

The greatest amount of care has been taken while scanning this thesis,

and the best possible result has been obtained.
ABSTRACT

The aim of this thesis is to analyse the economic model of urban residential location through the application of duality methods. Whilst some dual methods have been used in urban economic modelling in the past this paper proposes alternative dual approaches which appear to be novel, but are complementary to existing approaches to the urban model. As part of the application of dual techniques the paper proposes a method of application which is general enough to be applied to all Von Thünen type models and tests this proposition on the fundamental agrarian model of Von Thünen. As part of the dual analysis of the urban residential model the conditions for the traditional lot size hypothesis are examined in the light of conditional demand functions stemming from the dual analysis. The work also empirically tests the traditional residential lot size hypothesis for various Australian cities. The empirical method adopted involves estimation of density gradients utilising competing non-nested flexible form models and discrimination between these alternative models utilising semi-parametric non-nested tests based on an artificial regression model. Two of the three competing models have not been used in this context before, one of them being completely novel. Moreover, the artificial regression model has not been previously used in this context, requiring some modification to deal with the problem of competing models with dependent variable transformation.
DISCLAIMER

This work has been produced for the fulfilment of the degree of Doctor of Philosophy of the University of Western Sydney Nepean. Neither this manuscript nor any part of it has been submitted for a higher degree at any other institution.

Signed

R. Ham
January, 1999
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INTRODUCTION
Preface

The purpose of this work is to examine the equilibrium model of urban residential location using the methods of duality which have come to characterise the analysis of static optimising methods in modern microeconomic theory. The objective is to generalise and consolidate, using duality methods, the location model of centralising and countervailing decentralising forces utilised in urban economic modelling. An aspect of the urban model is an examination of those conditions leading to the hypothesis that residential lot size is increasing with radial distance from the city centre for large cities. Whilst the general method adopted in this work is applicable to all aspects of the comparative statics of location in the urban model, emphasis will be placed upon how the general method can be utilised to analyse variation in the demand for the location-fixed good in the model, such as land or built space, as necessary for the formulation of any hypotheses such as the residential lot size hypothesis. Following the theoretical model of Part A, the residential lot size hypothesis will be tested in Part B using population and household density gradients for the six Australian state capitals. Because of data requirements it is not possible to directly test the rich models stemming from the theoretical analysis of part A. However part B comprises a secondary and separate thesis, where the urban density model is used as a vehicle for the examination of the use of alternative flexible form methods and tests to discriminate between these alternative specifications in urban empirical modelling. For brevity the hypothesis will be referred to henceforth as the lot size hypothesis.
Whilst the work involves the melding of concepts and ideas from areas such as urban economics, duality theory and estimation and testing of non-linear functional forms, it should not be construed only as a synthesis. It goes further than synthesising methods from these fields by not only using them in a novel way, but also developing and enhancing some analytical methods in theoretical and empirical location economics. The main thesis, of Part A, is that the urban residential model is a special case of a general model of economic centralisation and countervailing decentralising forces in location. It is further argued that it is possible to set up a general economic model to analyse locational outcomes in such a generic model which can then be applied to specific location issues which might involve a production or consumption problem in location. Whilst the origins of such a general location model are old (Von Thünen, 1826) and well established in the literature, it is argued that new insights into the economics of location can be gained through the utilisation of duality theory in a formal general model which can then be applied to specific location problems with centralising and decentralising forces. It not only involves the application of standard duality tools, such as profit (utility) maximisation and cost (expenditure) minimisation, but also the development of new alternative dual problems in location which lead to a much more general and efficient way of undertaking comparative statics in location. Specifically these new dual problems involve:

- the minimisation of the difference between offer rent and market rent for location fixed goods such as land or built space, subject to a market rent constraint; and

- the direct choice of minimum or maximum radial distance from the point of central attraction, again subject to a market constraint.
Utilisation of envelope theorems around the novel optimal value functions associated with these problems allow new insights into locational issues in the model. Particularly useful comparative statics can be undertaken directly on location through the optimal distance function.

In order to lay the foundations for the two location problems already referred to, it is useful to derive the offer rent as a function of location in a way which is different from the route already established in urban modelling. This is done by specifying the primal location problems of profit (utility) maximisation and their duals, cost (expenditure) minimisation, as two stage problems. First, find the solution to each problem at every given location, resulting in optimal value functions which are functions of location measured by radial distance from the point of economic attraction. At this stage the problem of location has not been solved and these optimal value functions are referred to as ‘aspatial’ optimal functions in the thesis. The second stage involves solving these aspatial functions for optimal location. Offer rent functions for location fixed goods such as land or built space can be derived from these parent aspatial functions. These offer rent functions are maximum offers as a function of location conditioned on levels in the parent functions. Utilising envelope theorems it is possible to easily derive first and second order properties of the offer rent function.

The offer rent function is not only useful in setting up the new dual location problems, but also the first order properties yield a useful conditional demand for the location fixed good as a function of distance. It is contended in this thesis that economic
actors purchase a location through purchases of the location fixed good. In a general model of location where there are centralising and decentralising economic forces the demand for that location good, either land or built space, as a function of location is important. In the specific urban model of residential location variation in the demand for residential land or built space with location is an important theme. This thesis not only generalises this argument, but also utilises the conditional demand derived from the offer rent function to decompose the location factors determining the demand for the location good. This decomposition is possible because the demand derived from the offer rent function is abstracted from the price of the location fixed good and conditioned on levels in the aspatial parent function of the offer rent function.

The general model of centralised location is set up in chapter one section four. The general method derived from this model is then tested by applying it to the fundamental location model of Von Thünen in chapter two. This is useful because it is argued that the Von Thünen model is the antecedent to many modern models of centralisation in location with countervailing decentralising forces, such as the urban residential model. In chapters three and four the general location method developed in chapter one is then applied to the urban residential model. In both cases the general model is successfully applied to these two specific models in a way which allows straightforward location comparative statics. Because an objective is to apply the general method to the specific urban residential model, sections one, two, and three of chapter one are given over to placing that model in the context of location economics and the historical development of the model.
Wherever possible any hypothesis stemming from economic theory should be held subject to the test of disproof. Earlier, reference was made to the lot size hypothesis of the urban residential model and the testing of this hypothesis has been a consistent theme in empirical urban economics. The lot size hypothesis is usually tested through population density via the equivalent proposition that density is a monotone decreasing convex function of radial distance from urban centre. The hypothesis has mostly been tested through the estimation of log linear density gradients. More recent developments have used the flexible functional form method attributable to Box and Cox (1964) which allows the data to find the functional form and then test for linear versus log linear specifications through parameter restriction.

A secondary thesis, taken up in Part B, chapters five and six, is the further development of functional forms with similar characteristics to the Box Cox form in applications to the estimation of the urban density gradient. Novelty is introduced through the use of two other flexible functional forms as alternatives to Box and Cox. These are a flexible functional form attributed to Wooldridge (1992) and a log linear version of this flexible functional form introduced in this thesis. Just as in the Box and Cox transformation, both alternatives nest within them the linear and log linear forms. However, both alternatives offer different stochastic specifications to Box and Cox. Discrimination between these flexible functional form specifications is undertaken through the artificial regression model of Davidson and Mackinnon (1981 and 1993). Because the competing specifications involve discrimination between models with transformed response variables, the specification test has been modified to encompass this. Thus the empiricism incorporates a two stage testing down procedure. This involves first, testing down between the three alternative flexible
forms using a specification test, and then testing for functional form using parameter restriction. Empirical testing of the urban lot size hypothesis utilises density data from the capital cities of the six Australian states. This empirical work is taken up in chapters five and six, with chapter five being devoted to issues in method and chapter six given over to results.

Whilst the empirical test of the urban lot size hypothesis, via the density gradient, introduces novel and complex estimation and testing procedures, at this early stage it is important to point out that this empirical work is, like many others in testing the urban model, inherently weak. The absence of suitable data inevitably leads to a sophisticated theoretical model partnered by a data set which only indirectly contains the theoretical variables. Despite this, because empirical tests of theory in economics are tests of "disproof", density gradients are useful empirical tests of the urban lot size hypothesis, where the search is for density gradients which are, or are not, consistent with the lot size hypothesis.

To summarise, the main thesis, of Part A, chapters two to four, is that the analysis of the theoretical model of centralisation in location, stemming from the seminal work of Von Thünen can be generalised in method. This involves the derivation of offer rents as a function of distance in a novel way, which leads to more efficient comparative statics of offer rent and the derivation of a conditional demand function for the location fixed good which allows for the useful decomposition of location influence on demand. Further, the offer rent function can be used to construct problems in location which are dual to the problems of profit (utility) and cost (expenditure) minimisation in location. These alternatives involve choice of location to minimise
the difference between offer rent and market rent and choice of minimum and maximum radial distance from some economically attractive central focal point. Envelope theorems around the optimal value functions to these problems lead to useful comparative statics in location. As a secondary thesis, in Part B, it is argued that empirical tests of location phenomena, such as the lot size hypothesis of the residential urban model, can benefit from the use of alternative flexible functional form methods. Additionally, specification tests based on the artificial regression model can be used to discriminate between these competing alternatives, so that the data can be allowed to determine the functional form of the phenomena in location.

1.1 Introduction

The main thesis is that analysis of models like the urban residential model, which involve location through balancing centralising and decentralising forces, can be generalised. Part of this generalisation involves the utilisation of standard duality methods, but in a location context. Further to this it is possible to specify alternative problems which are dual to the application in location of standard dual methods. These alternatives lead to a useful general comparative static method in location. Because this general method will be applied to the urban residential model, it would be appropriate to trace the development of the urban economic model and place it within the context of location economics. Urban economics within the framework of location economics is examined in section two of this chapter. Section three deals with the historical development of the urban residential model. It is only then in section four of this chapter that the general analytical model is outlined.

Moreover, as a secondary thesis, an empirical test of the urban residential lot size hypothesis will be undertaken, utilising Australian density data, but importantly using competing flexible functional forms and using a non-nested test to discriminate between these flexible functional forms. Such an empirical method could be used to test other hypotheses about economic phenomena and location relative to some central focal point stemming from a generic model of location in the Von Thünen mould,
such as the relationship between urban land values, if data were available, and
distance from city centre. Thus section five of this chapter is given over to an
examination of the lot size hypothesis, urban density gradients and the literature of
empirical work on that hypothesis.

1.2 Location Economics and Urban Economics.

All economic activities have to take place somewhere. If these economic activities are
footloose, then why activities take place at certain locations and not others is an
interesting question. In the context of why economic activities occur in certain places,
location economics is catholic enough to contain a variety of approaches. One
approach is that adopted by those models which fall into the category of regional
economics. The characteristic of regional economics is that, even though they address
the question of why activities occur in one region and perhaps not in another, the
regions themselves are usually treated without any real reference to geographical
location. What is important is the relationship between regions in terms of:

- the flows of goods and services between regions and their specialisation in
  production; and

- the flows of factors between regions.

Examples of the former are inter-regional income multiplier models, especially with
import “feedback” (Brown et al., 1991), and inter-regional input-output models
(Gilchrist and St. Louis, 1994). The latter tend to be long term dynamic models with the objective of explaining the growth and development of regions. This involves the application of neo-classical growth theory in explaining regional convergence, (Borts and Stein, 1968 and Barro and Sala-i-Martin, 1992 and 1995, p. 28-32). A counterpoise to this is the regional divergence ideas of Kaldor (1970) taken from the general view of cumulative causative disequilibrium first mooted by Myrdal (1957, Chapter 3) and continued later by Thirlwall (1986). A good overview of regional economic theory can be gained from chapters 1 to 5 of Armstrong and Taylor (1993). In all cases explicit reference to location in geographical space is limited and in most cases references are made to a region or system of related regions within a benchmark economy. For example, this could be a sub-national region within a national economy or a country or region within a block of trading countries. If urban economics is the study of resource allocation at the level of the city and cities can be regarded as regions or the economic foci of geographical regions, then the analytics of regional economic theory are contained in urban economics.

Within location economics, location reference can be made in two ways:

- through reference to northings and eastings on a geographical grid. That is, relating two or more points on that grid in terms of distance (strength) and direction; or

- in terms of direct distance between two or more points without reference to direction of location.
The first way of referencing location has not been adopted within economics but it is a strong theme in economic geography. It can involve the mapping of contours of economic phenomena on a grid, usually to describe the process of geographic centralisation. For example see Goddard (1975, p. 16-19) in his review of Swedish studies of office contact potential to identify those cities which have the highest accessibility to offices. These methods tend to be descriptive and empirical, although clearly there must be some underlying economic explanation of the spatial patterns of economic phenomena highlighted. An example of using geographical trend surfaces as estimated complex polynomials in northings and eastings is Schroeder and Sjoquist (1976).

The second way of referencing location, through relative distance, has been highly utilised in the economics of location. Theories of industrial location tend to measure location through measures of direct physical or even time distance between activities. Location theory attributed to Weber (1929) falls within this category, where location of productive activity is initially examined from the point of view of minimising transport cost of inputs. Models where there is variation in demand with location measured by distance also fall into this category, such as Lösch (1967) in determining optimal market area. Locational interdependence models following Hotelling (1929) again reference location through distance and not direction. For a good overview of industrial location see Smith (1981). For the application of some of these ideas in the location of urban business see Stahl (1987). The location model of Von Thünen (1826) measures location through distance from a central focal point with economic forces being equal in every direction from that central focal point. Thus distance now becomes radial distance. For lucid and non-technical introductions to the elements of
the models of Weber, and Von Thünen, see Beckmann (1968, p. 14-19 and p. 60-62, respectively). Wheaton (1979, p. 108) rightly indicates that all urban residential models which are monocentric, so that employment opportunities are centralised, are Von Thünen type models. In this sense, Wheaton is characterising a generic Von Thünen model as a location model where:

1. location is relative to some central point with location forces being equal in every direction so that location refers to radial distance from a centre;

2. for the locators there is some economic attraction to proximity to the central place (centripetal force);

3. in competition for central location, land prices, as a decreasing function of radial distance, act to disperse locators around the central place (centrifugal force).

This has implications for the way that analysis of the model is undertaken. Beckmann (1968, p. 10-13) in discussing plant location within the profit paradigm, suggests that whether the location problem is best treated as a cost minimising, revenue maximising or profit maximising issue is dependent on the configuration of input and output prices in location. Thus, if some input price varies with location, but output price is constant in location, cost minimisation is appropriate. Conversely, if output price varies with location and input prices are constant in location, the problem can be dealt with by revenue maximisation. Finally where both input price(s) and output price vary with location, profit maximisation is appropriate. Clearly, in a Von Thünen
model the middle case of revenue maximisation cannot occur. If output price varies with location relative to a focal point, so that location proximate to the focal point is desirable, then competition for location will result in land prices acting as a dispersive force around that location. That is, through competition for location, input prices are bound to vary. This issue has important implications for the treatment of the residential building problem and will be raised in chapter 4. Indeed, because of this, whilst it is possible to treat Von Thünen type production (consumption) models in a profit (utility) maximising or cost (expenditure) minimising fashion, in production models revenue maximisation treatment will be difficult for a single product firm.

However, urban models, which are monocentric, are Von Thünen models.

1.3 The Historical Development of the Urban Residential Model

The urban residential model owes its origins to Von Thünen and falls within the characteristics of the generic Von Thünen model. Whilst the urban residential model owes much to Von Thünen, (Richardson, 1977a, p. 6-11), the study of "urban economics" owes as much to Ratcliff (1949) and Thompson (1965). However, even though these works are important in determining the legitimacy of studying urbanism within an economic context, they are far removed from the application of neo-classical economic modelling to intra-urban location issues which characterises the modern urban residential model. Ratcliff's focus of interest was urban real estate prices whilst Thompson focused more on the application of economic theory to explain issues of urban development and issues within cities, such as poverty,
employment, and local public sector allocation. Indeed, Thompson *op cit.* pages 11-18 still provides one of the most lucid accounts of the urban development process.

The urban residential model has been developed since the 1960’s and centres on the application of optimising techniques associated with neo-classical economics to urban housing. The following taxonomy of urban residential models is not unique and other reviewers of the literature may identify a different classification. The residential model has been developed in three areas:

1. the equilibrium model of the urban residential market, which utilises static optimisation methods;

2. optimal models of intra-urban location, which utilise dynamic optimisation methods, but in a non-temporal fashion. Optimal spatial patterns of development, according to some normative criterion, are analysed using say optimal control theory and replacing time with a spatial measure such as radial distance;

3. dynamic models of residential development, which look at optimisation issues over time. These can be further subdivided into two types:

   a). models which are time separable, where issues such as the optimal timing of development and replacement are analysed. These therefore deal with lumpy changes in the residential stock and even though they are dynamic, because they deal only in the beginning or end of a building, utilise
classical static optimisation methods. Thus building capital is durable with no physical depreciation;

b). models which deal with continuous investment in the building stock, the so called maintenance decision, because building capital depreciates, and normally utilise optimal control theory in identifying optimal investment paths.

Dealing with the third group first, examples of 3a) can be found in Arnott and Lewis (1979), Arnott (1980), Brueckner (1980) and are summarised in Henderson (1985, p. 85-87). These durable capital models are somewhat artificial by virtually abstracting issues to the optimal time of starting and ending a building. In a model of building investment where capital depreciates these would be the transversality conditions Chiang (1992, chapter 4). Models within category 3b) are more realistic because they deal with the temporal path of investment in non-durable building capital. Good examples of these are Arnott et al. (1983) and Arnott et al. (1986). The latter is particularly useful in that, not only does it examine optimal time paths of investment in buildings, but it also examines how those paths vary over radial distance in a Von Thünen type urban residential model. A summary of these models can be found in Henderson (op cit., p. 73-77).

Examples of optimising models of category 2 are Oron et al. (1973) and Sheshinski (1973) which optimise spatial structure from the point of view of balancing urban economies against transport diseconomies in congestion. Both papers treat the problem as one of optimal control in space.
The remaining category, category 1, appears to be the largest in terms of the literature. Seminal works are due to Alonso (1960 and 1964), Mills (1967 and 1972, chapters 5 and 6) and Muth (1969 and 1971). Again, the development of the equilibrium model falls into two broad categories:

- "Alonso type" models, where the household makes a choice between all other goods, which are footloose and residential land which is naturally fixed in location. The household deals directly in the land market as if it is an owner builder and residential land appears in the household’s utility function. For a good exposition of the Alonso model see Straszheim (1987, p. 718-725).

- "Muth type" models, where the household chooses between all other goods and housing. Housing consists of homogeneous units of space or services in a dwelling produced by residential builders out of building capital and land. Rather than land, which does not figure directly in the household’s utility function, the household derives utility from consumption of homogeneous housing units in a dwelling. For a terse, but rigorous overview of the Muth type model see Brueckner (1987).

Both model types are structured within the Von Thünen monocentric form, where household transport cost is increasing equally in every direction from a single employment centre, to which all relevant journeys take place. There are two main differences in the way these two model types are treated analytically.
• Any reference to the quantity of land used in a dwelling in the Muth type model has to refer to the residential builder’s behaviour in production.

• The Alonso type model directly addresses the issue of land prices and location through competitive bidding. Alternatively, Muth type models make inferences about the price of housing units in a dwelling and its location through the conditions for household equilibrium location.

In this thesis it is contended that the bidding model can be used to establish these prices in a Muth type model because, like land, housing is fixed in location. In this sense builders undertake the normal speculation that many producers make by producing something to then sell. If builders produced only under contract from households, that is sale was guaranteed prior to the commencement of production, then the Alonso model would apply. Throughout this thesis, to distinguish between the two different approaches to the urban residential model, the epithets Alonso type and Muth type will be maintained.

Since the late 1960’s there have been significant developments around the equilibrium residential model. It is possible to identify four areas within which the model has been developed. At the outset it must be made clear that analysts have not necessarily confined their contribution to one particular area and that these developments have occurred in both Alonso and Muth type models. These four areas are summarised briefly below.
1. The first area of development involves the introduction of a greater complexity into the analysis. A good example here is the introduction of leisure choice into the model where the opportunity cost of leisure time is seen as an important cost of transport. This development has been undertaken in two ways. Firstly the disutility of commuting has been introduced directly into the utility function, so that commuting time is a choice variable. See for example Beckmann (1969 and 1987), Brown (1985), Evans (1973, p. 38), Montesano (1972), Straszheim (1987, p. 718-724) and Wheaton (1977). These are difficult models to deal with because they produce unusually shaped level curves in the utility function. The alternative is to introduce leisure time as a choice variable where utility is increasing in leisure time, but locations with increasing radial distance erode available leisure time. Clearly, these models must include a time constraint as well as the usual budget constraint. Good examples of these models are Fujita (1989, p. 32-38) and Turnbull (1992 and 1995, p. 18-23).

Further complexity is introduced by additional choice variables in location. Yamada (1972) introduces neighbourhood quality. The consumption of local public goods which vary in provision by location is examined by Schuler (1974) and Yang and Fujita (1983). Local planning measures and the effect on location can be included in this area (Pasha, 1992). One might argue that some of these modifications move away from the monocentric structure. This is done explicitly in works where some decentralisation of employment has been introduced, for example, Curran et al. (1982) and White (1977). But even here this is mostly done within an overall monocentric structure with local variation on a broad regional trend.
2. A standard treatment of equilibrium in economic theory is to undertake comparative static analysis of change through change in the parameters determining equilibrium. The literature in the urban residential model has followed this standard. As indicated earlier, comparative statics falls into two areas. Firstly, there are those works which have not benefited from duality and the use of the envelope theorem. These make inferences about changes in the parameters to the consumer problem from the bordered Hessian matrix to the direct utility problem and include Brown (1985), De Salvo (1977 and 1985), Montesano (1972), and Sasaki (1987). Alternatively, there are those who use the envelope theorem with the indirect utility function of the consumer problem, such as Pines and Sadka (1986) and Pasha (1992), although the latter appeals directly to the Lagrangean of the direct utility problem. Fujita (1989, p. 18-21) and Turnbull (1992 and 1994) both investigate comparative statics by utilising the envelope theorem and the consumer’s expenditure function. In all cases, inferences about the nature of consumption of residential land (Alonso type models) or consumption of residential space (Muth type models) and radial distance are made within this comparative static framework.

It is also appropriate to mention here models which have attempted to analyse change in a continuous time setting, but within the equilibrium framework. Examples are Capozza and Helsley (1989), Fujita (1982) and Miyao (1987, p. 901-921). Here the objective is to identify continuous change in structural patterns within monocentricity, given such things as population and income growth. In this case the lot size hypothesis is translated into an outward wave of increasing
consumption in residential land or housing space. For a graphic illustration of this outward wave in terms of rising and then falling land prices see Miyao (*op cit.*, Figure 2.2, p. 914).

3. The third area of development concerns the existence and uniqueness of spatial equilibrium. This has been an important theme in Alonso type models. Following the Von Thünen tradition, spatial equilibrium requires competitive conditions in the bidding process for residential land or space. Stable outcomes from this process are characterised by specific relationships with radial distance. Thus, equilibrium land price in the Alonso model is a function of radial distance, where at each given location the market land price is an equilibrium land price and the land price distance function is a locus of points each one of which is an equilibrium price. Similarly, equilibrium residential land consumption is a locus of equilibrium quantities. It is only natural then that analysts identify those conditions which lead to stable outcomes in terms of price and distance and quantity and distance relationships. The early approach of Alonso (1960, p. 155-157 and 1964 chapter 5) tended to use heuristic reasoning in appealing to the characteristics of competition between households for location. Others followed, such as, for example Mills (1972, chapter 4), culminating in the work of Fujita, (1985 and 1989 chapters 3 and 4). Fujita’s 1989 work resulted in the identification of the conditions for the existence of a monotone decreasing piecewise, smooth continuous convex function of land prices in distance from centre and is the classic work on the existence and uniqueness of urban equilibrium as spatial equilibrium paths of prices and quantities.
4. Finally the urban residential model has benefited from the increasing application of duality. Very early in the development Evans (1973, p. 30-37), in a Muth type model, intimated that the consumer problem could be represented as a problem of expenditure minimisation as well as utility maximisation, but did not formalise it as a standard expenditure minimisation problem. Solow (1973) introduced the idea of examining the nature of equilibrium using the envelope theorem with the indirect utility function and retrieved the demand for residential land and the nature of offer prices in radial distance in an Alonso type model through Roy’s identity. Since that time, even though some still take a more traditional path, others have used this route to residential space demands and offer prices. However, recourse to the envelope theorem is not fully utilising duality. Representing the primal problem in an alternative and realistic way in which to gain further insight into the issue at hand is the real mark of duality. Elements of this can be seen in Wheaton (1979, p. 110) who considers the problem of what is the maximum offer price at a location conditioned on a level of utility. He attributes this rightly to Alonso’s (1964) development of the bid rent curve. Indeed this latter work is a remarkable diagrammatic achievement in three dimensions (see Alonso, op cit., p. 63-67) but does not use the traditional techniques of optimisation. Sasaki (1987, p. 55) writes out the Wheaton problem formally and identifies the choice variables as the arguments of the direct utility function. This approach is taken up by Fujita (1989), who also uses more traditional routes in duality by examining the expenditure minimisation problem as well as utility maximisation and explicitly retrieves compensated as well as uncompensated demands for land in an Alonso type model. Turnbull (1992, 1994,
and 1995), in a Muth type model, uses the expenditure function as well as the indirect utility function in retrieving demand functions.

These four themes represent continuing development in a significant literature on the equilibrium residential urban model. This thesis aims at continuing the development of the equilibrium residential model through the use of dual methods. The objective is to identify dual methods which are alternative and complementary to those of Wheaton, Sasaki, Fujita and Turnbull. It is argued that models which are generically Von Thünen type models can be treated in this way and these methods are not confined to the residential model. Generic Von Thünen models are defined as location models where there is economic attraction to some central focal point or central place (centripetal force), but competitive bidding for items that are fixed in location, such as land or built space, results in the prices for these items acting as a dispersive device (centrifugal force) in the allocation of location.

1.4 A General Dual Approach to Von Thünen Type Models.

In this section a generalised dual approach to Von Thünen location models is introduced. This approach is the method adopted in chapters 2, 3 and 4 where specific optimising issues in location are examined with a view to replicating the lot size hypothesis, or its equivalent, so long associated with the equilibrium residential model. Some novelty is introduced in this approach through:
- the fact that this is perceived as a general approach to all Von Thünen models, whilst to date duality methods have been confined to the urban residential model;

- the derivation of the bid rent function from the optimal value function of the primal problem. This is different from the usual route taken in the literature;

- the utilisation of alternative dual problems such as choosing a location to minimise the difference between bid rent and market rent, or choosing a location directly to minimise or maximise radial distance, all subject to the constraint that bid rent must be at least equal to market rent.

Because the objective is to produce a general approach to Von Thünen location models, objective functions, definitions of choice variables and parameters have been left suitably general to include production and consumption problems.

The method outlined owes as much if not more to the literature on duality and the envelope theorem in economics as it does to that on urban modelling. Lambert (1985), Madden (1986), and Takayama (1985 and 1994) have been utilised heavily as textbooks of mathematical method. Luenberger (1995), Mas-Colell, et al. (1995) and Varian (1992) provided useful overviews of the applications of duality in microeconomics. Blackorby, et al. (1978, chapter 2) and Diewert (1982) provided very useful rigorous and terse accounts of duality. Shephard (1981) and the early chapters of Chambers (1988) were used in terms of duality and production and cost problems. Dixit (1990) provided a good overview of non-linear programming, the envelope theorem and maximum value functions and applications to microeconomics.
Comes (1992) provided a book length rigorous exposition of the envelope theorem and duality, utilising useful diagrams and logical verbal explanations as well as traditional mathematics. All have been used heavily in this section and in chapters 2, 3 and 4, where more detailed references to these works are given.

The typical Von Thünen problem can characterised in the following way:

\[
\max_r \{ F(a(r), b, w) \} = \max_x \{ f(x, a(r), b) : g(x, a(r), b, w) \leq 0 \} \quad ; \quad (1.4.1)
\]

or by a dual to (1.4.1) as:

\[
\min_r \{ G(a(r), b, w) \} = \min_x \{ g(x, a(r), b, w) : f(x, a(r), b) \geq 0 \} \quad ; \quad (1.4.2)
\]

where:

- \( x \) is a vector of choice variables one of which will be land (Alonso type) or built space (Muth type).
- \( r \) is radial distance.
- \( a \) is a vector of parameters each of which is a function of distance in such a way that they are a centripetal force. That is they make central locations attractive.
- \( b \) is a vector of non-location parameters.
- \( w \) is land (Alonso type) or built space (Muth type) price and is an unknown function of distance. Land or space price has been isolated from the other parameters in that it will act as a centrifugal force in competition for
location. That is land or space price will act as a dispersive location force
and will counteract the other location parameters.

The functions \( f \) and \( g \) could refer to production and cost if the issue is a producer
problem or utility and expenditure if the issue is a consumer problem.

In both cases it must be made clear that the distinction between the inner and outer
problem is a heuristic device to isolate the location issue from other aspects of a
production or consumption problem. The separation into the inner and outer problem
is attributable to Anas (1982). It does not indicate that the relevant functions are
separable as in say two stage budgeting and a separable utility function, see Gorman
(1959, and 1987) and Blackorby et al. (1978, chapter 3). However, adapting the
nomenclature of intertemporal consumer theory, see McLaren and Cooper (1987), the
functions \( F \) and \( G \), which are the optimal value functions for the dual inner problems,
will be referred to as aspatial, because the problem of location has not been solved in
the inner optimisation. When \( r \) is optimised out of \( F \) and \( G \) the resulting optimal
value functions will be referred to as spatial functions \( F_r \) and \( G_r \). Similarly, in
alternative formulations of the optimisation problem, the relevant optimal value
functions where \( r \) has been optimised out are spatial functions and will be signified by
the subscript \( r \) meaning that the variable \( r \) has been optimised out. All optimal value
functions where \( r \) is to be determined, that is are still functions of \( r \), are aspatial
functions in the sense that they evaluate an optimisation in which location has not
been optimally chosen.
The analysis of the Von Thünen problem can then proceed through the following stages.

Firstly, establish that the optimal value functions for the alternative inner problems, $F$ and $G$, are monotone functions of $w$ and invert them for $w$. For $F$ denote the inverse as:

$$\omega_{(a(r),b,f)} ;$$

(1.4.3)

and for $G$,

$$\omega^g_{(a(r),b,g)} .$$

(1.4.4)

The result (1.4.3) or (1.4.4) is a maximum value function, known as the bid rent function, conditioned on levels in $f$ or $g$ and is the maximum at any given location $r$ a locator can bid. This route to the bid rent curve should be compared to that of Wheaton (1979, p. 110), Sasaki (1987, p. 55) and Fujita (1989, p. 14). Using the notation here, these works invert the function $g$ for $w$ in (1.4.1) and then pose the problem:

$$\omega_{(a(r),b,f)} = \max_x \{ w(x,a(r),b) : f(x,a(r),b) = f \} .$$

Returning to the proposed general method, it is useful to form identities (1.4.5) or (1.4.6) below, through substitution of (1.4.1) and (1.4.2) into (1.4.3) and (1.4.4) respectively. It is then possible to employ envelope theorems to establish important first and second order properties of the bid rent function from these identities:

$$\omega_{(a(r),b,F(a(r),b,w))} = w ;$$

(1.4.5)
\( \omega^g(a(r), b, G(a(r), b, w)) = w \). \hspace{1cm} (1.4.6)

For example, differentiating (1.4.5) with respect to \( w \) gives:

\[
\frac{\partial \omega(a(r), b, F)}{\partial f} = \frac{1}{\partial F/\partial w}.
\hspace{1cm} (1.4.7)
\]

Now, using the envelope properties of the function \( F \), say through Hotelling's lemma or Roy's identity, identifies the denominator on the right hand side (or some appropriate variation of it) as the demand for land or built space (albeit conditioned on \( w \)). For illustration here, it can be supposed that \( \partial F/\partial w \) gives a relevant demand expression, say \( x''(a(r), b, w) \).

Now, however, inspection of (1.4.7) shows that, in this case, demand can be retrieved as:

\[
x'(a(r), b, f) = \frac{1}{\partial \omega(a(r), b, f)/\partial f}.
\hspace{1cm} (1.4.8)
\]

The importance of (1.4.8) is that it asserts that a relevant demand such as the demand for land or built space retrieved in this way will be a function of the arguments of \( \omega \) and therefore will be conditioned on \( f \) and abstracted from own price. This opens up the prospect of signing the derivatives of land with respect to its arguments (conditioned on \( f \)) even if, in the conventional own-price conditional functions, there may be ambiguities. Whether this prospect can be realised will depend upon the
structure of the optimal value function from which \( \omega \) is obtained by inversion. The options and their consequences are explored in this thesis.

Continuing to outline the general structure of the research agenda, the remainder of the first derivative properties of \( \omega \) in its other arguments can now be examined by straightforward differentiation and substitution. That is for example, take any parameter \( a \) which is an element of \( \mathbf{a} \) and any parameter \( b \), an element of \( \mathbf{b} \), and differentiate \( \omega \) with respect to these parameters.

Properties 1.4.1 (further first derivative properties of \( \omega \))

\[
\frac{\partial \omega}{\partial a} = -\frac{\partial F}{\partial w} a \quad \text{and} \quad \frac{\partial \omega}{\partial b} = -\frac{\partial F}{\partial w} b.
\]

Using the notation here, this route should be compared to say Sasaki (1987, p. 55). He uses the first order conditions to the problem of choosing \( x \) to maximise \( \omega \) subject to a level in \( f \) to establish the first order properties of \( \omega \).

A similar procedure, but using the envelope properties of \( G \), through say Shephard's lemma, establishes the envelope properties of \( \omega^G \). The demand for land or built space can be retrieved through the envelope properties of \( \omega^G \). Again this will be conditioned on levels in \( g \) and abstracted from own price.
It might seem unusual that the demand for land or built space derived from the bid rent function is conditioned on levels of the functions \( F \) or \( G \) and abstracted from own price. Moreover, because they are highly conditioned, such demand functions may be of limited empirical use. Nevertheless they will serve a useful function in that they can be used to analyse the relationship between the demand for land or built space and radial distance \( r \). Particularly, abstracting from own price effects on demand allows concentration on the effects of parameters \( a \) on the demand in radial distance. Additionally, and most importantly from an empirical point of view, tests of the lot size hypothesis may need to be conducted under conditions in which the “own-price” is not available. The alternative conditional functions, which are the subject of the theoretical component of this thesis, hold out attractive possibilities for alternative empirical specifications.

Returning to the issue of the second order properties of \( \omega \), the discussion following (1.4.8) shows clearly that the signs of the first and second derivatives of \( \omega \) or \( \omega^g \) are determined by the characteristics of the functions \( F \) or \( G \). The first derivative signs are straightforward. The second derivative signs depend on the curvature characteristics of the function \( F \) or \( G \), because the functions \( \omega \) and \( \omega^g \) are expressions for the level curves of \( F \) and \( G \). The function \( \omega \) is convex if \( F \) is quasi-convex in \( a \) and \( b \) and \( \omega \) is the expression for the lower contour function, that is \( \omega \) is decreasing in levels of \( f \). The function \( \omega^g \) is convex if \( G \) is quasi-concave in \( a \) and \( b \) and \( \omega^g \) is the expression for the upper contour function, that is \( \omega^g \) is increasing in the levels of \( g \).

Next, it is necessary to establish the curvature properties of \( \omega \) or \( \omega^g \) in \( r \). With minimum qualification this can be done when the functions \( a(r) \) play an exclusively
centripetal role. In this case using the fact that $\omega$ or $\omega^G$ are expressions for the lower or upper contour functions of $F$ or $G$, so that determining the curvature characteristics of $F$ or $G$ and that $F$ or $G$ are decreasing or increasing functions of $w$ and $a(r)$, establishes the curvature properties of $\omega$ or $\omega^G$ in $r$. Alternatively, one can use the first and second derivative properties of $\omega$ and $\omega^G$ to establish their curvature characteristics, but again these are ultimately determined by the first and second derivative properties of $F$ and $G$. When $a(r)$ plays a centripetal role $\omega$ and $\omega^G$ are monotone decreasing convex functions of $r$.

Now one may use (1.4.3) or (1.4.4) in competitive bidding to establish that market price, $w(r)$ is a monotone decreasing continuous convex function. This is determined by the fact that $w$ is supported from below (competitive bidding) by a family of convex functions $\omega$ or $\omega^G$. It must be made clear here, however, that when $w$ is supported from below by a family of bid rent functions differentiated by their degree of slope in $r$, the convexity of those bid rent functions is sufficient, but not necessary, for the convexity of $w$. This observation needs to be elaborated upon in order to allow for the empirically plausible case where not all the parameters $a(r)$ play a centripetal role or, alternatively, where the conditioning variables of the $\omega$ function are, unlike $f$ and $g$, not able to be directly linked in such a way as to infer convexity (resp. quasi-concavity) of $\omega$ in $a$ if $\omega$ is decreasing (resp. increasing) in the conditioning variable.

To handle the more general situation of potential non-convexity (perhaps even concavity) of $\omega$ in $r$, it is helpful to introduce the general notion of F convexity. A careful examination of the definition of generalised F convexity (see Avriel et al., 1988, p. 294-295) will reveal that convexity of the support functions is not a necessary condition for the decreasing F convexity of the function $w(r)$. 
Definition 1.4.1 (after Avriel, et al., op cit.)

\( w(r) \) is a real function on the set \( S \) where \( S \subset \mathbb{R}_+ \).

\( w \) is F convex on the set \( S \) if for every \( r^0 \in S \) there is a function \( \omega(r) \) such that:

\[ w(r^0) = \omega(r^0) \quad \text{and} \quad w(r) \geq \omega(r) \quad \forall r \in S. \]

In the definition of F convexity, let \( r^0 \) be a particular distance under examination. Further, let \( \omega(r^0) \) be that bid rent function which is successful at \( r^0 \). Then this implies \( w(r^0) = \omega(r^0) \), but it also implies \( w(r) \geq \omega(r) \). Then definition 1.4.1 says that \( w \) is F convex. Thus a land or built space rent function which is a monotone decreasing F convex function of radial distance could even be derived from a family of concave bid rent curves as long as:

1. the bid rent curves are monotone decreasing in \( r \); and
2. there is complete non-coincidence of the slopes of the family of support functions, so that \( w(r^0) = \omega(r^0) \) and \( w(r) > \omega(r) \), for \( \forall r \in S, \ r \neq r^0 \).

Whether or not an F convex market rent function defined in this way is in fact convex in the standard sense does not depend on the curvature properties of the support function (although convexity of these would be sufficient). In general, it would be an empirical matter whether convexity applied, and this is pursued later. The point needs to be made here though, that the convexity or otherwise of conditional market rent functions (such as the \( f \) and \( g \) conditional functions which are used here for
illustration) depend upon the characteristics of the "parent" optimal value functions from which they have been derived, as well as upon the characteristics of the distance parameter functions \( a(r) \).

If one were to continue in the tradition of the Von Thünen problem, the next logical step, having established the characteristics of the market rent function \( w(r) \), would be to substitute the market rent function into the optimal value functions for the inner problems of (1.4.1) and (1.4.2) to give a clear specification of the outer problems:

\[
F_r(b) = \max_r \{ F(a(r), b, w(r)) \} \quad \text{and} \quad (1.4.9)
\]

\[
G_r(b) = \min_r \{ G(a(r), b, w(r)) \} . \quad (1.4.10)
\]

The existence of a solution to (1.4.9) or (1.4.10) turns on the fact that one must establish that the functions \( F \) or \( G \) are continuous in \( r \) and that the domains of \( F \) or \( G \) are compact sets of \( r \). The former is determined by the continuity of \( F \) or \( G \) in \( a \) and \( w \) and the continuity of \( a \) and \( w \) in \( r \). The latter is determined by two boundary conditions. Firstly, the inner boundary is guaranteed by the fact that \( r \) must be non-negative so that the inner boundary is \( r = 0 \). The outer boundary from the central place is more problematical. One has to appeal to the idea that there must be some finite number which marks the outer limit to location, \( r = \bar{r} \), that is, radial distance cannot go on forever.

An interesting alternative to this approach would be to exploit further the insights of duality theory by considering the choice of \( r \) under conditions which are logically equivalent but which require consideration of alternative optimisations to the outer
problems (1.4.9) and (1.4.10). Consider, for example, dual problems to (1.4.9) or (1.4.10) which may be represented as:

\[ \Omega_r (b, f) = \min_r \left\{ \omega (a(r), b, f) - w(r) : \omega (a(r), b, f) - w(r) \geq 0 \right\} ; \quad (1.4.11) \]

\[ \Omega^g_r (b, g) = \min_r \left\{ \omega^g (a(r), b, g) - w(r) : \omega^g (a(r), b, g) - w(r) \geq 0 \right\} . \quad (1.4.12) \]

There is no evidence in the literature reviewed thus far that either of these problems have been posed. However, they have interesting characteristics which are now discussed. Importantly, they lead to the optimal \( r \) for the initially posed problems. Moreover, they are logically equivalent to \( F \) and \( G \) in the sense that they require choosing location to minimise the difference between bid rent and market rent subject to the constraint that successful bids cannot be lower than market rent at any location.

The existence of a solution to either (1.4.11) or (1.4.12) is determined by establishing the compactness of the constraint set in \( r \) and the continuity of the objective function in \( r \). The latter is established through the continuity of \( \omega \) or \( \omega^g \) and \( w \) in \( r \). The former is determined by the formation of \( w(r) \) from \( \omega(r) \) or \( \omega^g(r) \). The subscript \( r \) on the \( \Omega \) functions is meant to denote, as defined for \( F \) and \( G \), that these are optimal value functions for "outer" problems in which \( r \) is optimised out. They are "spatial" functions in the sense that the problem they evaluate has involved spatial choice.

Following the methodology introduced earlier, substitution of the spatial functions \( F \) or \( G \) into the solutions of (1.4.11) or (1.4.12) gives the relevant identities for evaluation of derivatives. In this case due to the structure of the objective functions of (1.4.11) and (1.4.12) these are implicit functions:
\[ \Omega_r(b, F_r(b)) = 0 ; \]
\[ \Omega_r^C(b, G_r(b)) = 0 . \]

The envelope properties of these implicit functions are interesting and can be retrieved by utilising the envelope properties of the functions \( F \) or \( G \). For example differentiating \( \Omega \) with respect to any parameter \( b \), which is an element of the vector \( b \), gives:

\[
\frac{\partial \Omega_r}{\partial b} + \frac{\partial \Omega_r}{\partial f} \frac{\partial F_r}{\partial b} = 0 \Rightarrow \frac{\partial \Omega_r}{\partial f} = \frac{\partial \Omega_r}{\partial F_r} \frac{\partial b}{\partial b} .
\]

Further duals to either (1.4.7) or (1.4.8) respectively are:

\[
\mathcal{R}(b, f) = \min_r \left\{ r \in \mathfrak{R}_+ : \omega(a(r), b, f) - w(r) \geq 0 \right\} ; \quad (1.4.13)
\]
\[
\overline{\mathcal{R}}(b, f) = \max_r \left\{ r \in \mathfrak{R}_+ : \omega(a(r), b, f) - w(r) \geq 0 \right\} ; \quad (1.4.14)
\]

\[
\mathcal{R}^G(b, g) = \min_r \left\{ r \in \mathfrak{R}_+ : \omega^G(a(r), b, g) - w(r) \geq 0 \right\} ; \quad (1.4.15)
\]
\[
\overline{\mathcal{R}}^G(b, g) = \max_r \left\{ r \in \mathfrak{R}_+ : \omega^G(a(r), b, g) - w(r) \geq 0 \right\} . \quad (1.4.16)
\]

These give the closest or furthest distance, \( r \), from the central focal point of the Von Thünen system that the entity can locate. There appears to be no reference in the literature to such problems. This is surprising given that the issues essentially turn on location.
Figure 1.4.1 illustrates problem (1.4.11), (1.4.13) and (1.4.14) for a homogeneous class of locators, such as the producers of a specific crop in the traditional agrarian problem, a class of households (say of the same income and tastes) as in the urban residential problem or a specific industry for urban industrial location.

Figure 1.4.1

In Figure 1.4.1, $\omega(f,r)$ is a bid rent function for the homogeneous class of locators and that group face a market rent function $w(r)$. The shading in the figure shows the feasible set of bids for problems (1.4.11), (1.4.13) and (1.4.14), that is the set of distance/bid price co-ordinates where the difference between bid rent and market rent is non-negative. The solution to problem (1.4.11) is the interval of locations $r^1$ to $r^2$, where $\omega(f,r)=w(r)$ The solution to problem (1.4.13) is $r^1$ and that for (1.4.14) is $r^2$.

Returning to the optimal value functions of problems (1.4.13) to (1.4.16), these are spatial functions because $r$ has been optimised. They are not signified by the $r$ subscript, because by definition these optimal value functions are distances. Again the envelope properties of these functions can be explored by utilising the envelope
properties of previous functions. For example this can be done by forming an identity by substituting the aspatial function \( F \) from (1.4.9) into (1.4.13):

\[
R(b, F(a(r), b, w(r))) = r
\]

To interpret this identity, note that the aspatial function \( F \) contains the market rent function as its argument. The optimal distance is found where bid rent equals market rent, and the \( R \) function conditioned on optimal bid rent (for a given \( r \)) clearly returns the same \( r \) as optimal.

Differentiating this identity with respect to \( r \) gives:

\[
\frac{\partial R}{\partial f} = \frac{1}{\frac{\partial F}{\partial a} \frac{da}{dr} + \frac{\partial F}{\partial w} \frac{dw}{dr}};
\]

where the denominator of the right hand side indicates the fundamental centripetal and centrifugal forces of location in the Von Thünen location model. Differentiating the same identity with respect to \( b \) gives:

\[
\frac{\partial R}{\partial b} = \frac{\partial R}{\partial f} \frac{\partial F}{\partial b} ;
\]

and therefore:

\[
\frac{\partial R}{\partial b} = -\frac{\frac{\partial F}{\partial b}}{\frac{\partial F}{\partial a} \frac{da}{dr} + \frac{\partial F}{\partial w} \frac{dw}{dr}} .
\]
The gradient vector $\partial F / \partial \mathbf{b}$ on the right hand side is known from the envelope properties of the function $F$. Usually the sign of the denominator on the right, that is $\partial R / \partial f$, is indeterminate, so that only conditional comparative statics can be undertaken.

Finally the lot size or its equivalent hypothesis in built space can be examined by retrieving the relevant land or built space demands, via the envelope theorem, from the aspatial functions $F, G, \omega$ or $\omega^0$. These demand functions will be functions of radial distance $r$ and their relationship with $r$ can be examined by differentiation with respect to $r$.

However, recall that demand for land or built space retrieved from $\omega$ or $\omega^0$ are conditioned on levels in $F$ or $G$ and are abstracted from own-price. This will allow some decomposition into the various effects on the demand for land and built space and radial distance. For example, take the demand for land or built space retrieved from the function $\omega$ and the function $F$. Clearly one element in the vector $\mathbf{x}$ will refer to land or built space. Call this $x$. Then the two relevant demand functions are:

$$x^w = x(a(r), b, w(r)) \quad ; \text{and} \quad (1.4.17)$$

$$x' = x'(a(r), b, f) \quad . \quad (1.4.18)$$

The function (1.4.17) is the unconditional demand from the function $F$ and is a function of own-price, hence the superscript $w$. The function (1.4.18) is the
conditional demand from $\omega(a(r), b, f)$ and because it is conditioned on $f$ it has been
given the superscript $f$.

Differentiating (1.4.17) with respect to $r$ gives:

$$\frac{\partial x^w}{\partial r} = \frac{\partial x^w}{\partial a'} \frac{da}{dr} + \frac{\partial x^w}{\partial w} \frac{dw}{dr} . \tag{1.4.19}$$

Alternatively substituting the aspatial function $F$ from (1.4.9) into (1.4.18) gives the
identity:

$$x^w = x^f \left(a(r), b, F(a(r), b, w(r))\right) . \tag{1.4.20}$$

Differentiating (1.4.20) with respect to $r$ gives:

$$\frac{\partial x^w}{\partial r} = \frac{\partial x^f}{\partial a'} \frac{da}{dr} + \frac{\partial x^f}{\partial F} \frac{df}{dr} \frac{da}{dr} + \frac{\partial x^f}{\partial F} \frac{dw}{dr} \frac{dw}{dr} . \tag{1.4.21}$$

Comparison of (1.4.19) and (1.4.21) suggests the following decomposition:

$$\frac{\partial x^w}{\partial a} = \frac{\partial x^f}{\partial a} + \frac{\partial x^f}{\partial F} \frac{df}{da} ; \quad \text{and} \quad \frac{\partial x^w}{\partial w} = \frac{\partial x^f}{\partial F} \frac{dw}{da} ;$$

so that ambiguity in the term $\frac{\partial x^w}{\partial a}$, if it arises, may be explained in terms of the
components made up of the conditional effect of $a$, a substitution like effect and the
"level" effect operating through $F$. Where effects remain ambiguous, there is the
option to make use of the previously discussed relationships between $F$ and dual
functions such as $\Omega$ and $R$, in order to split up effects further into terms which may be separately capable of unambiguous interpretation. Whilst these issues are taken up in following chapters, it is appropriate at this point to summarise the relationship within which context the issues are to be discussed. By developing approaches such as these for the particular cases which are to be discussed in the following chapters, a clear statement of what may or may not be deduced from theory and what must be left to empirical examination can be obtained.

1.5 The Lot size Hypothesis and Empirical Tests.

The two main propositions stemming from the urban residential equilibrium model are that, in the large modern cities of the developed world, residential land prices will be a monotone decreasing convex function of radial distance and residential lot size will be a monotone increasing function of radial distance. These propositions come from both Alonso and Muth type models. The lot size hypothesis requires some restriction placed on the nature of the residential demand for land in the Alonso type model and builders’ derived demand for residential building land in the Muth type model. The lot size hypothesis is a strong theme in the comparative statics of the urban residential model. De Salvo (1977, esp. p. 8-10 and 1985), following a Muth (1969) urban residential model, in a comparative static exercise identifies those conditions for increasing housing consumption with radial distance. Muth op cit., shows that, in monocentric cities, that is cities with centralised employment, of modern developed economies, it is reasonable to expect, as a generalisation, increasing housing consumption with radial distance from city centre (p. 29-30). He
further shows that an increase in the quantity of land relative to building capital used in producing dwellings with radial distance is also a reasonable expectation (p. 54-60). Later work in the area continues this lot size theme. Fujita (1986, p. 79-80 and 1989, p. 18-22), using a model where households produce housing out of land, examines the lot size hypothesis directly in terms of increasing land consumption per household with radial distance. Turnbull (1992, 1994 and 1995) again examines the relationship between the demand for housing and radial distance through the price effects of radial distance on the Marshallian and Hicksian (compensated) demands for housing space. There are two main differences between the approach of the earlier to later works on the urban residential model of the lot size hypothesis.

- In the early works the route to household demand for residential space or land is through the bordered Hessian stemming from the conditions for the optimal solution of the direct utility problem. Later works utilise the envelope characteristics of the indirect utility function as a function of radial distance. For example, compare De Salvo (1977, p. 4) to Fujita (1989, p. 19). Perhaps an analogy here from consumer theory is a comparison of Hicks’ (1939, Mathematical Appendix, p. 307-314) approach to retrieving the Marshallian demands from the direct utility problem and that of Deaton and Muellbauer (1980, p. 41, summarised in Figure 2.10).

- Early work tends to be more definitive about the lot size hypothesis. That is, it is almost treated as axiomatic that a natural consequence of the market model of urban residential location is that housing consumption or residential lot size will be increasing in radial distance from urban centre. In later work the emphasis
changes to the demand and supply conditions which would lead to the lot size hypothesis, recognising that the lot size hypothesis is only one of several structures that could stem from the market conditions of the model, dependent on the characteristics of the demand and supply functions as functions of radial distance.

Neither of these points should be taken as a criticism of the earlier works on the urban residential market. They simply trace out the development of a model as it becomes more established. The first point illustrates the fact that later researchers could take advantage of developments within the analytics of the mainstream neo-classical economics of optimisation. Indeed what is surprising is that these developments were not taken up earlier in the exposition of the urban residential model. For example, De Salvo (1985, p. 162) was still undertaking comparative statics using the bordered Hessian. Sasaki (1987, p. 55), even though he utilises a dual to the direct utility problem of the urban residential consumer by posing the problem maximise bid rent for residential land for a given level of utility, still analyses outcomes using the bordered Hessian.

The second issue is perhaps a natural consequence of the environment within which the early models where developed. Casual empiricism of North American cities in the post war period indicated the traditional lot size hypothesis to be the norm. Moreover the two points are related. Retrieving demand functions as functions of distance utilising the envelope theorem gives the analyst a broader picture of those demand functions as functions of radial distance compared to more traditional methods of comparative static analysis.
In terms of comparative statics, it is also argued that changes in the values of parameters determining equilibrium outcome will be reflected in changes in these relationships with radial distance. Particularly in terms of lot size, increases in household real incomes or increases in commuting transport efficiency, resulting in a decline in real commuting cost, will lead to the suburbanisation of residential activity, reflected in the slopes of the lot size gradient. This then constitutes a third proposition stemming from the theoretical urban model.

These three propositions appear to be testable. However there are difficulties. The main issue in land price studies is observability of unimproved land prices. Sales of residential properties which consist of land only (without added capital) will be infrequent and will be biased to non-central locations. Empirical methods therefore require the isolation of structural effects on sale price from the price of the unimproved land. Building capital is ignored in the Alonso model and is assumed to be perfectly malleable homogeneous units in the Muth model. Observed sales of residential property will be of properties which are highly differentiated in fixed capital. This points the researcher in the direction of hedonic price indices, the construction of which is based around the relationship between price and the quality characteristics of classes of goods where product differentiation occurs within classes. For the early general use of hedonic price indices see Griliches (1971). There are important empirical issues in utilising such methods in studies of residential land prices. One important issue, highlighted in Rosen (1974), is the simultaneity of structural outcomes through demand and supply forces. Estimation methods must therefore address the problem of simultaneous equation bias. There is also a quality
issue linked to the market identification problem, which if ignored leads to
multicollinearity problems in estimation. An early example is Wilkinson (1973) which
used principal components analysis. This is not a method normally embraced by
econometrics because of the serious problem of lack of theoretical content in terms of
the principal components identified through the data. This contrasts with more
traditional methods associated with standard econometrics, see Horowitz (1987),
Murray (1983), and Witte et al. (1979). Can (1992) raises yet another rather
specialised issue, that of the problem of spatial autoregression, which also requires
attention in this context.

Another serious issue with hedonic price indices has been that of choice of functional
form, where clearly theory gives little direction. The practice of imposing arbitrary
functional forms which “work” according to some criteria, such as goodness of fit, has
increasingly given way to methods which allow the data to choose functional form in
estimation, such as the Box Cox transformation. For example compare Herrin and

However they are used, hedonic price studies of housing require a considerable
amount of data in terms of characteristics per observation. This in turn requires large,
formal property data bases which are in their infancy (at different stages of adequacy
between States) in Australia.

Price studies have not been confined to hedonic price equations. Where land price
data has been available, it has been used. A famous data set is the much used Hoyt’s
historical real estate series for Chicago, used in Kau and Sirmans (1979) and
McMillen (1990). These studies, albeit using quite historical data, were able to concentrate on the land price radial distance relationship. This might suggest utilisation of non-market estimates of land value as a surrogate for land market price. A candidate here would be estimates of land value for tax purposes. However this begs the questions, what methods were used in arriving at these estimates and why should tax assessors have more insight into land prices than economists? This is not a criticism of those involved in land valuation and property but just a recognition that they face the same problems in observation as the economist. An example of the use of land value estimates for Chicago is McMillen and McDonald (1991).

Lot size, notwithstanding the issue of multiple dwellings per lot, is directly observable. However data collection and availability is a problem. The reciprocal of land per household is household land density. Lot size studies have therefore tended toward density studies rather than pure lot size. If lot size is a monotone increasing function of distance, then density will be a monotone decreasing function of distance (Richardson, 1977a, p 23-24). However, testing the lot size hypothesis through density gradients, which is standard in urban economics, is not without its problems.

- Firstly, population and not household density is generally used. Most works use population census as the data source, where number of households per census tract should be utilised in principle, as well as persons per census tract.

- Secondly, data usually take the form of gross density, that is including non-residential land, whereas the reciprocal of lot size is really a net density measure, excluding non-residential land. If the relationship between net and gross density
remains unchanged with radial distance then the distinction between the two measures is unimportant. This is an unlikely circumstance. Given the geometry of the circle, it is more likely that gross density relative to net density increases with radial distance from centre. This has implications for the nature of testing lot size through density. Clearly these tests are forced to be truly Popperian in the sense that the search must be for density gradients which contradict the lot size hypothesis, that is density gradients increasing in distance from centre.

Early population density studies used the negative exponential function as a matter of course. This was the case prior to any formal urban modelling. See for example the influential early work of Clark (1951) and his later summary of density studies Clark (1982, chapter 14). Both Mills, (1967, p. 208-209) and Muth (1969, p 70-77) derive negative exponential density gradients from their equilibrium models and refer directly to Clark on this. However such outcomes are the result of important restrictions on the builder’s production function (a Cobb Douglas with constant returns) and the household’s demand function for housing, particularly an own price elasticity of unity. For an accessible review of Mills’ work and the derivation of the population density gradient see Kau et al. (1986). This work is interesting in itself because it shows (pages 209-224) the importance of population density gradients to empiricism in the residential model. Perhaps the best review on the literature of population density gradients can be found in McDonald (1989). Three elements can be discerned in the literature.

1. Many early studies simply test the exponential population density gradient for different cities, that is, attempt to replicate empirical outcomes stemming from the

2. Other studies use population density gradients to identify change in cities. A good example here is estimating gradients over time and analysing temporal change in slope parameters to examine such things as residential suburbanisation. See for example Mieszkowski and Smith (1991) and Mills (1972, chapter 3). These two studies both presume the negative exponential form, but approach the issue in two different ways. Mills uses an elegant interpolative method which is based on limited density information, the so called “two point estimate” of population density. Mieszkowski and Smith estimate a log-linear specification using ordinary least squares.

3. Some studies estimate population density gradients which are more flexible than just the exponential form. There are two reasons for this:

a.) Exponential gradients reinforce monocentricity. Alternative forms can be used to identify the development of sub-centres or even polycentric forms of cities. Some examples here are Alperovich (1980), Brueckner (1986) and Griffith, (1981). Within this category one might also include those studies which contrast the cities of less developed economies with those of the developed economies. See for example Mills and Tan (1980).

b.) Even though theoretical models might suggest a certain functional form, estimating empirical models from the outset using that functional form does not allow a thorough test of the theory. Only by using methods which
are flexible enough to encompass alternative functional forms can theory be truly tested.

Concentrating on point 3b, an extremely interesting series of contributions has been made around the issue of using estimation methods which allow for flexible functional forms and allow tests for specifications such as linear or log-linear. Kau and Lee (1976 a and b) estimated density gradients for various cities using the Box Cox transformation which allows for flexible functional form and testing between the linear and log-linear specification. For a good review of this work see Kau et al. (1986). An issue with Box Cox estimation is the problem of interpreting the function of the transformation parameter in the presence of heteroskedasticity. Lahiri and Numrich (1983) re-estimated the Kau and Lee density gradients using Box Cox estimation but using a criterion (the log-likelihood function) which is weighted for heteroskedastic errors. Another issue is the problem of error truncation in the Box Cox estimation. This issue is addressed by Numrich (1990) who again re-estimates the Kau and Lee gradients but with a criterion that reflects one sided error truncation.

The empirical thesis in this work follows a similar methodology in testing lot size through population density, using census data for the six Australian states' capital cities. Some novelty is introduced in the estimation through the use of two alternative flexible forms to the Box Cox transformation as well as Box Cox estimation itself. One of these alternatives is that proposed by Wooldridge (1992) and the other is a logarithmic version of the Wooldridge alternative. These alternatives, like the Box Cox transformation, nest within them the log-linear and linear specifications. Further, the thesis proposes the utilisation of a non-nested testing method based on the
artificial regression model of Davidson and Mackinnon (1981 and 1993, p. 381-384) to discriminate between Box Cox and the two alternatives. This specification test falls within the testing down procedure normally associated with general to specific methodology which was developed out of the Davidson et al. (1978) paper. Whilst the Wooldridge alternative to the Box Cox transformation has received some limited use elsewhere in econometric modelling, such as by Berndt et al. (1993) and Gencay and Yang (1996), it has not been used in conjunction with the third logarithmic form alternative, nor has the method of discriminating between models utilised here been employed in this context.

1.6 Structure of the Work.

This introductory chapter is followed by the theoretical model of part A. Chapter two of part A analyses the fundamental Von Thünen agricultural model using the generalised procedure outlined in section 1.4. Despite the fact that the focus of this work is urban it is appropriate to start with the fundamental agricultural model of Von Thünen because it forms the basis to the urban model. If the argument is that the method of section 1.4 is a general analytical method applying to all models which involve a trade-off between centralising and decentralising forces in location relative to some central focal point, then it must be demonstrated that it applies to the fundamental Von Thünen model as well as the urban residential model. Chapters three and four are taken up with the analysis of the urban residential model within a Muth framework. Chapter three deals with the urban consumer problem in residential location and chapter four is taken up by an examination of the residential builder’s
production problem. The theoretical thesis is then followed by part B, dealing with the empirical thesis. This comprises of two chapters where the lot size hypothesis is tested using population and household density gradients on the six Australian states' capital cities. Chapter five concentrates on methodological issues and chapter six is given over to estimation and results. Finally, outcomes of parts A and B are summarised in a brief concluding section.
PART A

THEORETICAL DEVELOPMENTS
2.1 Introduction

Samuelson (1983, p. 1468-1469) rightly indicates that Von Thünen’s “Isolated State” anticipated marginalism and general equilibrium. Moreover, Samuelson shows that these contributions have not been fully recognised in mainstream economics. In urban economics the story is different. Von Thünen’s contribution to location theory is well recognised and his agricultural model with its familiar rings or concentric zones of cultivation is seen as an important antecedent to the development of intra-urban models of location equilibrium. By the mid 1970s the rapid development of urban residential models had led to a readily accepted generalisation of residential location theory based on a Thünen type monocentric city. For the elements of such a model, see Richardson (1977b, p. 254-258). More recent developments of the urban economic model have been couched in terms of duality in optimising issues associated with residential location. Perhaps the most comprehensive application of duality in intra-urban location is Fujita (1989, chapter 2). Not only does he recast the standard Alonso type model in terms of duality, utilising the indirect utility and expenditure functions, but he also introduces an important dual problem in the issue, that of maximising offer price (bid rent) conditioned on utility. Not that Von Thünen has rested untouched by the urban theorists. Fujita and Thisse (1986) undertake an integration of a Von Thünen model and a Hotelling location interdependence model in the intra-urban location of residence and business.
The purpose of this chapter is to review Von Thünen's location model in terms of modern duality theory. It aims to take a different direction in the application of duality compared to that of Fujita, \textit{op cit.} Not only can Von Thünen's agricultural model be analysed in using the profit and cost function, but also alternatives to spatial profit maximisation and spatial cost minimisation, through the constrained minimisation of the difference between bid rent and market rent and the constrained minimisation of radial distance in location can be used. These alternatives are useful in that they can be used, with some modification, in analysing outcomes in urban residential consumption and production.

The remainder of this section will be devoted to setting up the elementary Von Thünen profit maximising paradigm. This will be followed by section 2.2 which analyses outcomes within this profit maximising paradigm and produces alternatives to the basic problem in terms of minimising the difference between offer rent and market rent and minimising distance as a direct, but constrained, choice. The next section recasts the Von Thünen model in terms of cost minimisation. Not only is the cost function examined but also similar alternatives in terms of minimising rent difference and minimising location as a direct choice are offered. Section 2.4 summarises the results of the two previous sections in terms of relating dual optimisation problems to each other within profit maximisation and cost minimisation objectives. Section 2.5 concentrates on the derived demand for land stemming from the Von Thünen model. In particular, the agricultural equivalent to the urban residential lot size hypothesis, which relates demand for land to radial distance, is examined.
Take the typical Von Thünen location problem. Farmers produce a crop, $i$, which they sell at a central market place for a unit price over which they have no power. All land is equally as fertile around the central place, and labour and capital inputs are mobile, so that labour and capital input prices will be invariant with location relative to the central place. Farmers have to bear the cost of transporting the crop to the central place. Transport cost is increasing in distance and marginal transport cost in distance is either constant or decreasing. Transport cost is identical in each direction from the central place so that location can be measured by radial distance from the market. Farmers' profit is defined as:

$$p^i f^i(K, L, N) - w_K K - w_L L - w_r^i N - t^i(r) f^i(K, L, N) \;; \quad (2.1.1)$$

where:

- $i$ is a superscript referring to the crop type
- $p^i$ is output price
- $f(\cdot)$ is a continuous monotone increasing strictly concave function of capital ($K$), labour ($L$) and land ($N$) inputs.
- $w$ is input price with relevant factor subscript.
- $t(r)$ is the transport cost per unit of output and is a function of radial distance, $r$, from central place, with $\frac{dr}{dt} > 0$ and $\frac{d^2t}{dr^2} \leq 0$.

Definition (2.1.1) could be factorised around output to give net receipt per unit of output, that is, to express output price as net of transport cost. However, it is
convenient to retain transport cost as a separate and explicitly distinct element in the following analysis.

2.2 The Profit Function

Profit maximisation can be set up as a two stage problem:

$$\max_r \left\{ \max_{K,L,N} \left\{ p^r f^r(K, L, N) - w_K K - w_L L - w_N N - t^r f^r(K, L, N) \right\} \right\}. \quad (2.2.1)$$

The solution to the inner problem of (2.2.1) is aspatial and results in a conditional profit function the arguments of which are output price, input prices and transport cost, which is itself a function of distance. Thus the outer problem can be written in terms of the aspatial profit function:

$$\max_r \left\{ \Pi^r(p^r, w, t^r(r)) = \max_{K,L,N} \left\{ p^r f^r(K, L, N) - w_K K - w_L L - w_N N - t^r f^r(K, L, N) \right\} \right\}; \quad (2.2.2)$$

where, for brevity, \( w \) is the vector of input prices.

The objective function for problem (2.2.2) will have all the properties that have been well identified for any non-location profit function in the literature, see Chambers, (1988, p. 124) and Cornes (1992, 116-117), in that it will be a decreasing convex function of input prices and an increasing convex function of output price. In the current context \( t' \) acts to reduce the net output price. Maintaining separate notation for
the gross output price and per unit transport cost, we state the conditions in extended form by including the property that the function is decreasing and convex in transport cost and the aspatial profit function is decreasing in distance from central place. The solution to this problem is an optimal location, \( r^* \), and substituting \( r^* \) into the aspatial profit function results in the spatial profit function:

\[
\Pi'_c = \Pi'_c(p', w) = \Pi'(p', w, t'(r^*)) .
\]  

(2.2.3)

At present the solution to (2.2.2) is trivial because under these conditions the location issue collapses to the profit maximising location, through minimising transport cost, adjacent to the central place.

Interesting and sensible solutions to (2.2.2) require that within the objective function the centripetal force of transport cost is counterbalanced by some centrifugal force in the aspatial profit function to be maximised. The route to this in Von Thünen type analysis is by demonstrating that offer prices for land (bid rents) are decreasing in distance. Thus land prices are an important part of the location process in the Von Thünen model and it would be worth viewing their relationship to the other parameters in the aspatial profit function.

Figure 2.2.1 a

- \( \Pi \) 
- \( w_n \) 
- \( w_k \) 
- \( w_L \) 
- \( t \) 

Figure 2.2.1 b

- \( \Pi \) 
- \( w_n \) 
- \( p \)
Figures 2.2.1 a and b show level curves of the aspatial profit function in the land price/capital price (or labour price or unit transport cost) plane and the land price/output price plane respectively. In these figures:

1. the heavy arrow shows direction of increasing profit; and
2. the shaded area shows the lower contour set which must be a convex set (see Madden, 1986, p. 51-52 ) and this implies that the elements of the principal diagonal of the Hessian matrix of the profit function must be non-negative (see Novshek, 1993, p. 98-99).

To demonstrate that land prices act as a centrifugal force and are decreasing in radial distance it is useful to solve the aspatial profit function for land price conditional on a given level of profit \( \pi' \), resulting in an offer price function conditioned on profit. To distinguish them from market prices, offer prices will be represented by:

\[
\omega'_K = \omega' m'(p', w_K, w_L, t'(r), \pi') .
\] (2.2.4)

Equation (2.2.4) is the bid rent function so important in location models based on the Von Thünen paradigm. The bid rent function is itself a maximum value function, because it is derived from the profit function. The bid rent function is the maximum bid rent that can be paid conditioned on profit and is the equation to the level curves of Figures 2.2.1 a and 2.2.1 b. The bid rent function is important, yet little has been said about it, particularly its first derivative properties. An exception is Sasaki (1987, p. 55), but he uses a quite different route to these properties. Most models proceed quickly to establish that the bid rent function is decreasing in profit level and a continuous monotone decreasing convex function of distance. A full examination of
the function's first derivative properties would be useful. The following properties can be generalised for all crops so it is possible to drop the crop superscript of (2.2.4). Substituting the aspatial profit function of (2.2.2), again dropping the crop superscript, into (2.2.4) gives the identity:

$$\omega_N(p, w_K, w_L, t(r), \Pi(p, w_K, w_L, w_N, t(r))) = w_N.$$  

(2.2.5)

It is now possible to establish the first derivative properties of $\omega_N$ by using the envelope theorem.

Differentiating (2.2.5) with respect to $w_N$ gives:

$$\frac{\partial \omega_N}{\partial \pi} \frac{\partial \Pi}{\partial w_N} = 1.$$

Utilising Hotelling's Lemma and rearranging gives:

$$\frac{\partial \omega_N(p, w_K, w_L, t(r), \pi)}{\partial \pi} = -\frac{1}{N}.$$

Thus it is possible to retrieve the derived demand for land through the bid rent function by:
\( N^\pi(p, w_K, w_L, t(r), \pi) = -\frac{1}{\partial \frac{\omega_N(p, w_K, w_L, t(r), \pi)}{\partial \pi}}. \)

Note that even though Hotelling's Lemma was used to find the expression for the derivative of offer price with respect to profit, the derived demand retrieved is not unconditional. The derived demand retrieved is conditional on profit level, hence the \( \pi \) superscript, and is also abstracted from own price. This must be so as the bid rent function is conditioned on profit and by definition cannot be a function of land price.

Similarly differentiating (2.2.5) with respect to \( p, w_K, w_L \) and \( t \), and utilising Hotelling's Lemma gives the first derivative properties of \( \omega_N \). Once these have been established the second order properties are established by further differentiation.

Properties 2.2.1 - (the properties of \( \omega_N \)):

1. a) Non-decreasing in \( p \).

\[
\frac{\partial}{\partial p} \omega_N(p, w_K, w_L, t(r), \pi) = -\frac{\partial \omega_N}{\partial \pi} \frac{\partial \Pi}{\partial p} = \frac{\nu}{N^\pi}(p, w_K, w_L, t(r), \pi) \geq 0 ; \text{ and}
\]

b) non-increasing in \( w_K, w_L, t, \) and \( \pi \).

\[
\begin{align*}
\frac{\partial}{\partial w_K} \omega_N(p, w_K, w_L, t(r), \pi) &= -\frac{\partial \omega_N}{\partial \pi} \frac{\partial \Pi}{\partial p} = -\frac{K^\pi}{N^\pi}(p, w_K, w_L, t(r), \pi) \\
\frac{\partial}{\partial w_L} \omega_N(p, w_K, w_L, t(r), \pi) &= -\frac{\partial \omega_N}{\partial \pi} \frac{\partial \Pi}{\partial p} = -\frac{L^\pi}{N^\pi}(p, w_K, w_L, t(r), \pi) \\
\frac{\partial}{\partial t} \omega_N(p, w_K, w_L, t(r), \pi) &= -\frac{\partial \omega_N}{\partial \pi} \frac{\partial \Pi}{\partial p} = -\frac{y^\pi}{N^\pi}(p, w_K, w_L, t(r), \pi) \\
\frac{\partial}{\partial \pi} \omega_N(p, w_K, w_L, t(r), \pi) &= -\frac{1}{N^\pi}(p, w_K, w_L, t(r), \pi)
\end{align*}
\[
\leq 0
\]
2. Recalling Figures 2.2.1 a and 2.2.1 b, $\omega_n$ is a convex function of $p, w_k, w_L$ and $t$, implying,

\[
\begin{align*}
\frac{\partial^2 \omega_N}{\partial p^2} &= \frac{\partial}{\partial p} \left( \frac{y^\pi}{N^\pi} \right) \\
\frac{\partial^2 \omega_N}{\partial w_k^2} &= \frac{\partial}{\partial w_k} \left( -\frac{K^\pi}{N^\pi} \right) \\
\frac{\partial^2 \omega_N}{\partial w_L^2} &= \frac{\partial}{\partial w_L} \left( -\frac{L^\pi}{N^\pi} \right) \\
\frac{\partial^2 \omega_N}{\partial t^2} &= \frac{\partial}{\partial t} \left( -\frac{y^\pi}{N^\pi} \right)
\end{align*}
\]

It is therefore possible to retrieve output and derived demands from $\omega_n$, but expressed as ratios to land input, abstracted from the price of land and conditioned on profit. Supply and input demands so conditioned carry the $\pi$ superscript.

Properties 2.2.1 may have implications for production issues other than the spatial allocation of land. Whilst the concept of offer prices for inputs seems to be well developed in land market and location issues, analysis of offer prices need not be limited to land input or spatial allocation problems. Properties 2.2.1 indicate that it is possible to relate optimal input ratios to change in relevant factor price. For example, in the agricultural profit maximising paradigm, through bid rents for land, it is possible to relate capital to land input ratios to change in capital input price. For those interested in input price influence on input ratios in production problems, analysing outcomes in terms of relevant bid rent or offer price might be useful. Moreover, this is not restricted to profit maximisation issues. Cost minimisation in section 2.3 will have a bid rent function analogue, but now conditioned on cost and output, rather than profit.
In the offer price/distance from central place plane, bid rent curves are level curves in
Because transport cost is an increasing concave function of distance these level curves
in the offer price/distance plane will have negative slopes and will be convex. That is,
for a given level of profit, any increase in transport cost must be balanced by a
compensatory reduction in offer price and the change of the compensatory offer price
diminishes with distance.

Recalling properties 2.2.1

\[
\frac{\partial \omega_N}{\partial t} < 0 \quad \text{and} \quad \frac{\partial^2 \omega_N}{\partial t^2} \geq 0;
\]

and from these inequalities it can be inferred that bid rents are a continuous monotone
decreasing convex function of distance. To see this, recall from the discussion
following (2.1.1) the basic assumptions:

\[
\frac{dt}{dr} > 0 \quad \text{and} \quad \frac{d^2 t}{dr^2} \leq 0,
\]

which lead to the following two corollaries:

Corollary 2.2.1

\[
\frac{\partial \omega_N}{\partial r} = \frac{\partial \omega_N}{\partial t} \frac{dt}{dr} < 0;
\]

Corollary 2.2.2

\[
\frac{\partial^2 \omega_N}{\partial r^2} = \frac{\partial \omega_N}{\partial t} \frac{d^2 t}{dr^2} + \frac{\partial^2 \omega_N}{\partial t^2} \left( \frac{dt}{dr} \right)^2 \geq 0.
\]
The corollaries give the inverse relationship between offer price and radial distance and the second derivative characterisation of a convex function respectively (Leonard and Van Long, 1992, p. 8, definition 1.1.4). Alternatively, using the rules on composite functions (Madden, p. 200-202 and Avriel, et al., p. 153-158), and recalling the concavity of the function \( l(r) \), because \( \omega_N \) is a monotone decreasing convex transformation of a monotone increasing, concave function of \( r \), \( \omega_N \) is a monotone decreasing, convex function of \( r \).

**Proposition 2.2.1**

*Offer prices or bid rents are decreasing in profit and are a monotone decreasing convex function of radial distance \( r \) from central place.*

Figure 2.2.2 illustrates this proposition with a level curve in profit \( \pi' \) which is convex in the offer price/distance plane. The heavy arrow shows the direction of increasing profit. The shaded area is the lower contour set \( L(\pi') \) and is the set of distance/offer price co-ordinates which result in profit levels where \( \pi \leq \pi' \).

**Figure 2.2.2**
In terms of the spatial price allocation mechanism it is now necessary to introduce \( m \) different agricultural goods, each crop type having a different set of offer prices implied by their profit function.

\[
\omega_N^i = \omega_N(p', w_K, w_L, t'(r), \pi^i) \quad \{i = 1, \ldots, m\}.
\] (2.2.6)

Under competitive conditions these \( m \) different land uses will bid against each other to capture land. Farmers will make normal profits only and market rent as a function of distance will consist of the normal profit bid rent of that crop which offers the highest bid rent at any given radial distance \( r \). Land uses with steep offer curves in distance will capture central locations and peripheral sites will be dominated by crop types which have shallow bid rents.

**Figure 2.2.3**

Market rent for agricultural land is a monotone decreasing convex function of distance from central place. It will be a continuous and piecewise smooth function, where each piecewise smooth segment is that portion of the normal profit offer gradient for that use which is willing to pay the highest market rent at a particular interval of location. Figure 2.2.3 illustrates the normal profit bid rent curves for three different crops. Locations to the left of distance 1 will be dominated by land use superscript 1 and at
these locations \( w_N = \omega_1 \), where price without a superscript refers to market price and with a superscript refers to offer price of a particular land use. Similarly the interval of locations 1 to 2 will be dominated by use 2 and \( w_N = \omega_2 \). Finally locations to the right of distance 2 will be taken up by crop 3 and \( w_N = \omega_3 \). The following definition, after Avriel et al., (1988, p. 294-295) formalises a general definition of convexity (F convexity) applied to the market rent function for land in distance \( w_N(r) \).

**Definition 2.2.1**

\( w_N(r) \) is a real function on the set \( S \) where \( S \subset \mathbb{R}_+ \).

\( w_N \) is F convex on the set \( S \) if for every \( r^0 \in S \) there is a function \( \omega(r) \) such that:

\[
\omega_N(r^0) = \omega(r^0) \quad \text{and} \quad w_N(r) \geq \omega(r) \quad \forall r \in S.
\]

In the definition of F convexity, let \( r^0 \) be a particular distance under examination. Further, let \( \omega(r^0) \) be that bid rent function which is successful at \( r^0 \). Then this implies \( w_N(r^0) = \omega(r^0) \), but it also implies \( w_N(r) \geq \omega(r) \) for \( r \neq r^0 \). Then definition 2.2.1 says that \( w_N \) is F convex.

In this definition of F convexity of market rents for land the family of support functions \( \omega^i(r) \), which support the F convex function \( w_N(r) \) from below need not be convex. The fact that the family of support functions, which consist of the normal profit bid rent curves of each crop \( i \), are convex ensures that market rent is a convex function of distance \( r \). Convex bid rent curves are not a necessary condition. The
definition implies that monotone decreasing bid rents are necessary for decreasing F
covex market rents in distance. Interestingly Alonso (1960, Figure 4) in his
geometric exposition defines residential land market rent as a strictly convex function
of radial distance, where market rent is supported from below at each location by a
different bid rent function. That is he anticipates a complete non-coincidence of
competing household bid rent. This results in a smooth continuous function of \( r \) and
the weak inequality in the second condition of definition 2.2.1 would be replaced by a
strong inequality. This contrasts strongly with the Von Thünen type market rent
gradient where piecewise smooth convexity allows for significant concentric zones of
land use. Similarly, Fujita (1989, chapter 4) in his residential model allows for classes
of households, where within classes there is complete coincidence of bid rent and
between classes there is non-coincidence of bid rent. The result is a convex
residential market rent gradient which is piecewise smooth, and significant concentric
residential zones for different household classes. This distinction will become
important in the residential model of chapter three. For the present, according to
definition 2.2.1, convex bid rent curves supporting the market rent function from
below are sufficient, but not necessary for a convex market rent function in radial
distance.

**Proposition 2.2.2**

*Land market rents determined through competitive bidding, where
bid rents of differentiated land uses are a monotone decreasing
convex function of radial distance \( r \), will be a monotone decreasing
convex function of distance \( r \).*
If we are prepared to ignore the issue of derivatives at turning points in piecewise smooth functions, utilising proposition 2.2.2, it is now possible to rewrite problem (2.2.2) with an aspatial profit function where land market price as well as transport cost is a function of distance $r$.

$$ \Pi_r(p, w_k, w_L) = \max_r \left\{ \Pi(p, w_k, w_L, w_N(r), t(r)) \right\} . \quad (2.2.7) $$

In (2.2.7), having established the convexity of market rents in distance, it is now sensible to drop the $i$ superscript. Under competitive conditions land users are not restricted to the specialist production of a particular crop. Zero barriers to entry would imply that land users are free to take up that crop which is most profitable at any location so that farmers face a general aspatial profit function (2.2.7). The solution to the problem will be a spatial profit function, because $r$ has been optimised out. That is the spatial problem has now been addressed and in this case the outcome is the spatial profit function. Following the convention of chapter 1 such functions are denoted by the subscript $r$.

The first order condition for (2.2.7) is:

$$ \max_r \left\{ \Pi(p(r), w_k, w_L, w_N(r), t(r)) \right\} ; \quad (2.2.7n) $$

with first order condition:

$$ \frac{\partial \Pi}{\partial w_k} \frac{d w_k}{d r} = \frac{\partial \Pi}{\partial t} \frac{d t}{d r} + \frac{\partial \Pi}{\partial p} \frac{d p}{d r} . \quad (2.2.8n) $$

Note that $p(r)$ will be a 'step' function. Price $p$ will not vary with distance $r$ when crops are not 'switched' so that $dp/dr = 0$ for most locations and (2.2.8) applies. At crop transference points such as distances 1 and 2 in Figure 2.2.3 there may be a discontinuous upward or downward shift in price so that $dp/dr$ will not exist at these locations. Thus the boundary between crops may be characterised by the non-existence of $dp/dr$ if the relevant crops have different market prices. Fortunately this issue
\[
\frac{\partial \Pi}{\partial w_N} \frac{d w_N}{d r} = \frac{\partial \Pi}{\partial t} \frac{d t}{d r}.
\] 

(2.2.8) is the agrarian restatement of the now classic urban trade-off equation (Muth, 1969, p. 23, equation (3') and Turnbull, 1995, p. 10, equation (2.5)), where optimal location(s) is that (are those) location(s) where the increase in profit through reduction in land cost (left hand side of (2.2.8)) is just counterbalanced by the reduction in profit due to the increase in transport cost (right hand side of (2.2.8)) with relocations further from the central place.

A solution to (2.2.7) is ensured if the profit function is a continuous concave function of radial distance \( r \), so that it at first increases and then decreases with \( r \) for \( r \geq 0 \). A requirement for this is that the reduction in profit brought about by the increase in transport cost through relocations away from centre is at first overcome by the increase in profit brought about by the reduction in land cost. However, with further relocations away from the central market, the reduction in profit through increased transport cost outweighs the increase in profit through reduction in land cost.

Given the formation of bid rents and market rents for all locations it is possible to make the following proposition:

---

*does not occur in the urban residential consumer or producer problem, where land uses are not ‘switched’ involving output price changes. Because we are interested in location relative to \( r \) and not crop locations relative to each other, we can ignore this issue and continue with (2.2.7) and (2.2.8).*
Proposition 2.2.3

At locations close to the centre:

\[ \left| \frac{\partial \omega^i_N}{\partial r} \right| < \left| \frac{\partial w_N}{\partial r} \right| ; \]

and at locations further away from centre:

\[ \left| \frac{\partial \omega^i_N}{\partial r} \right| \geq \left| \frac{\partial w_N}{\partial r} \right| . \]

These inequalities are determined by the fact that market rent is supported from below by bid rent. This is illustrated in Figure 2.2.4., where for land use \( i \) to the left of \( r_1 \) the first inequality holds and to the right of \( r_2 \) the second applies.

![Figure 2.2.4](image)

Proposition 2.2.3 combined with \( \partial \omega^i_N / \partial \pi^i < 0 \) ensures the concavity of profit with distance.

Returning to the first order conditions for profit maximisation, using Hotelling's Lemma one can substitute the derived demand for land in (2.2.8) for \( -\frac{\partial \Pi}{\partial \omega^i_N} \). Solving
for the slope of the land market rent gradient gives the agricultural version of Muth's classic urban housing rent equation, (Muth, 1969, p. 22, equation (3) and Turnbull, 1995, p. 10, equation (2.6)):

\[
\frac{dw_N}{dr} = \frac{\frac{\partial \Pi}{\partial t}}{\frac{\partial t}{\partial r}} \frac{dt}{dr} \frac{1}{N(p, w_K, w_L, w_N(r), t(r))}; \tag{2.2.9}
\]

where the function \( N \) is the derived demand for land.

By a further use of Hotelling's Lemma one can retrieve the negative of the supply function as:

\[
\frac{\partial \Pi}{\partial t} = -\gamma(p, w_K, w_L, w_N(r), t(r)) \tag{2.2.10}
\]

Substituting into 2.2.9 gives:

\[
\frac{dw_N}{dr} = \gamma(p, w_K, w_L, w_N(r), t(r)) \frac{dt}{dr} \frac{1}{N(p, w_K, w_L, w_N(r), t(r))} \leq 0 \tag{2.2.11}
\]

Equation (2.2.11) confirms that market rent diminishes with distance from central place.

Because of the construction of market rents from bid rents it follows that land uses with shallow bid rent curves capture peripheral locations. Given this, (2.2.11) leads to the proposition:
Proposition 2.2.4

Those agricultural land uses with high ratios of output to land input will take up central locations and those agricultural land uses with low ratios of output to land input will take up peripheral locations.

It is possible to go further in relating land allocation to radial distance by relating the derived demand for land to distance \( r \). This important issue will be deferred to section 2.5 pending outcomes from dual problems to profit maximisation.

Bid rents for a crop (2.2.4) were arrived at by solving the aspatial profit function in (2.2.2) for land prices and treating the result as offer prices conditioned on profit level. Clearly the solution to (2.2.7) gives a profit level and a bid rent which is actually paid and therefore is the market price for land. The relationship of land market prices to radial distance \( r \) was predicated on the fact that a market price paid by a land user at a location \( r \) is the same as that land user’s bid rent at location \( r \) conditioned on profit received.

A realistic alternative to the profit maximising problem, which would fully replicate the solution to (2.2.7), is to change the objective to that of choosing location to minimise the difference between bid rent and market rent. This objective is sensible in that it conforms with the objective of profit maximisation. Farmers are constrained in this objective by the condition that bids must be at least equal to market rent at a location.

The solution to this problem is;
\( \Omega_N(p, w_K, w_L, \pi) \)
\[ = \min_r \left\{ \omega_N(p, w_K, w_L, t(r), \pi) - w_N(r) : \omega_N(p, w_K, w_L, t(r), \pi) - w_N(r) \geq 0 \right\} . \]

(2.2.12)

Note that the function \( \Omega \) carries the \( r \) subscript and is a spatial optimal value function because \( r \) has been optimised out. A solution to the problem exists because:

- the constraint set \( \omega_N(p, w_K, w_L, t(r), \pi) \geq w_N(r) \) is a convex set \((\omega_N(\_ ) - w_N(r) \) is a concave function of \( r \)); and

- \( \omega_N(p, w_K, w_L, t(r), \pi) \geq w_N(r) \) is continuous in \( r \);


In terms of Figure 2.2.4 the solution to problem (2.2.12) for crop \( i \) is the interval \( r_1 \leq r \leq r_2 \). The solution is characterised by the following first order conditions:

\[
\frac{\partial \omega_N}{\partial t} \frac{dt}{dr} = \frac{dw_N}{dr} ; \text{ and } \omega_N(p, w_K, w_L, t(r), \pi) = w_N(r) .
\]

(2.2.13)

Recalling properties 2.2.1 and substituting into (2.2.13) for \( \partial \omega_N/\partial t \) gives:

\[
\frac{d w_N}{dr} = - \frac{y^\pi}{N^\pi} (p, w_K, w_L, t(r), \pi) \frac{dt}{dr} \leq 0 .
\]

(2.2.14)
Proposition 2.2.4 follows directly from (2.2.14), except that this result is now conditioned on profits. Thus proposition 2.2.4 is approached from the point of view that in order to maintain profit levels farmers, when faced with increasing transport cost with distances further from the market centre, will have to produce output with a higher ratio of land to output.

By its construction $\Omega_{n}$ is the implicit profit function. This can be seen by forming an identity by substituting the spatial profit function of (2.2.7) into $\Omega_{n}$:

$$\Omega_{n}(p, w_{K}, w_{L}, \Pi_{r}(p, w_{K}, w_{L})) = 0 \quad (2.2.15)$$

This must be identically equal to zero as optimal locations require the difference between bid rents and market rents to be zero. Differentiating (2.2.15) with respect to $p, w_{K}$ and $w_{L}$ and utilising Hotelling’s Lemma gives:

$$\frac{\partial \Pi_{r}}{\partial p} = -\frac{\partial \Omega_{n}}{\partial \Omega_{n}} \frac{\partial p}{\partial \pi} \geq 0 \quad ; \quad \frac{\partial \Pi_{r}}{\partial w_{K}} = -\frac{\partial \Omega_{n}}{\partial \Omega_{n}} \frac{\partial w_{K}}{\partial \pi} \leq 0 \quad \text{and} \quad \frac{\partial \Pi_{r}}{\partial w_{L}} = -\frac{\partial \Omega_{n}}{\partial \Omega_{n}} \frac{\partial w_{L}}{\partial \pi} \leq 0 \quad .$$

These appear to be similar to Roy’s theorem but unfortunately they cannot be used to fully establish the first derivative properties of the function $\Omega_{n}$.

Problem (2.2.12) suggests a dual to (2.2.7) is:

$$R(p, w_{K}, w_{L}, \pi) = \min_{r} \{ r \in \mathcal{R}_{+} : \Omega_{n}(p, w_{K}, w_{L}, r(\pi)) - w_{n}(r) \geq 0 \} \quad . \quad (2.2.16)$$
That is, for a given bid rent function, what is the closest to the central place that cultivation of the crop associated with that bid rent function can take place? This objective is again subject to the constraint that at least the payment of market rent at that location must be made. A solution to this problem exists because the constraint set, conditioned on profit, is a convex set \((\omega_N - w_N)\) is a concave function of \(r\) and \(r\) is continuous. The solution to this is \(r\), in Figure 2.2.4 when \(\pi=0\), which is the closest to the central place that the farmer can locate cultivating a given crop.

An alternative to (2.2.16) is:

\[
\overline{R}(p, w_K, w_L, \pi) = \max \{ r \in \mathbb{R}_+ : \omega_N(p, w_K, w_L, t(r), \pi) - w_N(r) \geq 0 \} . \quad (2.2.17)
\]

The solution to this is \(0r_2\) in Figure 2.2.4 when \(\pi=0\), which is the furthest from the central place that the farmer can locate cultivating a given crop. If the solutions to (2.2.16) and (2.2.17) are identical then the extent of a crop is a single point in radial distance from centre. This would suggest that crop types are so highly differentiable that land market rents are a smooth continuous convex function of radial distance. This might produce some difficulties with a competitive market structure. However, smooth continuous convexity would be reasonable in an urban residential land market with competing but highly differentiated households on the basis of time transport cost. Under these conditions the sensible economic problem would be (2.2.16) given the centripetal nature of urban agglomerative forces.
Both \( \mathcal{R} \) and \( \mathcal{R} \) are dual to the outcome of (2.2.7) and the spatial profit function of (2.2.7) can be retrieved from either of them by inversion. Both of the optimal value functions of (2.2.16) and (2.2.17) are spatial analogues to the distance or transformation function, (Shephard, 1981, p 5-8, esp. equation (2) and Blackorby et al., 1978, chapter 2, sections 2.3.2 and 2.3.3). The transformation function in standard production models completely describes the production function, see Shephard op cit.. In the consumer problem preference ordering is completely described by the distance function, Blackorby et al., op cit.. Similarly for a given crop and conditioned by profit the functions \( \mathcal{R} \) and \( \mathcal{R} \) transform bid rents into minimum and maximum radial distances respectively.

One can make comparative statics around optimal distance. Substituting the aspatial profit function of (2.2.7) into (2.2.16) and (2.2.17) forms the identities:

\[
\mathcal{R}(p, w_K, w_L, \Pi(p, w_K, w_L, w_N(r), t(r))) = r ; \quad \text{and} \quad \mathcal{R}(p, w_K, w_L, \Pi(p, w_K, w_L, w_N(r), t(r))) = r .
\]

(2.2.18)  
(2.2.19)

Dropping the under and over bars for the moment and differentiating (2.2.18) or (2.2.19) with respect to \( r \) gives:

\[
\frac{\partial R}{\partial \pi} = \frac{1}{\frac{\partial \Pi}{\partial t} + \frac{\partial \Pi}{\partial w} \frac{1}{\frac{\partial t}{dr} + \frac{\partial w}{dr}}} \leq 0 ; \quad \text{and} \quad \mathcal{R}(p, w_K, w_L, \Pi(p, w_K, w_L, w_N(r), t(r))) = r .
\]

(2.2.20)
the sign of which depends on the relative values of the two terms of the denominator on the right hand side. This reflects the inherent forces in the trade-off effects of location.

Differentiating (2.2.18) or (2.2.19) with respect to \( p, w_k \) and \( w_L \) gives:

\[
\frac{\partial R}{\partial p} = \frac{\partial R \partial \Pi}{\partial \pi \partial p}, \quad \frac{\partial R}{\partial w_k} = \frac{\partial R \partial \Pi}{\partial \pi \partial w_k} \quad \text{and} \quad \frac{\partial R}{\partial w_L} = \frac{\partial R \partial \Pi}{\partial \pi \partial w_L}.
\]

The sign of these derivatives could be determined if the common term on the right hand side was known. It is possible to sign this term if the nature of the function \( R \) is identified. If it is the minimum distance, problem (2.2.16), then recalling Figure 2.2.4, for small perturbations around the optimal location:

\[
\frac{\partial \Pi}{\partial t} \frac{dt}{dr} + \frac{\partial \Pi}{\partial w} \frac{dw}{dr} \geq 0.
\]

The converse is true for the function \( R \) in the maximisation problem (2.2.17) and for small perturbations around optimal \( r \):

\[
\frac{\partial \Pi}{\partial t} \frac{dt}{dr} + \frac{\partial \Pi}{\partial w} \frac{dw}{dr} \leq 0.
\]

With these conditions it is now possible to undertake comparative statics:
\[ \frac{\partial \bar{R}}{\partial p} = \frac{\partial R \partial \Pi}{\partial \pi \partial p} \leq 0 \quad \text{and} \quad \frac{\partial \bar{R}}{\partial p} = \frac{\partial R \partial \Pi}{\partial \pi \partial p} \geq 0 , \]

so that an increase in output price increases the extent of crop location. Similarly:

\[ \frac{\partial \bar{R}}{\partial w_k} = \frac{\partial R \partial \Pi}{\partial \pi \partial w_k} \geq 0 \quad \text{and} \quad \frac{\partial \bar{R}}{\partial w_k} = \frac{\partial R \partial \Pi}{\partial \pi \partial w_k} \leq 0 ; \quad \text{and} \]

\[ \frac{\partial \bar{R}}{\partial w_L} = \frac{\partial R \partial \Pi}{\partial \pi \partial w_L} \geq 0 \quad \text{and} \quad \frac{\partial \bar{R}}{\partial w_L} = \frac{\partial R \partial \Pi}{\partial \pi \partial w_L} \leq 0 , \]

so that an increase in capital or labour input prices for a particular crop decreases the extent of crop location.

Unfortunately it is not possible to undertake comparative statics on the effect of an increase or decrease of transport cost at each location through (2.2.18) or (2.2.19). However, the effects of an upward or downward shift of \( l(r) \) can be intuitively derived through utilising the first order properties of the bid rent function, properties 2.2.1 1b.

\[ \frac{\partial \omega}{\partial t}(p, w_k, w_L, l(r), \pi) \leq 0 , \]

means an increase in transport cost at all locations decreases bid rents conditioned on a profit level at all locations. Combining this downward shift with the competitive determination of market rents (recall Figure 2.2.3) implies an increase in transport cost at all locations reduces the extent of crop location. The converse is true for a decrease in transport cost at all locations.
However the effect of a change in marginal transport cost in radial distance can be derived from (2.2.18) and (2.2.19) through (2.2.20). Rearranging this latter equation gives the condition for optimal $r$ whether it is minimum or maximum radial distance as:

$$\frac{\partial R}{\partial \pi} \left[ \frac{\partial \Pi}{\partial t} \frac{dt}{dr} + \frac{\partial \Pi}{\partial w} \frac{dw}{dr} \right] = 1 .$$

The sign of the first product in the bracket is negative and the second product sign is positive. A decrease in marginal transport cost in radial distance therefore implies that optimal $r$ increases. That is in the agricultural case the inner and outer boundaries of the concentric zone of cultivation for this crop shift outwards. The converse applies to an increase in marginal transport cost.

### 2.3 The Cost Function

A brief examination of the profit problem (2.1.1) quickly reveals that the spatial allocation problem of Von Thünen is tractable using traditional dual methods between cost and profit. This is because the location variable $r$ is restricted to cost and does not appear in revenue in (2.1.1). This means that all of the preceding outcomes can be derived from the farmer’s cost function without resorting to the strong assumption of profit maximisation. Thus, once the aspatial cost function has been established, it is then possible to identify dual problems of minimising the difference between bid and market rents and minimising distance, similar to those introduced in the previous section.
The first stage in the argument requires that the relationship between land market prices and radial distance is established as a centrifugal force countering the centripetal attractions of lower transport cost. To initiate this argument it is necessary to define the cost dual to (2.2.2) for a specific crop:

\[
\min_r \left\{ C'(w, t'(r), y') = \min_{k, l, n} \left\{ w_k K + w_l L + w_n N + t'(r) f'(K, L, N) : f'(K, L, N) \geq y' \right\} \right\}; \tag{2.3.1}
\]

where notation takes on the same meaning as in section 2.2 and \( y' \) is the crop production constraint.

In its non-location aspects (2.3.1) is a standard cost function and therefore is an increasing monotone concave function of input prices and is non-decreasing in output \( y \). Further, it is a positive linear homogeneous function in input prices. Given the definition of transport cost, (2.3.1) will be a positive linear homogeneous concave function of transport cost.

It is not possible to proceed in the derivation of market rents without saying something about the motivations of farmers associated with problem (2.3.1). Clearly profit maximising objectives will imply the cost minimising behaviour specified in (2.3.1), together, of course, with optimal choice of output. However, cost minimising behaviour may be determined by motivations other than profit maximisation. In the following argument it is assumed, for all crops, that farmers wish to minimise cost for the same reason, which may not be profit maximisation. Having said this, the cost function can be utilised in exactly the same way as the profit function. If land input
prices are invariant in distance cost minimising location collapses to the central place. Land prices therefore must act as a dispersive force.

Again because land prices are important in the location decision through competition it would be useful to look at land price and other input price and land price and output more closely.

Figure 3.2.1

Figure 3.2.1 shows the level curve of the aspatial cost function in the land price/capital price (or labour price or unit transport cost or output) plane, where:

1. the heavy arrow shows direction of increasing cost; and
2. the shaded area shows the upper contour set which must be a convex set.

For a three dimensional view of the level curve of the cost function see Chambers (1988, p. 88). Solving the aspatial cost function in (2.3.1) for land price:

\[
\omega^c_N = \omega^c_N(w_K, w_L, t'(r), y', c')
\]  

(2.3.2)
gives the equation to the level curves of the cost function. At the risk of making even more complex notation it is necessary to distinguish the bid rent function and those other functions derived from it in cost minimisation from similar functions stemming from profit. This is done through use of the superscript $c$. Like (2.2.4), (2.3.2) is a maximum value function in that it defines the maximum bid for land units conditioned on cost.

The first derivative properties of (2.3.2) can be retrieved from the identity resulting from the substitution of the aspatial cost function of (2.3.1) into the bid rent function, (2.3.2). Again this is a generalisation, so that crop superscripts can be dropped:

$$
\omega^c_N(w_K, w_L, t(r), y, C(w_K, w_L, w_N, t(r), y)) = w_N .
$$

(2.3.3)

Differentiating (2.3.3) with respect to $w_N$ gives:

$$
\frac{\partial \omega^c_N}{\partial c} \frac{\partial c}{\partial w_N} = 1 .
$$

Utilising Shephard’s Lemma and rearranging gives:

$$
\frac{\partial \omega^c_N(w_K, w_L, t(r), y, c)}{\partial c} = \frac{1}{N} .
$$

Again it is possible to retrieve the derived demand for land through the bid rent function by:

$$
N^\sigma(w_K, w_L, t(r), y, c) = \frac{1}{\frac{\partial \omega_N(w_K, w_L, t(r), y, c)}{\partial \pi}} .
$$
Note, that even though Shephard’s Lemma was used to find the expression for the derivative of offer price with respect to cost, the derived demand retrieved is not just conditional on output. The derived demand retrieved is also conditional on cost level and is abstracted from own price. This must be so as the bid rent function is conditioned on cost and by definition cannot be a function of land price. This derived demand is denoted by the superscript \( \psi \).

Differentiating (2.3.3) with respect to \( w_K, w_L \) and \( \tau \), and \( y \) and utilising Shephard’s Lemma gives the first derivative properties of \( \omega_N^\psi \). Then the second order properties can be established by further differentiation.

**Properties 2.3.1 - (properties of \( \omega_N^\psi \))**

1. a) Non-decreasing in \( \tau \).

\[
\frac{\partial \omega_N^\psi(w_K, w_L, \tau(r), y, \tau)}{\partial \tau} = \frac{1}{N^\psi(w_K, w_L, \tau(r), y, \tau)} \geq 0
\]

b) Non-increasing in \( w_K, w_L, \tau \) and \( y \).

\[
\begin{align*}
\frac{\partial \omega_N^\psi(w_K, w_L, \tau(r), y, \tau)}{\partial w_K} &= -\frac{\partial \omega_N^\psi}{\partial c} \frac{\partial \tau}{\partial w_K} = -\frac{K^\psi}{N^\psi}(w_K, w_L, \tau(r), y, \tau) \\
\frac{\partial \omega_N^\psi(w_K, w_L, \tau(r), y, \tau)}{\partial w_L} &= -\frac{\partial \omega_N^\psi}{\partial c} \frac{\partial \tau}{\partial w_L} = -\frac{L^\psi}{N^\psi}(w_K, w_L, \tau(r), y, \tau) \\
\frac{\partial \omega_N^\psi(w_K, w_L, \tau(r), y, \tau)}{\partial \tau} &= -\frac{\partial \omega_N^\psi}{\partial c} \frac{\partial \tau}{\partial \tau} = -\frac{y}{N^\psi}(w_K, w_L, \tau(r), y, \tau) \\
\frac{\partial \omega_N^\psi(w_K, w_L, \tau(r), y, \tau)}{\partial y} &= -\frac{\partial \omega_N^\psi}{\partial c} \frac{\partial \tau}{\partial y} = -\frac{\partial C^\psi}{N^\psi}(w_K, w_L, \tau(r), y, \tau)
\end{align*}
\]
2. Recalling Figure 2.3.1 $\omega^c_N$ is a convex function of $w_k$, $w_l$, $t$ and $y$, this implies:

\[
\begin{align*}
\frac{\partial^2 \omega^c_N}{\partial w_k^2} &= \frac{\partial}{\partial w_k} \left( -\frac{K^{cy}}{N^{cy}} \right) \\
\frac{\partial^2 \omega^c_N}{\partial w_l^2} &= \frac{\partial}{\partial w_l} \left( -\frac{L^{cy}}{N^{cy}} \right) \\
\frac{\partial^2 \omega^c_N}{\partial t^2} &= \frac{\partial}{\partial t} \left( -y/N^{cy} \right) \\
\frac{\partial^2 \omega^c_N}{\partial y^2} &= \frac{\partial}{\partial y} \left( -(\partial y/\partial C)/N^{cy} \right) \\
\end{align*}
\]

It is therefore possible to retrieve output and derived demands from $\omega^c_N$, but expressed as ratios to land input, abstracted from the price of land and conditioned on cost as well as output. Supply and input demands so conditioned carry the $cy$ superscript.

In the offer price/radial distance plane bid rent curves are level curves associated with a level of total cost $c$ and not profit level $\pi$ as in (2.2.4). The other difference between (2.3.2) and (2.2.4) is due to the different parameters associated with problems (2.2.2) and (2.3.1). That is output price in (2.2.4) is replaced with output level in (2.3.2). In both (2.2.4) and (2.3.1) the location characterisation is determined by transport cost. Therefore, in terms of location, (2.3.1) can be utilised in much the same way as (2.2.4).

If the cost function is an increasing concave function of input prices and transport cost, the upper contour set is convex. The following corollaries stem from the fact that the cost function is an increasing concave function of input prices and transport cost and transport cost is an increasing concave function of $r$. 

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Corollary 3.2.1

\[
\frac{\partial \omega^c_N}{\partial r} - \frac{\partial \omega^c_N}{\partial t} \frac{dt}{dr} < 0
\]

Corollary 3.2.2

\[
\frac{\partial^2 \omega^c_N}{\partial r^2} = \frac{\partial \omega^c_N}{\partial t} \frac{d^2 t}{dr^2} + \frac{\partial^2 \omega^c_N}{\partial t^2} \left(\frac{dt}{dr}\right)^2 \geq 0
\]

The corollaries give the inverse relationship between offer price and radial distance and the second derivative characterisation of a convex function.

Proposition 3.2.1

Offer prices or bid rents are increasing in cost and are a monotone decreasing convex function of radial distance r from central place.

Moreover, even though these corollaries lead to a well behaved family of continuous convex bid rent curves in distance, this is not determined by strictly convex level curves in the production function. Diewert (1982, p. 539-541) and Cornes (1992, p. 17-19) demonstrate that the cost function is a well behaved continuous concave function in its arguments if the objective function (producer cost) is concave in its arguments. Clearly producer cost is concave in input prices, and is concave in distance through transport cost by assumption. Even though it is not necessary to assume a highly regular production function to obtain well behaved bid rents, the assumption of a smooth continuous quasi-concave production function ensures the existence and uniqueness of spatial equilibrium in the agrarian land market, by ensuring that the derived demand for land is continuous in price (and therefore radial.
distance). These arguments will be revisited in more depth in the urban residential
consumer model of chapter three

Finally like (2.2.4), (2.3.2) is a maximum value function in that it defines the
maximum bid for land units conditioned on cost.

The dual to (2.2.6) is:

$$\omega_{\tilde{c}}^i(\omega^i_{\tilde{c}}(w_k, w_L, t^i(r), y^i, c^i) \quad \{i = 1, \ldots, m\} \quad (2.3.4)$$

Introducing a competitive land market results in a market rent gradient which is a
piecewise smooth monotonically decreasing convex function of distance from central
place. Each piecewise smooth segment is that portion of the offer curve for that use
which is willing to pay the highest market rent at a particular interval of location.
However, each segment will not now be associated with the same level of profit, but
will be associated with the same level of cost, $c$. Thus market rent, $w_N$, is a convex
function on $r$ which is supported from below by a family of convex functions $\omega_{\tilde{c}}^i(r)$
(recall definition 2.2.1). Even though $w_N(r)$ is a convex function of distance it will
not replicate the market rent gradient of section 2.2 if cost minimisers are not
motivated by profit maximisation. Only if output level $y$ in (2.3.4) is that which is
associated with profit maximising output will the rent gradient, derived from cost
minimisation, be identical to that rent gradient derived from profit maximisation.

If we are prepared to ignore the issue of derivatives at turning points in piecewise
smooth functions it is now possible to substitute the market land rent function $w_N(r)$
into (2.3.1) and dropping crop superscript the general cost minimising location problem is:

\[ C_r(w_K, w_L, y) = \min_r \{ C(w_K, w_L, w_N(r), t(r), y) \} \]  

(2.3.5)

where \( C \) is the aspatial cost dual to the aspatial profit function (2.2.7) and \( C_r \) is the spatial cost function.

The first order condition for (2.3.5) gives the familiar trade-off equation but now written in terms of cost and not profit:\(^2\):

\[ \frac{\partial C}{\partial w_N} \frac{dw_N}{dr} = \frac{\partial C}{\partial t} \frac{dt}{dr} \]  

(2.3.6)

(2.3.6) is the agrarian version of urban trade off, but in terms of farmers’ cost. Optimal location(s) is that (are those) location(s) where the reduction in land cost (left hand side of (2.3.6)) is just countered by the increase in cost through increased transport cost (right hand side of (2.3.6)) with relocations further from central place.

\(^2\) Recalling footnote 1, (2.3.6) is true almost every where. However, even though output \( y \) is measured in quantity, if quantity of output varies amongst crop types, and crop types vary with distance then (2.3.5) should be written as;

\[ \min_r \{ C(w_K, w_L, w_N(r), t(r), y(r)) \} \]  

(2.3.5n)

and 2.3.6 as;

\[ \frac{\partial C}{\partial w_N} \frac{dw_N}{dr} = \frac{\partial C}{\partial t} \frac{dt}{dr} + \frac{\partial C}{\partial y} \frac{dy}{dr} \]  

(2.3.6n)

\( y(r) \) will be a 'step' function. Output \( y \) will not vary with distance \( r \) when crops are not 'switched' so that \( dy/dr = 0 \) for most locations and (2.3.6) applies. At crop transference points there may be a discontinuous upward or downward shift in output so that \( dy/dr \) will not exist at these locations. An alternative characteristic of a crop boundary therefore is the non-existence of \( dy/dr \).
A solution to (2.3.6) is ensured if the cost function is a continuous convex function of radial distance \( r \), so that it at first decreases and then increases with \( r \) for \( r \geq 0 \). A requirement for this is that the increase in cost brought about by the increase in transport cost through relocations away from the centre is at first overcome by the reduction in cost brought about by the reduction in land cost. However, with further relocations away from the central market the increase in cost through increased transport cost outweighs the decrease in cost through reduction in land cost. Proposition 2.2.3 combined with \( \frac{\partial \omega_N}{\partial c} > 0 \) ensures the convexity of cost with distance.

Solving (2.3.6) for the first derivative of land rent with respect to distance and utilising Shephard’s Lemma results in the Muth equation;

\[
\frac{dw_N}{dr} = -\frac{\frac{\partial C}{\partial t} \frac{dt}{dr}}{N'\left(w_K,w_L,w_N(r),t(r),y\right)} ; 
\]

(2.3.7)

where, in comparison to (2.2.10), the derived demand for land is now conditioned on output level, denoted by superscript \( y \).

Moreover, recalling that in (2.3.1) output occurs in total transport cost, and cost minimisation fixes output at the constrained level \( y \), then \( \frac{\partial C}{\partial t} = y \). Equation (2.3.7) can therefore be rewritten as:
\[
\frac{dw_N}{dr} = -\frac{\gamma}{N^\gamma(w_K, w_L, w_N(r), t(r), y)} \frac{dt}{dr} \leq 0. \tag{2.3.8}
\]

If the market price for land is a monotonic decreasing convex function of distance, supported by bid rent curves conditioned on cost, proposition 2.2.4 follows from (2.3.8).

A dual to (2.3.5) is:

\[
\Omega^c_N(w_K, w_L, y, c) = \min_r \left\{ \omega^c_N(w_K, w_L, t(r), y, c) - w_N(r) : \omega^c_N(w_K, w_L, t(r), y, c) - w_N(r) \geq 0 \right\}. \tag{2.3.9}
\]

The difference between (2.2.12) and (2.3.9) is that the former is a function of output price and profit level whilst the latter is a function of output and cost level. Like (2.2.12) the problem is to minimise the difference between bid rent and market rent subject to the constraint that bids must be at least equal to market rent. The only difference now is that bids are conditioned on \( c \) and not \( \pi \). A solution exists because the convex properties of the constraint and the continuity properties of the objective function that applied in (2.2.12) also occur for (2.3.9)

The first order conditions are:

\[
\frac{\partial \omega^c_N}{\partial t} \frac{dt}{dr} = \frac{dw_N}{dr}; \text{ and } \omega^c_N(p, w_K, w_L, t(r), \pi) = w_N(r). \tag{2.3.10}
\]
Recalling properties 2.3.1 and substituting for $\frac{\partial \omega^c_N}{\partial t}$ gives:

$$\frac{dw_N}{dr} = -\frac{y}{N^y} (w_N, t(r), y, c) \frac{dt}{dr} \leq 0 \quad . \tag{2.3.11}$$

Note that unlike (2.3.7), (2.3.11) is conditioned on $c$ as well as $y$. Thus proposition 2.2.4 is approached from the point of view that in order to maintain cost levels farmers, when faced with increasing transport cost with distances further from the centre, will have to produce output with a higher ratio of land to output.

By its construction $\Omega_{mn}$ is the implicit cost function. This can be seen by forming an identity by substituting the spatial cost function of (2.3.5) into $\Omega_{mn}^c$:

$$\Omega_{mn}^c(w_K, w_L, y, C_r(w_K, w_L, y)) = 0 \quad . \tag{2.3.12}$$

This must be identically equal to zero as optimal locations require the difference between bid rents and market rents to be zero. Differentiating (2.3.12) with respect to $w_K$, $w_L$ and $y$ and utilising Shephard’s’s Lemma gives:

$$\frac{\partial C_r}{\partial w_K} = \frac{\partial \Omega_{mn}^c}{\partial \omega_N} \frac{\partial w_K}{\partial c} \geq 0 \quad , \quad \frac{\partial C_r}{\partial w_L} = \frac{\partial \Omega_{mn}^c}{\partial \omega_N} \frac{\partial w_L}{\partial c} \geq 0 \quad \text{and} \quad \frac{\partial C_r}{\partial y} = \frac{\partial \Omega_{mn}^c}{\partial \omega_N} \frac{\partial y}{\partial c} \geq 0 \quad .$$

Again these appear to be similar to Roy’s theorem but unfortunately they cannot be used to fully establish the first derivative properties of the function $\Omega_{mn}^c$. 

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Recalling (2.2.16) and (2.2.17) duals to (2.3.5) are:

\[ R^c_r(w_K, w_L, y, c) = \min_r \{ r \in \mathbb{R}_+ : \omega_N^c(w_K, w_L, t(r), y, c) - w_N^c(r) \geq 0 \} \quad \text{and} \quad (2.3.13) \]

\[ \overline{R}^c_r(w_K, w_L, y, c) = \max_r \{ r \in \mathbb{R}_+ : \omega_N^c(w_K, w_L, t(r), y, c) - w_N^c(r) \geq 0 \} \quad . \quad (2.3.14) \]

That is, for a given bid rent function, what is the closest (2.3.13) to and furthest from (2.3.14) the central place that cultivation of the crop associated with that bid rent function can take place, both subject to at least the payment of market rent at that location. Solutions to these problems exist because the constraint set, conditioned on cost, is a convex set (\( \omega_N^c - w_N^c \) is a concave function of \( r \)) and \( r \) is continuous. The solution to (2.3.13) is \( r_1 \) and the solution to (2.3.14) is \( r_2 \) in Figure 2.2.3. Because it is a dual to (2.3.5) the spatial cost function can be retrieved from it by inversion.

Comparative statics can be undertaken around optimal distance. Substituting the aspatial cost function of (2.3.5) into (2.3.14) and (2.3.15) forms the identities:

\[ R^c_r(w_K, w_L, y, C(w_K, w_L, w_N(r), t(r), y)) = r \quad ; \quad \text{and} \quad (2.3.16) \]

\[ \overline{R}^c_r(w_K, w_L, y, C(w_K, w_L, w_N(r), t(r), y)) = r \quad . \quad (2.3.17) \]

Dropping the under and over bars for the moment and differentiating (2.3.16) or (2.3.17) with respect to \( r \) gives:
\[
\frac{\partial R}{\partial c} = \frac{1}{\frac{\partial C}{\partial t} \frac{dt}{dr} + \frac{\partial C}{\partial w} \frac{dw}{dr}} \leq 0 ;
\]

(2.3.18)

the sign of which depends on the relative values of the two terms of the denominator on the right hand side. This reflects the inherent forces in the trade-off effects of location.

Differentiating (2.3.16) or (2.3.17) with respect to \(w_K\), \(w_L\) and \(y\) gives:

\[
\frac{\partial R}{\partial w_K} = \frac{\partial R}{\partial c} \frac{\partial C}{\partial w_K} , \quad \frac{\partial R}{\partial w_L} = \frac{\partial R}{\partial c} \frac{\partial C}{\partial w_L} \quad \text{and} \quad \frac{\partial R}{\partial y} = \frac{\partial R}{\partial c} \frac{\partial C}{\partial y} .
\]

The sign of these derivatives could be determined if the common term on the right hand side was known. It is possible to sign this term if the nature of the function \(R\) is identified. If it is the minimum distance, problem (2.3.13), then for small perturbations around the optimal location:

\[
\frac{\partial C}{\partial t} \frac{dt}{dr} + \frac{\partial C}{\partial w} \frac{dw}{dr} \leq 0 .
\]

The converse is true for the function \(R\) in the maximisation problem (2.3.14) and for small perturbations around optimal \(r\):

\[
\frac{\partial C}{\partial t} \frac{dt}{dr} + \frac{\partial C}{\partial w} \frac{dw}{dr} \geq 0 .
\]

With these conditions it is now possible to undertake comparative statics.
\[ \frac{\partial \bar{R}}{\partial w_k} = -\frac{\partial \bar{R}}{\partial c} \frac{\partial C}{\partial w_k} \geq 0 \quad \text{and} \quad \frac{\partial \bar{R}}{\partial w_L} = -\frac{\partial \bar{R}}{\partial c} \frac{\partial C}{\partial w_L} \leq 0 ; \text{ and} \]

\[ \frac{\partial \bar{R}}{\partial y} = -\frac{\partial \bar{R}}{\partial c} \frac{\partial C}{\partial y} \geq 0 \quad \text{and} \quad \frac{\partial \bar{R}}{\partial y} = -\frac{\partial \bar{R}}{\partial c} \frac{\partial C}{\partial y} \leq 0 . \]

This implies that an increase in capital or labour input prices or output for a particular crop decreases the extent of crop location.

Finally the effects of an upward or downward shift of \( \iota(r) \) can only be intuitively derived through utilising the first order properties of the bid rent function, properties 2.3.1 1b.

\[ \frac{\partial}{\partial t} \omega_{\iota}(w_k, w_L, \iota(r), y, c) \leq 0 , \]

means an increase in transport cost at all locations decreases bid rents conditioned on a cost level at all locations. Combining this downward shift with the competitive determination of market rents (recall Figure 2.2.3) implies an increase in transport cost at all locations reduces the extent of crop location. The converse is true for a decrease in transport cost at all locations.

Again the effect of a change in marginal transport cost in distance can be derived from (2.3.16) and (2.3.17) through (2.3.18). Rearranging this latter equation gives the condition for optimal \( r \) whether it is minimum or maximum radial distance as:
\[
\frac{\partial R}{\partial C} \frac{\partial C}{\partial t} + \frac{\partial C}{\partial w} \frac{\partial w}{\partial r} = 1.
\]

The sign of the first product in the bracket is positive and the second product sign is negative. A decrease in marginal transport cost in distance therefore implies that optimal \( r \) increases. That is, in the agricultural case the inner and outer boundaries of the concentric zone of cultivation for this crop shift outwards. The converse applies to an increase in marginal transport cost.

2.4 Summary of Results

Five functions characterise the profit maximising location problem. Maximising the aspatial profit function:

\[
\Pi(p, w, w_N(r), t(r))
\]

(2.4.1)

gives the spatial profit function, where spatial functions are those functions which have solved the fundamental location problem, so that \( r \) is not an argument in these functions:

\[
\Pi_r(p, w, w_L).
\]

(2.4.2)

Further, inverting the aspatial profit function (2.4.1) gives the bid rent function, which again is aspatial because optimal \( r \) has not yet been found:
\[ \omega_N(p, w_K, w_L, t(r), \pi) . \]  

(2.4.3)

Minimising the difference between bid rents and market rents subject to a market constraint gives the spatial function:

\[ \Omega_N(p, w_K, w_L, \pi) . \]  

(2.4.4)

Finally, for the profit maximising agrarian firm, minimising (or maximising) radial distance, subject to a market constraint gives the spatial function:

\[ R(p, w_K, w_L, \pi) . \]  

(2.4.5)

Optimal radial distance, bid rent and profit level can be retrieved from these functions by appropriate substitution.

Firstly, substituting (2.4.1) into (2.4.5) for profit gives radial distance for arbitrary optimal profit:

\[ R(p, w_K, w_L, \Pi(p, w_K, w_L, w_n(r), t(r))) = r . \]  

(2.4.6)

Substituting (2.4.2) into (2.4.4), recalling that market rent is exogenous and this function is the outcome of minimising the difference between bid and market rent, gives the implicit profit function:

\[ \Omega_{nN}(p, w_K, w_L, \Pi_r(p, w_K, w_L)) = 0 . \]  

(2.4.7)
Substituting (2.4.1) into (2.4.3) gives bid rent:

$$\omega_N(p, w_K, w_L, t(r), \Pi(p, w_K, w_L, w_N(r), t(r))) \equiv w_N(r) . \quad (2.4.8)$$

Substituting 2.4.5 into 2.4.1 gives profit level:

$$\Pi(p, w_K, w_L, w_N(R_r(p, w_K, w_L, \pi)), t(R_r(p, w_K, w_L, \pi))) \equiv \pi . \quad (2.4.9)$$

Substituting (2.4.5) into (2.4.3) gives bid rent:

$$\omega_N(p, w_K, w_L, t(R_r(p, w_K, w_L, \pi), \pi)) \equiv w_N . \quad (2.4.10)$$

Profit can also be retrieved by substituting 2.4.3 into 2.4.1 for land input price, recalling that at the optimum land market price and bid rent are the same:

$$\Pi(p, w_K, w_L, \Omega_{nk}(p, w_K, w_L, \pi), t(r)) \equiv \pi . \quad (2.4.11)$$

Differentiating (2.4.1) with respect to land price retrieves the unconditional derived demand for land. Alternatively differentiating (2.4.8) with respect to land price retrieves the derived demand for land conditioned on profit and abstracted from own price. That is:

$$N^*(p, w_K, w_L, t(r), \pi) = -\frac{1}{\partial \omega_N} \frac{\partial \pi}{\partial \pi} . \quad (2.4.12)$$
The relations (2.4.1) to (2.4.12) are illustrated in Figure 2.4.1.

Similarly five functions characterise the cost minimising location problem. Minimising the aspatial cost function:

\[ C(w_K, w_L, w_N(r), t(r), y) ; \]  \hspace{1cm} (2.4.13)

results in the spatial cost function:

\[ C_s(w_K, w_L, y) . \]  \hspace{1cm} (2.4.14)

Inverting the aspatial cost function (2.4.13) gives the aspatial bid rent function conditioned on cost:

\[ \omega^s_N(w_K, w_L, t(r), y, c) . \]  \hspace{1cm} (2.4.15)
Minimising the difference between bid rents and market rents subject to a market constraint gives the spatial function:

\[ \Omega_{cM}^\text{c}(w_K, w_L, y, c) \]  
(2.4.16)

Finally, for the cost minimising farmer, minimising (or maximising) radial distance, subject to a market constraint gives the spatial function:

\[ R^c(w_K, w_L, y, c) \]  
(2.4.17)

Optimal radial distance, bid rent and cost can be retrieved from these functions by appropriate substitution.

Substituting (2.4.13) into (2.4.16) gives:

\[ R^c(w_K, w_L, y, C(w_K, w_L, w_N(r), t(r), y)) \equiv r \]  
(2.4.18)

Substituting (2.4.14) into (2.4.15) gives:

\[ \Omega_{cM}^\text{c}(w_K, w_L, y, C(y, w_K, w_L)) \equiv 0 \]  
(2.4.19)

Substituting (2.4.12) into (2.4.14) gives:

\[ \omega_{cN}^\text{c}(w_K, w_L, t(r), y, C(w_K, w_L, w_N(r), t(r), y)) \equiv w_N \]  
(2.4.20)
Substituting (2.4.16) into (2.4.12) gives:

\[ C(w_K, w_L, \omega^c_N(R^c(y, w_K, w_L, c)), t(R^c(y, w_K, w_L, c)), y) \equiv c \].  

(2.4.21)

Substituting (2.4.16) into (2.4.14) gives:

\[ \omega^c_N(w_K, w_L, t(R^c(y, w_K, w_L, c)), y, c) \equiv w_N. \]  

(2.4.22)

Finally, substituting (2.4.15) into (2.4.12) gives:

\[ C(w_K, w_L, \omega^c_N(y, w_K, w_L, c), t(r), y) \equiv c \].  

(2.4.23)

Differentiating (2.4.13) with respect to land price retrieves the derived demand for land conditioned on output. Alternatively differentiating (2.4.20) also retrieves the derived demand for land but conditioned on cost as well as output and abstracted from own price. That is:

\[ N^c(y, w_K, w_L, t(r), y, c) = \frac{1}{\partial \omega^c_N / \partial c} \]  

(2.4.24)

The identities (2.4.13) to (2.4.23) are summarised in Figure 2.4.2.
2.5 The Derived Demand for Land

In chapter 1 an important relation identified in the urban residential model was the lot size hypothesis. This stated that the quantity of land consumed by households increased with distance from centre. The agricultural analogue to the lot size hypothesis is that farm size, measured by land units in production, increases with distance from the market. That is, the derived demand for land is an increasing function of distance.

In this chapter two different derived demand for land functions have been established, namely:

\[ N(p, w_K, w_L, w_N(r), t(r)) \] \; ; \; \text{and} \; \tag{2.5.1}

\[ N^p(y, w_K, w_L, w_N(r), t(r)) \] \tag{2.5.2}
The obvious and well known distinction to be made between these derived demands is that (2.5.2) is conditioned on output and stems from cost minimisation. This derived demand is distinguished by the superscript \( y \). The derived demand of (2.5.1) is unconditional and stems from the profit maximising problem.

Further, two alternative derived demands have been retrieved from the bid rent function conditioned on profit and the bid rent function conditioned on cost. These are respectively:

\[
N^x(p, w_K, w_L, t(r), \pi) \quad \text{and} \quad N^y(y, w_K, w_L, t(r), c)
\]  \hspace{1cm} (2.5.3)

Taking (2.5.2) and differentiating with respect to distance gives:

\[
\frac{\partial N^y}{\partial r} = \frac{\partial N^y}{\partial w_N} \frac{dw_N}{dr} + \frac{\partial N^y}{\partial t} \frac{dt}{dr} < 0
\]  \hspace{1cm} (2.5.5)

Recalling that:

\[
\frac{dw_N}{dr} < 0
\]

and, if the Hessian matrix of the cost function is negative semi-definite, so that:

\[
\frac{\partial N^y}{\partial w_N} = \frac{\partial^2 C}{\partial w_N^2} \leq 0;
\]
leads to:

**Proposition 2.5.1**

\[
\frac{\partial N^\gamma}{\partial t} > 0 \Rightarrow \frac{\partial N^\gamma}{\partial r} > 0 ;
\]

\[
\frac{\partial N^\gamma}{\partial t} < 0 \Rightarrow \begin{cases} 
\frac{\partial N^\gamma}{\partial r} > 0 & \iff \frac{\partial N^\gamma}{\partial w_N} \frac{dw_N}{dr} > \frac{\partial N^\gamma}{\partial t} \frac{dt}{dr} \\
\frac{\partial N^\gamma}{\partial r} < 0 & \iff \frac{\partial N^\gamma}{\partial w_N} \frac{dw_N}{dr} < \frac{\partial N^\gamma}{\partial t} \frac{dt}{dr}
\end{cases}
\]

However, proposition 2.5.1 can be further decomposed. Taking the aspatial cost function and substituting into (2.5.4) gives the identity:

\[
N^\gamma = N^c(y, w_K, w_L, t(r), C(w_K, w_L, w_N(r), t(r), y)) .
\]  

(2.5.6)

Differentiating (2.5.6) with respect to distance gives:

\[
\frac{\partial N^\gamma}{\partial r} = \frac{\partial N^\gamma}{\partial t} \frac{dt}{dr} + \frac{\partial N^\gamma}{\partial c} \frac{dc}{dt} \frac{dt}{dr} + \frac{\partial N^\gamma}{\partial w_N} \frac{dw_N}{dr} .
\]  

(2.5.7)

Equations (2.5.5) and (2.5.7) imply:

\[
\frac{\partial N^\gamma}{\partial t} = \frac{\partial N^\gamma}{\partial t} + \frac{\partial N^\gamma}{\partial c} y ,
\]

which can be substituted into proposition 2.5.1. Properties 2.3.1 1.b) imply:
\[
\frac{\partial N^c}{\partial t} \geq 0 ;
\]

so that:

\[
\frac{\partial N^c}{\partial t} \leq 0 \Rightarrow \frac{\partial N^c}{\partial c} \leq 0 .
\]

Turning to the unconditional derived demand of the profit maximiser, differentiating (2.5.1) with respect to \( r \) gives:

\[
\frac{\partial N}{\partial r} = \frac{\partial N}{\partial w_N} \frac{dw_N}{dr} + \frac{\partial N}{\partial t} \frac{dt}{dr} \geq 0 . \tag{2.5.8}
\]

Again, the sign of the first derivative of derived demand with respect to transport cost is indeterminate. Now this problem is exacerbated by the indeterminacy of the sign of the first derivative of the unconditional derived demand for land with respect to land price. It is important to establish this sign to establish the agrarian equivalent of the urban lot size hypothesis. If a change in land price can be decomposed into a factor substitution and an output effect, then the proposition on land input and radial distance is fairly complex.

**Proposition 2.5.2**

\[
\begin{align*}
\frac{\partial N}{\partial t} > 0 & \quad \left\{ \begin{array}{l}
\frac{\partial N}{\partial r} > 0 \iff \frac{\partial N}{\partial w_N} \frac{dw_N}{dr} \leq \frac{\partial N}{\partial t} \frac{dt}{dr} \\
\frac{\partial N}{\partial r} < 0 \iff \frac{\partial N}{\partial w_N} \frac{dw_N}{dr} \geq \frac{\partial N}{\partial t} \frac{dt}{dr} \\
\frac{\partial N}{\partial w_N} > 0 \Rightarrow \frac{\partial N}{\partial r} > 0
\end{array} \right.
\end{align*}
\]

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\[
\begin{align*}
\frac{\partial N}{\partial w_N} < 0 & \Rightarrow \frac{\partial N}{\partial r} > 0 \\
\frac{\partial N}{\partial w_N} > 0 & \Rightarrow \frac{\partial N}{\partial t} > \frac{\partial N}{\partial t} \frac{\partial w_N}{\partial t} \frac{\partial w_N}{\partial t} \\
\frac{\partial N}{\partial r} < 0 & \Rightarrow \frac{\partial N}{\partial t} < \frac{\partial N}{\partial t} \frac{\partial w_N}{\partial t} \frac{\partial w_N}{\partial t}
\end{align*}
\]

Alternatively if the Hessian matrix of the aspatial profit function is positive semi-definite, which was a requirement to derive the properties of the bid rent function, then:

\[
\frac{\partial N}{\partial w_N} = -\frac{\partial^2 \Pi}{\partial w_N^2} \leq 0 ;
\]

which simplifies proposition 2.5.2 to:

**Proposition 2.5.2 a**

\[
\begin{align*}
\frac{\partial N}{\partial t} > 0 & \Rightarrow \frac{\partial N}{\partial r} > 0 \\
\frac{\partial N}{\partial r} < 0 & \Rightarrow \frac{\partial N}{\partial t} < \frac{\partial N}{\partial t} \frac{\partial w_N}{\partial t} \frac{\partial w_N}{\partial t}
\end{align*}
\]

Thus 2.5.2a assumes a well behaved production function, so that any output effect of an input price change conforms with factor substitution. Whilst this is a common assumption in producer theory, such an assumption cannot be made in consumer theory. In chapter three therefore the Slutsky decomposition into substitution and income effect of a price change will be an important factor in determining propositions about location aspects of the demand for residential space.
However, propositions 2.5.2 and 2.5.2a can be further decomposed. Substitute the aspatial profit function into (2.5.3) gives the identity:

\[ N \equiv N^x(p, w_K, w_L, t(r), \Pi(p, w_K, w_L, w_N(r), t(r))) \]  \hspace{1cm} (2.5.9)

Differentiating (2.5.9) with respect to \( r \) gives:

\[ \frac{\partial N}{\partial r} = \frac{\partial N^x}{\partial t} \frac{dt}{dr} + \frac{\partial N^x}{\partial \pi} \frac{d\pi}{dr} + \frac{\partial N^x}{\partial w_N} \frac{dw_N}{dr} \]  \hspace{1cm} (2.5.10)

Equations (2.5.8) and (2.5.10) imply:

\[ \frac{\partial N}{\partial t} = \frac{\partial N^x}{\partial t} - \frac{\partial N^x}{\partial \pi} y. \]

This can be substituted into propositions 2.5.2 and 2.5.2. Properties 2.2.1 1.b) imply:

\[ \frac{\partial N^x}{\partial t} < 0 \]

so that:

\[ \frac{\partial N}{\partial t} \geq 0 \Rightarrow \frac{\partial N^x}{\partial \pi} \geq 0. \]

Both propositions 2.5.1 and 2.5.2 are testable. Particularly 2.5.2a leads to the testable hypothesis.
For the derived demand for land to be non-decreasing in location $r$, the price effect of location on the derived demand for land must be reinforced by or outweigh the transport cost effect of location on the derived demand for land.

2.6 Concluding Remarks

It could be argued that the outcomes so far are a result of the restriction that cost minimisers and profit maximisers do not mix in the same land market unless cost minimisers have a profit maximising objective. However, it is interesting to ask the question, what would happen if cost minimisers and profit maximisers were allowed to mix in the same land market, but initially without restricting cost minimisers to a profit maximising objective? Nothing definitive can be said about outcomes until some structure is placed on markets. Particularly if the land market and market for output are competitive then it is possible to say something reasonable about the outcome of mixed profit maximising and cost minimising behaviour.

1. In a competitive market it is not possible for cost minimisers to make more profit than that earned by profit maximisers and in a competitive market profit will be normal.

2. It follows from 1, that if cost minimisers and profit maximisers also compete in the same land market, profit maximisers will set bid rents to normal profit level and determine land market rents.
3. In such a competitive land market cost minimisers can only acquire land for production if they are prepared to pay the market rent set by normal profit making profit maximisers.

4. It follows from 3 therefore that cost minimisers can only get land for production if they produce profit maximising output.

Other things being equal cost minimisers are *de facto* profit maximisers. This reverses the normal argument that profit maximisers will be cost minimisers. Given the market structure, the existence of profit maximisers forces cost minimisers to take on profit maximisation.

It is possible to go further. In section 2.2 the bid rent function lacks practical application in the sense that it is conditioned on profit level, which is not easily observable. Alternatively the bid rent function of section 2.3 is determined by things which are directly observable, cost and output level. This has implications for identifying land prices in a competitive agricultural land market where sales are infrequent. Market values can be estimated through that which is observable in the bid rent function conditioned on cost. Output, and therefore revenue are observable and so is cost. The difference between these will be land value (maximum feasible bid rent). Of course this is nothing new. Any undergraduate text on land allocation will arrive at this conclusion through a constant returns to scale production function, competitive market and the adroit use of Euler's theorem, (for example see Mills and Hamilton, 1994, 91-96). Nor has this escaped valuation practice (see Clark, 1973, p. 58-75 and Millington, 1994, p. 91-93).
It is possible to generalise the dual approach summarised in section 2.4 to spatial models other than agricultural land. Spatial allocation models where transport cost to some central place is the only exogenous spatial variable are generic Von Thünen models. Such models can be treated using dual methods because the true spatially determined variable is contained in cost. Thus urban models of spatial production allocation, where cost of transporting output is borne by the producer and is increasing in distance from the centre (circular market or transport node access), can be analysed in exactly the same way as the traditional agricultural model utilising the spatial cost or profit functions. Further, urban residential models can be treated in the same fashion if commuting cost increases with radial distance. In this case the spatial indirect utility function replaces the profit function and the spatial expenditure function replaces the traditional cost function of production problems. This applies whether the problem is an Alonso type problem, where the consumer is an “owner builder” and derives utility directly from quantity of residential land used in residential production (Fujita, 1989, p. 11-30), or whether it is a Muth/Mill type problem where the consumer derives utility from “housing space” purchased from a builder who produces this good out of land and non-land inputs (Turnbull, 1995, p. 7-18). Moreover all these problems can be replaced by the dual problem of minimising distance from the central place subject to a rent constraint. If land market rent is a smooth continuous convex function of radial distance then the outcome to this problem will be a single radial location, $r$. Alternatively, if land market rent is piecewise smooth and convex, the outcome will be the nearest $r$ to the centre out of a set of adjacent optimal locations to the primal problem.
Whether land market rent is continuous smooth convex or piecewise smooth convex is important in applying the other dual problem presented in this paper, that of choosing location to minimise the difference between bid and market rent subject to a market rent constraint. If land market rent is continuous smooth convex then choosing location to minimise bid rent subject to market rent will produce an identical solution to the primal problem. If market rent is piecewise smooth the solution to this problem is the furthest r from the centre out of a set of adjacent optimal locations to the primal problem. Alternatively, if the problem is specified as minimise difference between bid rent and market rent, then irrespective of the nature of the convexity of the market rent constraint, outcomes will always replicate the solution to the primal problem. The nature of the market rent function is important. Most models of residential location use continuous smooth convex market rent functions. But this need not be the case in residential location. Fujita (1985 and 1989, chapter 4) in establishing the existence and uniqueness of equilibrium urban residential land market rent opens up the possibility of piecewise smooth continuous market rent functions which are convex. The dual problem of minimising rental difference is tractable under these conditions. Thus the dual problems introduced in this paper are suitably general to be applied to urban models of the Von Thünen type. Very similar dual methods will be utilised in chapter three to analyse a Muth type residential model with leisure choice.

These arguments will not apply when exogenous spatial variables occur in revenue as well as or alternatively to occurring in cost (see Smith, 1981, p. 109-116); and Lösch type models with exogenous spatial variation in revenue (see Lösch, 1967, chapter 9). Models based on spatial variation in revenues naturally lend themselves to locational interdependence in spatial competition. This is associated with the pioneering work
of Hotelling (1929, p. 45-48). A particularly good analysis of locational interdependence and spatial oligopoly is Gabsewicz and Thisse (1986, esp. p. 23-44). Again Fujita and Thisse (1986) with the integration of Von Thünen and Hotelling produce a model with firm revenue varying with location. Most applications of the revenue function are aspatial and concentrate on efficient output mix in multi-product firms. See Färe and Grosskopf (1994, Chapter 2 and esp. p. 23-30) for the revenue function and Färe and Primont, (1995) for a complete analysis of duality, the revenue function and its applications. However, if costs are invariant with location, then an examination of the firms location relative to revenue maximisation would be a dual to profit maximisation if one could find a suitable specification of the firm’s revenue function in space.

Chapter four deals with the residential building problem where the exogenous spatial variable is contained within builders’ revenues. Under these conditions the analysis will be within a profit maximising framework, but this still allows the use of the dual problem of minimising the difference between bid rent and market rent for land input.
Chapter 3  The Household Problem: Residential Consumption and Choice of Residential Location

3.1 Introduction

The consumer problem is taken to be that of maximising utility by choosing optimal quantities of residential space, leisure time, and a Hicksian composite good, representing all non-location goods, the price of which is represented by a price index (Hicks, 1939, p. 33 and Mathematical Appendix p. 312-313, §10 and Deaton and Muellbauer, 1980, p. 120-122). Residential space in a dwelling, following the Muth/Mill type model is produced out of land and building capital by builders, (Muth, 1971, p. 244). Preferences over the arguments of the utility function are strictly convex. Utility is a monotonically increasing twice differentiable quasi-concave function of residential space, the composite good and leisure time. Choices are constrained by income and time, the first and second constraints respectively in problem (3.1.1). The first constraint is formed out of wage and non-wage income, which must be dispersed amongst expenditure on residential space, the composite good and pecuniary transport cost. Leisure choice in the criterion is subject to the time constraint of problem (3.1.1) where all time is dispersed amongst leisure, work and transport time. Issues of wage income/leisure time trade-off are ignored, so that work time can be regarded as either a rationed or a non-rationed good. For a good review of the literature on the income/leisure trade-off, combined with a diagrammatic exposition, see Bryant (1990, chapter 5). Becker (1976, chapter 5) deals with the issue of work/leisure trade-off and allocation of time in household production at a
more technical level. Both authors show that, through the general diminution of work relative to leisure time, understanding the allocation of leisure time is becoming increasingly important. The issue to be examined here is the trade-off between leisure and travel time which is resolved through the choice of location.

Problem (3.1.1) defines the residential consumption problem.

\[
\begin{align*}
\max_{y_s, y_z, t_l} & \quad U = U(y_s, y_z, t_l) \\
\text{st} & \quad p_s(r)y_s + p_zy_z + c(r) \leq Y + w_At_w, \text{ and} \\
\text{st} & \quad t_l + t_w + t_c(r) \leq t
\end{align*}
\]  

(3.1.1)

where:

- \(y_s\) is residential space (Muth/Mill definition).
- \(y_z\) is the composite good with price indexed to \(p_s\).
- \(p_s\) is the unit price of residential space which is some unknown function of radial distance \(r\) from city centre.
- \(Y\) is non wage income.
- \(w_A\) is the wage rate.
- \(c\) is pecuniary transport cost, which is a monotone increasing concave function of radial distance \(r\).
- \(t\) is time, which must sum to leisure time, \(t_l\), work time, \(t_w\) and travel time, \(t_c\), the latter of which is a monotonically increasing concave function of radial distance.

Problem (3.1.1) can be converted into a single constraint problem by solving the time constraint for \(t_w\) and then substituting into the budget constraint of (3.1.1). This
allows us to write the single constraint, (3.1.2), as income net of transport cost, \( I(r) \), which must be dispersed amongst expenditures on residential space, the composite good and the opportunity cost of leisure:

\[
I(r) - p_s(r)y_z - p_sy_z - wlt_i \geq 0 ; \quad (3.1.2)
\]

where:

\[
I(r) = Y + w_L(t - t_L(r)) - c(r) \quad (3.1.3)
\]

Note through (3.1.3) that net income is a monotone decreasing convex function of radial distance \( r \), due to the fact that pecuniary transport cost and the time opportunity cost of transport are both monotone increasing concave functions of \( r \).

Because all location determined variables occur in the single net income constraint it is now possible to re-write the consumer choice problem as (3.1.4).

\[
\max_r \left[ \max_{y_s,y_z} \begin{array}{c} U = U(y_s,y_z,t_i) \\ \text{st} \quad I(r) - p_s(r)y_z - p_sy_z - wlt_i \geq 0 \end{array} \right] \quad (3.1.4)
\]

Here it is envisaged that the consumer solves the inner problem by choosing \( y_s, y_z \) and \( t_i \), so as to maximise utility for any given location \( r \). The outer problem is then solved by choosing that \( r \) which maximises maximum utility. As shall be seen presently, this latter problem is best done through the indirect utility function of the inner problem which will be a function of distance through net income and the price of space. Anas (1982, p. 21, equation (1.8)) sees the consumer choice problem in this way, with a two stage solution, solving first the inner problem and then the outer
problem. Unfortunately, it is not followed through. Moreover Anas continues to write
the outer problem as a direct utility problem. Solow (1973, p. 5-6), uses the indirect
utility function in his land use model. By Straszheim (1987, p 720), the indirect
utility function approach had become well established in urban bidding models. In the
following analysis the objective function for the outer problem will be written as the
indirect utility function to the inner problem. This indirect utility function will have
some structure in radial distance which allows the household to make sensible
optimising choices over it. The indirect utility function is a function of $r$ but may be
regarded as aspatial. This is because it is a locus of points in radial distance where
utility has been maximised at each location without reference to other locations. It is
only when the aspatial indirect utility function is maximised in radial distance and the
location problem has been solved that it becomes spatial and it then ceases to be a
function of $r$, since $r$ has been optimised out. Even though the residential location
problem is not explicitly stated in this way elsewhere, such a two stage process is
implied by those traditional models which analyse outcomes through bid rent curves.

The solution to the outer problem of (3.1.4) necessitates saying something about the
relationship between the market price for residential space and radial distance $r$. Re-
writing (3.1.4) without making a priori judgement about residential space price and
distance gives:

$$
\max_{r} \left\{ V(p_s, p_z, w_z, I(r)) = \max_{y_s, y_z, \theta} \left\{ U(y_s, y_z, t_i) : p_s y_s + p_z y_z + w_z t_i \leq I(r) \right\} \right\} . \quad (3.1.5)
$$

The solution to the outer problem in (3.1.5) is trivial. Utility is increasing in net
income $I$ which is monotone decreasing and convex with radial distance $r$. Thus the
utility maximising location collapses to the central location bordering the central
business district. It is sensible therefore, just as in the agricultural model of chapter 2, that residential space prices act as a dispersive counter to the centralising force of commuting cost.

It is possible to go quickly to the relationship between residential prices and distance by using the first order conditions of the inner problem of (3.1.5), without resorting to the indirect utility function and offer prices. Muth (1969, p. 22-25), utilises the first order conditions to get to residential prices and distance. Similarly Brown (1985, p. 33) in her time augmented model goes directly to prices through the first order conditions of the direct utility problem. The first order conditions in Brown’s model are somewhat different, because, like Beckmann (1969, p. 62, and 1987, p. 124), she introduces time into the utility function as commuting time. Utility is decreasing in commuting time, which of course leads to unusually shaped indifference curves. Muth and Brown, as do most urban modellers, ignore the route to demand prices through the first order conditions long established in consumer theory by Hotelling and Wold.

From the first order conditions for the inner problem, the Hotelling-Wold identity (Hotelling, 1935, p. 71, equation (3.4), Wold and Jureen, 1953, p. 92, and Cornes, 1992, p. 37-38) gives the expression for the inverse demand curve in normalised prices for residential space as:

\[
\frac{p_s}{I(r)} = \frac{\partial U{\partial y_s}}{y_s(\partial U{\partial y_s}) + y_s(\partial U{\partial y_s}) + t_i(\partial U{\partial t_i})}.
\]

(3.1.6)

Re-write this in terms of monetary prices for space by multiplying through by \(I(r)\):
\[ p_s = \left( \frac{\partial U \beta y_s}{y_s \beta U \beta y_s + y_s \beta U \beta y_s + t_i \beta U \beta t_i} \right) I(r). \] (3.1.7)

Differentiating (3.1.7) with respect to distance \( r \) results in:

\[ \frac{\partial p_s}{\partial r} \Big|_{r, y_s, t_i} = \left( \frac{\partial U \beta y_s}{y_s \beta U \beta y_s + y_s \beta U \beta y_s + t_i \beta U \beta t_i} \right) \frac{dI}{dr} \leq 0. \] (3.1.8)

This is non-positive because the first derivative of \( I \) is less than zero, and the bracketed term of the right hand side must be non-negative if the utility function is a monotone increasing strictly quasi-concave function. Adding the further restriction that \( y_s > 0 \), results in:

\[ \frac{\partial p_s}{\partial r} \Big|_{r, y_s, t_i} = \left( \frac{\partial U \beta y_s}{y_s \beta U \beta y_s + y_s \beta U \beta y_s + t_i \beta U \beta t_i} \right) \frac{dI}{dr} < 0. \] (3.1.9)

The restriction that residents in the city, at least those in the formal housing market, must consume positive amounts of residential space is not unreasonable.

Further, differentiating again with respect to distance gives:

\[ \frac{\partial^2 p_s}{\partial r^2} \Big|_{r, y_s, t_i} = \left( \frac{\partial U \beta y_s}{y_s \beta U \beta y_s + y_s \beta U \beta y_s + t_i \beta U \beta t_i} \right) \frac{d^2 I}{dr^2} \geq 0. \] (3.1.10)

Equations (3.1.9) and (3.1.10) lead to the following proposition.
Proposition 3.1.1

Residential space prices will be a monotone decreasing convex function of radial distance $r$, holding the demand for residential space, the composite good and leisure time fixed in distance.

Proposition 3.1.1 is determined by the fact that income net of transport cost will be a monotone decreasing convex function of distance, which is itself determined by the fact that both commuting time cost and pecuniary transport cost are monotone increasing concave functions of $r$, by definition.

Not only has the restriction that residential consumption is positive been made, but also restrictions on the properties of the direct utility function have also been made. Implicit in the argument to proposition 3.1.1 is that a solution to the inner problem exists and that the first order conditions can be utilised. The Hotelling-Wold identity requires a well behaved direct utility function. Equation (3.1.8) is equivalent to the Muth trade-off equation (Muth, 1969, p. 23, equation (3') and Turnbull, 1995, p. 10, equation (2.5)). However, the terms contained in the right hand side are not as useful as the Muth equation. The bracketed term of the right hand side contains first derivatives of the unobservable direct utility function. Further, the equation does not help very much in determining outcomes to the outer problem of (3.1.5). A spatial solution requires that the aspatial indirect utility function of (3.1.5) is a concave function of radial distance. Recalling chapter 2, proposition 2.2.3 was used to relate the first derivative of bid rent to market rent in establishing the concavity of the profit function in distance. Similarly in the residential problem it is necessary to establish
the relationship between bid rent conditioned on utility and market rent in order to establish the concavity of the indirect utility function.

Finally the convexity of \( p(r) \) in (3.1.7) has been determined by the condition that the choice variables \( y_x, y_z, \) and \( t_i \) remain unchanged in distance \( r \). In terms of comparative statics, especially where the focus of interest is the relationship between demand for residential space and distance, this condition is hardly tenable. The alternative route of establishing the convexity of market price in distance through the nature of the underlying bid rent curves is advantageous if sensible comparative statics are to be made.

3.2 The Bid Rent Function.

Solving the aspatial indirect utility function of (3.1.5) for the price of residential space, \( p_z \), gives the bid rent function conditioned on utility level \( v \). Note that the "omega" convention from chapter 2, for denoting bid price, has been retained in (3.2.1):

\[
\omega_s = \omega_s(p_z, w_L, I(r), v) .
\]  

(3.2.1)

Because (3.2.1) was arrived at by solving a maximum value function, it is itself a maximum value function and is the maximum bid payable at a location conditional on utility level \( v \). Using an alternative route, Fujita (1989, p. 14, equation (2.7)) approaches maximum bid rent by solving the budget constraint for price and then maximising this objective by choosing quantity of space subject to a utility level. Equation (3.2.1) is the expression for the level curves of the indirect utility function.
Substituting the aspatial indirect utility function of (3.1.5) into (3.2.1) gives:

\[\omega_s(p_z, w_L, I(r), V(p_z, p_z, w_L, I(r))) = p_z.\]  

(3.2.2)

Some useful first derivative properties of \(\omega_s\) can be established from the partial derivatives of (3.2.2). Prior to this it must be recalled that the aspatial indirect utility function of (3.1.5) is monotone decreasing convex in its price arguments (including the opportunity cost of leisure, \(w_L\)) and that the lower contour function in these price planes will be a convex set. This is important in determining the curvature properties of level curves of the function \(\omega_s\) because it is the expression for level curves in the indirect utility function. The aspatial indirect utility function is a monotone increasing function of income net of transport cost, \(I\). The second derivative properties of the aspatial indirect utility function with net income are not known.

Differentiating (3.2.2) with respect to \(p_z\) gives:

\[\frac{\partial}{\partial v} \omega_s(p_z, w_L, I(r), v) = \frac{1}{\partial V/\partial p_z}.\]

Using Roy's identity on the right hand side gives:

\[\frac{\partial}{\partial v} \omega_s(p_z, w_L, I(r), v) = \frac{1}{(\partial V/\partial I)y_z}.\]

The demand for space can be retrieved by:
\[
y_i^v(p_z, w_L, I(r), \nu) = \frac{1}{(\partial V/\partial I)(\partial \omega_2/\partial \nu)}.
\]

Notwithstanding the fact that Roy's identity was used initially to get the demand for residential space, it is a conditional demand. It is conditioned on the fact that offer price must vary in such a way as to produce identical levels of maximum utility. Thus it is conditioned on indirect utility \(\nu\) and abstracted from own price. The superscript \(\nu\) is used to distinguish this demand.

Similarly differentiating (3.2.2) with respect to \(p_z\) and \(I\) and \(w_L\) and utilising Roy's identity gives some further useful properties.

**Properties 3.2.1 (properties of \(\omega_i\)).**

1. a) Non-increasing in \(\nu\), \(p_z\) and \(w_L\).

\[
\frac{\partial \omega_i(p_z, w_L, I(r), \nu)}{\partial \nu} = -\frac{1}{(\partial V/\partial I)y_i^v(p_z, w_L, I(r), \nu)},
\]

\[
\frac{\partial \omega_i(p_z, w_L, I(r), \nu)}{\partial p_z} = \frac{\partial V/\partial p_z}{(\partial V/\partial I)y_i^v(p_z, w_L, I(r), \nu)} - \frac{\nu_i^v}{y_i^v}(p_z, w_L, I(r), \nu),
\]

\[
\frac{\partial \omega_i(p_z, w_L, I(r), \nu)}{\partial w_L} = \left[\frac{\partial \omega_i}{\partial I} + \frac{\partial \omega_i}{\partial w_L}\left(\frac{\partial V}{\partial w_L} + \frac{\partial V}{\partial I} \frac{\partial I}{\partial w_L}\right)\right] = \frac{t_i^v}{y_i^v}(p_z, w_L, I(r), \nu)
\]

b) Non-decreasing in \(I\).

\[
\frac{\partial \omega_i(p_z, w_L, I(r), \nu)}{\partial I} = \frac{1}{y_i^v}(p_z, w_L, I(r), \nu) \geq 0
\]
2. Recalling that the function \( V \) is convex in its price arguments and lower contour sets are therefore convex, \( \omega_n \) is a convex function of \( v, p_z \) and \( w_L \), which implies:

\[
\frac{\partial^2 \omega_i}{\partial v^2} = \frac{\partial}{\partial v} \left( \frac{1}{(\partial V/\partial I) y^*_i} \right) \\
\frac{\partial^2 \omega_i}{\partial p_z^2} = \frac{\partial}{\partial p_z} \left( -y^*_i / y^*_i \right) \geq 0 \\
\frac{\partial^2 \omega_i}{\partial w_L^2} = \frac{\partial}{\partial w_L} \left( -t^*_i / y^*_i \right)
\]

The route adopted here to the properties of the bid rent function contrasts with that of Sasaki (1987, p.55), who uses the first order conditions to the problem of maximising bid rent to a utility constraint.

Particularly in the bid price/distance plane it is important to demonstrate that the bid price function is a monotone decreasing convex function of \( r \). The following corollaries would lead to monotone decreasing convex bid price in distance \( r \).

**Corollary 3.2.1**

\[
\frac{\partial \omega_i}{\partial r} = \frac{\partial \omega_i}{\partial I} \frac{dI}{dr} < 0 .
\]

**Corollary 3.2.2**

\[
\frac{\partial^2 \omega_i}{\partial r^2} = \frac{\partial \omega_i}{\partial I} \frac{d^2 I}{dr^2} + \frac{\partial^2 \omega_i}{\partial I^2} \left( \frac{dI}{dr} \right)^2 \geq 0 .
\]
Corollary 3.2.1 is straightforward and establishes $\omega$, as a monotone decreasing function of $r$. Corollary 3.2.2 is problematical because of the sign of the second product on the right hand side. If $\partial^2 \omega, \hat{B} I^2 \geq 0$ then the corollary would be ensured. However this is not tenable as it requires convexity of the parent aspatial indirect utility function in $I$. Alternatively if $\partial^2 \omega, \hat{B} I^2 \leq 0$ then:

$$\frac{\partial \omega}{\partial I} \frac{d^2 I}{dr^2} \geq \left| \frac{\partial^2 \omega}{\partial I^2} \left( \frac{dI}{dr} \right)^2 \right|$$

However, decreasing convex bid rents in radial distance can be arrived at without assuming differentiable functions. The following demonstration of the convexity of the indirect utility function in the offer price/distance plane is sufficient for the convexity of bid rent (see Diewert 1982, p. 539-541 and Cornes, 1992, p. 17-19). Particularly the demonstration closely follows that of Cornes op cit., p. 17-18, which considers the problem of an objective function of choice variables which is to be maximised subject to a concave constraint in parameters. The general demonstration shows that the maximum value function of such a problem will be quasi-convex in parameters. Clearly the inner problem of (3.1.4) falls into this general category. Straszheim (1987, p. 724-725, Figure 1) attempts a diagrammatic demonstration in an Alonso type model using a four quadrant diagram, but fails to complete the $360^\circ$ rotation. This should be compared with Cornes (op cit., p. 41-42 and Figure 2.3) who uses a four quadrant diagram to derive a well behaved indirect utility function of normalised prices in the standard consumer problem.
Choose two locations with associated space prices:

\[ r_1, p_{s1} \quad \text{and} \quad r_2, p_{s2} \]

so that:

\[ V(r_1, p_{s1}) = V(r_2, p_{s2}) = k \]

If solutions to the direct utility problem exist at \( r_1, p_{s1} \) and \( r_2, p_{s2} \), then they are optimal and:

\[ U(y_{s1}, y_{s2}, t_{s1}) = U(y_{s2}, y_{s2}, t_{s2}) = k \]

Take a convex combination of distance and price of space co-ordinates:

\[ \theta r_1 + (1 - \theta) r_2 \quad \text{and} \quad \theta p_{s1} + (1 - \theta) p_{s2} \quad \{0 < \theta < 1\} \]

Define \( y^*_s, y^*_z \) and \( t^*_t \) as the optimal solution to the direct utility problem at \( \theta r_1 + (1 - \theta) r_2, \theta p_{s1} + (1 - \theta) p_{s2} \). Since this must also be feasible, then:

\[ (\theta p_{s1} + (1 - \theta) p_{s2}) y^*_s + y^*_z + wt^*_t + c(\theta r_1 + (1 - \theta) r_2) - Y - w(t - t_c(\theta r_1 + (1 - \theta) r_2)) \leq 0 \]

By the assumptions for problem (3.1.1), both the pecuniary and time components of transport cost are increasing and concave functions of distance. Thus the budget constraint is concave with distance, so that:
\[
(\theta p_{s1} + (1-\theta)p_{s2})y^*_s + y^*_z + wt^*_t + c(\theta r_1 + (1-\theta)r_2) - Y - w(t-t_c(\theta r_1 + (1-\theta)r_2)) \\
\geq \theta \left( p_{s1}y^*_s + y^*_z + wt^*_t + c(r_1) - Y - w(t-t_c(r_1)) \right) \\
+ (1-\theta) \left( p_{s2}y^*_s + y^*_z + wt^*_t + c(r_2) - Y - w(t-t_c(r_2)) \right) .
\]

This implies that:

\[
\theta \left( p_{s1}y^*_s + y^*_z + wt^*_t + c(r_1) - Y - w(t-t_c(r_1)) \right) \\
+ (1-\theta) \left( p_{s2}y^*_s + y^*_z + wt^*_t + c(r_2) - Y - w(t-t_c(r_2)) \right) \leq 0 .
\]

Because \( \{0 < \theta < 1\} \), then at least one of the non-theta terms in the previous expression must be non-positive. Say that:

\[
\theta \left( p_{s1}y^*_s + y^*_z + wt^*_t + c(r_1) - Y - w(t-t_c(r_1)) \right) \leq 0 ;
\]

which implies \( y^*_s, y^*_z \) and \( t^*_t \) is feasible at \( r_1, p_{s1} \). Recall that \( y_{s1}, y_{z1}, t_{t1} \) is optimal at \( r_1, p_{s1} \), then:

\[
U(y^*_s, y^*_z, t^*_t) \leq U(y_{s1}, y_{z1}, t_{t1}) = k ; \text{ and}
\]

\[
V(\theta r_1 + (1-\theta)r_2, \theta p_{s1} + (1-\theta)p_{s2}) \leq V(r_1, p_s) = k .
\]

Applying to all \( r \) and \( p_s \), the level curves in \( r/p_s \) space are convex and \( k \) is a lower contour set which is convex and \( V(r, p_s) \) is quasi-convex.
This then leads to:

**Proposition 3.2.1**

Concave transport cost in distance $r$ will result in a monotone decreasing convex bid rent function in distance $r$ conditioned on utility and the aspatial indirect utility function is a quasi-convex function in the residential space price/distance plane.

Proposition 3.2.1 simply replicates the fact that the curvature characteristics of the indirect utility function are determined by the constraint. No restrictions were placed on the nature of the utility function in arriving at proposition 3.2.1. The only assumption made was that solutions to the direct utility problem existed at differing price distance co-ordinates. This echoes the comments made in chapter 2, section 2.3 on the derivation of well behaved bid rent curves from the cost function. Even though the demonstration leading to proposition 3.2.1 suggests that well behaved level curves in indirect utility can emanate from level curves in direct utility which are not strictly convex, it is still useful to assume a strictly quasi-concave utility function. This is necessary for continuous individual demand curves, which is sufficient for continuous market demands and aids in the establishment of unique spatial equilibria in the residential market.

Having established the existence of convex bid rent curves for residential space and the first and second derivative properties of the bid rent function, it is now possible to analyse the relationship between offer price and market price for residential space and then determine the concavity of the aspatial indirect utility function. Before proceeding in this direction it is useful to recall the conclusion to chapter 2. There it
was claimed that, because the urban residential problem was a generic Von Thünen
problem, bid rent curves for residential space could be derived by starting with the
expenditure minimisation problem as a dual to utility maximisation. This claim must
be substantiated.

Recalling (3.1.2) and (3.1.3) expenditure can be written as:

\[ p_s(r)y_z + p_zt_z + w_LT_z + w_LT_c(r) + c(r) \leq Y + w_LT_w . \]  

This simply states that expenditure on goods and transport and the time cost of leisure
and transport cannot exceed the summation of non-wage and potential wage income.

The model of Turnbull (1992, p. 65 and 1994, p. 256) appears to be the only Muth
type residential model utilising the expenditure function, but it does not account for
commuting time cost explicitly in the expenditure function. Further, it does not derive
bid rents from expenditure minimisation, which is done below.

The expenditure minimisation dual to problem (3.1.5) is:

\[ \min_r \{ E(p_s, p_z, w_L, t_c(r), c(r), u) \} = \]
\[ \min_{y_z, u_i, u} \{ p_s(r)y_z + p_zt_z + w_LT_z + w_LT_c(r) + c(r) : U(y_z, y_z, t_z) \geq u \} \].  

(3.2.4)

Solving the function \( E \) for \( p_s \) gives the bid rent function:

\[ \omega_s^e = \omega_s(p_s, w_L, t_c(r), c(r), u, e) ; \]  

(3.2.5)
where \( e \) is level of expenditure. As in (3.2.1), (3.2.5) can be regarded as a maximum value function, but now maximum bid conditioned on expenditure. The aspatial expenditure function (3.2.4) is concave in its arguments, Diewert (1982, p. 539) and Cornes (1992, p. 67). The upper contour set of this function will be a convex set and (3.2.5) is the expression for level curves of the expenditure function, so that (3.2.5) will be a convex function.

Useful properties of (3.2.5) can be retrieved from the identity formed by substituting the aspatial expenditure function of (3.2.4) into (3.2.5) for expenditure level \( e \):

\[
\omega^s_z(p_z,w_L,t_c(r),c(r),u,E(p_z,p_L,w_L,t_c(r),c(r),u)) = p_z . \quad (3.2.6)
\]

Differentiating (3.2.6) with respect to \( p_z \) gives:

\[
\frac{\partial}{\partial e} \omega^s_z(p_z,w_L,t_c(r),c(r),u,e) \frac{\partial E}{\partial p_z} = 1 ;
\]

and using Shephard's Lemma results in:

\[
\frac{\partial}{\partial e} \omega^s_z(p_z,w_L,t_c(r),c(r),u,e) = \frac{1}{y^e} .
\]

The Hicksian demand for living space can be retrieved by:
\[ y_s^{eH}(p_z, w_L, t_c(r), c(r), u, e) = \frac{1}{\partial \omega_s^e/\partial e} \cdot \]

Note that not only is the retrieved demand conditioned on \( u \), it is also conditioned on \( e \). It is that Hicksian (compensated) demand which results from offer price varying always to produce minimum expenditure. Thus, it is also abstracted from own price. To distinguish this space demand from others it is given the superscript \( e \) and the superscript \( H \) denotes it is Hicksian.

Similarly further properties can be obtained by differentiating (3.2.6) with respect to \( p_z, t_c, c \) and \( u \) and using Shephard's Lemma. The second order properties are established through a further differentiation.

**Properties 3.2.2 - (the properties of \( \omega_s^e \)):**

1. a) Non-decreasing in \( e \):

\[ \frac{\partial}{\partial e} \omega_s^e(p_z, w_L, t_c(r), c(r), u, e) = \frac{1}{\frac{\partial y_s^{eH}}{\partial e}(p_z, w_L, t_c(r), c(r), u, e)} \geq 0 ; \text{ and} \]

b) non-increasing in \( p_z, t_c, c \) and \( u \):

\[ \begin{align*}
\frac{\partial}{\partial p_z} \omega_s^e(p_z, w_L, t_c(r), c(r), u, e) &= -\frac{\partial}{\partial y_s^{eH}} \frac{\partial y_s^{eH}}{\partial p_z}(p_z, w_L, t_c(r), c(r), u, e) = -\frac{\partial y_s^{eH}}{\partial y_s^{eH}}(p_z, w_L, t_c(r), c(r), u, e) \\
\frac{\partial}{\partial t_c} \omega_s^e(p_z, w_L, t_c(r), c(r), u, e) &= -\frac{\partial}{\partial y_s^{eH}} \frac{\partial y_s^{eH}}{\partial t_c}(p_z, w_L, t_c(r), c(r), u, e) = -\frac{\partial y_s^{eH}}{\partial y_s^{eH}}(p_z, w_L, t_c(r), c(r), u, e) \\
\frac{\partial}{\partial c} \omega_s^e(p_z, w_L, t_c(r), c(r), u, e) &= -\frac{\partial}{\partial y_s^{eH}} \frac{\partial y_s^{eH}}{\partial c}(p_z, w_L, t_c(r), c(r), u, e) = -\frac{\partial y_s^{eH}}{\partial y_s^{eH}}(p_z, w_L, t_c(r), c(r), u, e) \\
\frac{\partial}{\partial u} \omega_s^e(p_z, w_L, t_c(r), c(r), u, e) &= -\frac{\partial}{\partial y_s^{eH}} \frac{\partial y_s^{eH}}{\partial u}(p_z, w_L, t_c(r), c(r), u, e) \\
\end{align*} \]

\[ \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
\end{bmatrix} \leq 0 \]
Recalling that the function $E$ is concave and upper contour sets are therefore convex, $\omega^e_t$ is a convex function of $p_t, t_e, c, u$ and $e$:

\[
\begin{align*}
\frac{\partial^2 \omega^e_t}{\partial p_t^2} &= \frac{\partial}{\partial p_t} \left( -\frac{y^e_t}{y^e_t} \right) \\
\frac{\partial^2 \omega^e_t}{\partial t_e^2} &= \frac{\partial}{\partial t_e} \left( -\frac{w_t}{y^e_t} \right) \\
\frac{\partial^2 \omega^e_t}{\partial c^2} &= \frac{\partial}{\partial c} \left( -\frac{1}{y^e_t} \right) \\
\frac{\partial^2 \omega^e_t}{\partial u^2} &= \frac{\partial}{\partial u} \left( -\frac{\partial e/\partial u}{y^e_t} \right) \\
\frac{\partial^2 \omega^e_t}{\partial e^2} &= \frac{\partial}{\partial e} \left( \frac{1}{y^e_t} \right)
\end{align*}
\geq 0
\]

The nature of bid rents conditioned on expenditure is established through:

**Corollary 3.2.3**

\[
\frac{\partial \omega^e_t}{\partial r} = \frac{\partial \omega^e_t}{\partial t_e} \frac{dt_e}{dr} + \frac{\partial \omega^e_t}{\partial c} \frac{dc}{dr} < 0.
\]

**Corollary 3.2.4**

\[
\frac{\partial^2 \omega^e_t}{\partial r^2} = \frac{\partial \omega^e_t}{\partial t_e} \frac{d^2 t_e}{dr^2} + \frac{\partial^2 \omega^e_t}{\partial t_e^2} \left( \frac{dt_e}{dr} \right)^2 + \frac{\partial \omega^e_t}{\partial c} \frac{d^2 c}{dr^2} + \frac{\partial^2 \omega^e_t}{\partial c^2} \left( \frac{dc}{dr} \right)^2 + \frac{\partial \omega^e_t}{\partial c^2} \frac{dc}{dr} \frac{d^2 t_e}{dr^2} \frac{dt_e}{dr} \geq 0.
\]

The sign of the cross partial in corollary 3.2.4 is contentious but is non-negative as indicated in the three dimensional Figure 3.2.1.
The surface described on the axes (dashed lines) in this figure is a surface of equal expenditure. The level curve (dotted line) is a level curve of expenditure and offer price. The figure demonstrates that the partial derivative of $\omega^e$ with respect to $t_c$ is negative and the absolute value of this becomes smaller as $c$ increases so that the sign of the cross partial is non-negative.

**Proposition 3.2.2**

*Bid rent conditioned on expenditure is increasing in expenditure and is a monotone decreasing convex function of radial distance.*

We do not need differentiability to determine convexity of bid rent in distance as long as the expenditure function is confirmed as increasing concave in distance and offer price. A demonstration of the concavity of the expenditure function can be found in Diewert (1982, p. 539-541) and Cornes (1992, p. 17-19) and will not be replicated here. A similar demonstration in terms of the quasi-convexity of the indirect utility
function has already been used in this chapter. Because the upper contour set is convex, and the bid rent function defines level curves of the expenditure function in the distance/space price plane and expenditure is increasing in \( \omega^c \) and \( r \), the bid rent function is a monotone decreasing convex function of distance.

An alternative demonstration of the convexity of bid rent in distance \( r \) is to use the rule on composite functions (Madden, p. 200-202), Avriel, et al., p. 153-158), as \( \omega^c \), is a monotone decreasing transformation of a monotone increasing concave function of \( r \), \( \omega^c \) is a monotone decreasing convex function of \( r \).

3.3 Bid Rent and Market Rent.

Because residential space is not mobile, under competitive conditions, households will bid against each other for residential space at differing locations \( r \). Each location will be captured by that household which is willing to pay the highest bid rent. Households with steep bid rent curves will capture central locations and those with shallow bid rent curves will capture peripheral locations. This is exactly the process envisaged in the agrarian model of chapter 2 and is that which is used by Alonso (1960, p.155-156 and 1964, chapter 5). Thus the market rent function for residential space will be a strictly convex function supported from below by a family of convex bid rent functions (recall definition 2.2.1, chapter 2, section 2.2). It is important to establish whether the market rent function is a continuous and piecewise smooth function versus a continuous function respectively. Alonso envisages the latter. The conditions for this are fairly strict. If market rents are supported from below by
convex bid rent functions, and market rents are a continuous strictly convex function of \( r \), then each household’s bid rent function will be unique and cannot coincide with others. In this way the market rent function will be a locus of points, where each point is taken from a bid rent curve which is unique to that household. Alternatively, a continuous monotone decreasing convex market rent function could be derived from a bidding regime where there is complete coincidence of household bid rent functions, see for example Solow (1973, p. 5). That is, households would have identical incomes and tastes. Under these conditions the market rent function would be a particular bid rent curve associated with a level of utility (or expenditure if we start from problem (3.2.3)). Utility (expenditure) levels would be set by the fact that residences compete either against other non-residential urban uses for built space at the edge of the non-residential core or against agrarian users of built space at the edge of the city.

Alternatively Fujita (1985 and 1989, chapter 4) introduces the idea of multiple classes of households. Within each class bid rent functions coincide, but there is complete non-coincidence between classes. Under these conditions Fujita, in an Alonso type model and using agricultural rents as an initial starting point, builds a market rent function, which is monotone decreasing piecewise smooth and convex, and where each piecewise segment is that portion of the bid rent curve for that class which is willing to pay the highest rent at a particular interval of location. The initial referee need not have been agricultural rent, but could have been business rent at the edge of the CBD. Because it is immobile the same process can be envisaged in terms of the formation of market prices for residential space.
The following analysis of equilibrium location assumes this competitive bidding process in the formation of the residential space price function and the following proposition echoes proposition 2.2.2 of chapter 2.

**Proposition 3.3.1**

Residential space rent, determined through competitive bidding and where bid rents of differentiated classes of households are convex in radial distance \( r \), will be a monotone decreasing piecewise smooth convex function of distance \( r \).

This now allows use of proposition 2.2.3 of chapter 2, rewritten in terms of residential prices and allowing for bid rents conditioned on utility or expenditure levels.

**Proposition 3.3.2**

At locations close to the centre:

\[
\frac{\partial \omega}{\partial r} \leq \frac{\partial p_s}{\partial r}, \quad \text{or} \quad \left| \frac{\partial \omega}{\partial r} \right| \leq \left| \frac{\partial p_s}{\partial r} \right|;
\]

and at locations further away from centre:

\[
\frac{\partial \omega}{\partial r} \geq \frac{\partial p_s}{\partial r}, \quad \text{or} \quad \left| \frac{\partial \omega}{\partial r} \right| \geq \left| \frac{\partial p_s}{\partial r} \right|.
\]

Combining the relevant set of inequalities with \( \partial \omega, \beta u < 0 \) ensures concavity of the aspatial indirect utility function of (3.1.5) where space prices are a monotone decreasing piecewise smooth convex function of \( r \). Combining the relevant set of inequalities with \( \partial \omega, \beta e > 0 \) ensures convexity of the aspatial expenditure function
of (3.2.3) where space prices are a monotone decreasing piecewise smooth convex function of \( r \). Thus solutions to (3.1.5) and (3.2.3) are assured.

Starting with the aspatial indirect utility function of (3.1.5) and writing it with residential prices as a monotone decreasing piecewise smooth convex function of \( r \), the household problem in utility is:

\[
V_r(p_z, w_L) = \max_r \left\{ V(p_z(r), p_z, w_L, I(r)) \right\} .
\]  

(3.3.1)

The solution to (3.3.1) gives the spatial indirect utility function denoted by subscript \( r \) because \( r \) has been optimised out. The first order condition for the solution to (3.3.1) is:

\[
\frac{dp_z}{dr} = -\frac{\partial V/\partial I}{\partial V/\partial p_z} \frac{dI}{dr} .
\]  

(3.3.2)

Recalling (3.1.3), then:

\[
\frac{dI}{dr} = -w_L \frac{dt_c}{dr} - \frac{dc}{dr} ;
\]

and using Roy’s identity, (see Straszheim, 1985, p. 720-722 and Turnbull, 1995, p. 10), gives:

\[
\frac{dp_z}{dr} = -\frac{w_L}{y'(p_z(r), p_z, w_L, I(r))} \left( \frac{dt_c}{dr} + \frac{dc}{dr} \right) < 0 .
\]  

(3.3.3)
Equation (3.3.3) is the leisure augmented version of Muth’s residential equation and the superscript \( M \) refers to the uncompensated (Marshallian) demand to distinguish it from the conditional demand functions developed thus far. Alternatively, starting with the aspatial expenditure function of (3.2.3) and writing it with residential prices as a monotone decreasing piecewise smooth convex function of \( r \), the household problem in expenditure is:

\[
E_r(p_z, w_L, u) = \min_r \left\{ E(p_z(r), p_z, w_L, t_c(r), c(r), u) \right\} .
\]  

(3.3.4)

The first order condition for this problem is:

\[
\frac{dp_z}{dr} = \frac{\frac{\partial E}{\partial t_c} \frac{dt_c}{dr} + \frac{\partial E}{\partial c} \frac{dc}{dr}}{\frac{\partial E}{\partial p_z}} .
\]

(3.3.5)

Recalling the inner problem of (3.2.3), then:

\[
\frac{\partial E}{\partial t_c} = w_L \text{ and } \frac{\partial E}{\partial c} = 1 ;
\]

and utilizing Shephard’s Lemma, (3.3.5) can be re-written as:

\[
\frac{dp_z}{dr} = -\frac{w_L \frac{dt_c}{dr} + \frac{dc}{dr}}{\frac{\partial E}{\partial p_z}} < 0 .
\]

(3.3.6)
Because (3.3.3) was arrived at through utility maximisation and (3.3.6) stemmed from expenditure minimisation, (3.3.3) utilises the Marshallian demand for residential space and (3.3.6) utilises the Hicksian compensated demand for residential space. This latter is distinguished by the $H$ superscript. Both are versions of the Muth equation and illustrate the trade-off between access (transport cost) and living space.

Whilst both (3.3.3) and (3.3.6) confirm that market rent for residential space diminishes with radial distance, as was the case in chapter 2, neither can say anything about the demand for space and distance without qualification. Because of the construction of market rents from bid rents it follows that households with steep bid rent curves capture central locations and households with shallow bid rent curves capture peripheral locations. The distinction between the Marshallian and Hicksian demands for residential space will enrich the discussion on the relationship between residential demand and location. However, prior to that discussion it is useful to look at location problems which are dual to utility maximisation and cost minimisation. Solutions to these problems contain residential space demands which are conditioned in such a way that they abstract from own price effect and will further enhance the analysis of the relationship between residential space demands and radial distance.

3.4 Equilibrium Location and Dual Problems to Utility Maximisation and Expenditure Minimisation.

In chapter 2 an interesting dual to profit maximisation and cost minimisation was presented as the problem of choosing $r$ to minimise the difference between bid rent
(conditioned on profit or cost) and market rent, subject to a market rent constraint. Similar problems can be utilised in the urban residential location problem, using the bid rent functions derived in section 3.2 of this chapter.

Starting with the utility maximising paradigm, it is perfectly feasible to represent this alternatively as, choose \( r \) to minimise the difference between bid rent conditioned on utility and market rent subject to the constraint that this difference must be non-negative. The solution to this problem is:

\[
\Omega_n(p_z, w_L, \nu) = \min_r \left\{ \omega_z(p_z, w_L, I(r), \nu) - p_z(r) : \omega_z(p_z, w_L, I(r), \nu) - p_z(r) \geq 0 \right\}.
\]

(3.4.1)

A solution to the problem exists, as the same Weierstrasse conditions that applied to problem (2.2.11) of chapter two apply here. The solution to the problem is characterised by the following first order conditions:

\[
\frac{dp_z}{dr} = \frac{\partial \omega_z}{\partial r} = \frac{\partial \omega_z}{\partial I} \frac{dI}{dr} ; \text{ and } \quad p_z = \omega_z.
\]

(3.4.2)

Recalling that:

\[
\frac{dI}{dr} = -w_L \frac{dt_c}{dr} \frac{dc}{dr} ; \text{ and }
\]

from properties 3.3.1:

\[
\frac{\partial \omega_z}{\partial I(r)} = \frac{1}{y_z} \left( p_z, w_L, I(r), \nu \right) ;
\]

and substituting into (3.4.2) gives:
\[
\frac{dp_z}{dr} = -\frac{w_i \frac{dt_c}{dr} + \frac{dc}{dr}}{y_i'} (p_z, w_L, I(r), v) < 0 .
\] (3.4.3)

Substituting the spatial indirect utility function of 3.3.1 into \( \Omega_n \) for \( v \) gives the implicit spatial indirect utility function:

\[
\Omega_n(p_z, w_L, V_r(p_z, w_L)) \equiv 0 .
\] (3.4.4)

Differentiating (3.4.4) with respect to \( p_z \) and \( w_L \) gives:

\[
-\frac{\partial \Omega_n}{\partial p_z} \frac{dp_z}{\partial \Omega_n} \frac{\partial V_r}{\partial p_z} \leq 0 \quad \text{and} \quad -\frac{\partial \Omega_n}{\partial w_L} \frac{dw_L}{\partial \Omega_n} \frac{\partial V_r}{\partial w_L} \leq 0 .
\]

The same problem in the expenditure minimisation paradigm is:

\[
\Omega^e_n(p_z, w_L, u, e)
\equiv \min_r \left\{ \omega^e_z(p_z, w_L, t_c(r), c(r), u, e) - p_z(r) : \omega^e_z(p_z, w_L, t_c(r), c(r), u, e) - p_z(r) \geq 0 \right\} .
\] (3.4.5)

The first order conditions for this problem are:

\[
\frac{dp_z}{dr} = \frac{\partial \omega^e_z}{\partial r} - \frac{\partial \omega^e_z}{\partial t_c} \frac{dt_c}{dr} + \frac{\partial \omega^e_z}{\partial c} \frac{dc}{dr} ; \text{ and} \quad p_z = \omega_z .
\] (3.4.6)
Recalling properties 3.2.2:

\[ \frac{\partial \omega_t}{\partial t_c} = -\frac{w_L}{y^{eh}_t} (p_z, w_L, t_c(r), c(r), u, e), \quad \text{and} \quad \frac{\partial \omega_t}{\partial c} = -\frac{1}{y^{eh}_t} (p_z, w_L, t_c(r), c(r), u, e); \]

and substituting into (3.4.6) and factorising gives:

\[ \frac{dp_z}{dr} = -\frac{w_L}{y^{eh}_t} \frac{dt_c}{dr} + \frac{dc}{dr} (p_z, w_L, t_c(r), c(r), u, e) < 0. \]  

(3.4.7)

Also substituting the spatial expenditure function of (3.3.4) into \( \Omega^e_{n} \) for \( e \) gives the identity:

\[ \Omega^e_{n}(p_z, w_L, u, E, (p_z, w_L, u)) = 0; \]  

(3.4.8)

which is the implicit spatial expenditure function. Differentiating (3.4.8) with respect to \( p_z, w_L \) and \( u \) gives:

\[ \frac{\partial \Omega^e_{n}}{\partial \Omega^e_{n}} \frac{\partial p_z}{\partial E} \geq 0, \quad \frac{\partial \Omega^e_{n}}{\partial w_L} \frac{\partial w_L}{\partial E} \geq 0 \quad \text{and} \quad \frac{\partial \Omega^e_{n}}{\partial \Omega^e_{n}} \frac{\partial u}{\partial E} \geq 0. \]

Importantly, equations (3.4.3) and (3.4.7) replicate the Muth equation in their respective utility maximising or expenditure minimising paradigms.
Further duals to (3.3.1) and (3.3.4) are:

\[ R(p_s, w_L, v) = \min_r \{ r \in \mathbb{R}_+: \omega_s(p_s, w_L, I(r), v) - p_s(r) \geq 0 \} \quad ; \quad (3.4.9) \]

\[ \overline{R}(p_s, w_L, v) = \max_r \{ r \in \mathbb{R}_+: \omega_s(p_s, w_L, I(r), v) - p_s(r) \geq 0 \} \quad ; \quad \text{and} \quad (3.4.10) \]

\[ R^\varepsilon(p_s, w_L, u, e) = \min_r \{ r \in \mathbb{R}_+: \omega^\varepsilon_s(p_s, w_L, t_e(r), c(r), u, e) - p_s(r) \geq 0 \} \quad ; \quad (3.4.11) \]

\[ \overline{R}^\varepsilon(p_s, w_L, u, e) = \max_r \{ r \in \mathbb{R}_+: \omega^\varepsilon_s(p_s, w_L, t_e(r), c(r), u, e) - p_s(r) \geq 0 \} \quad . \quad (3.4.12) \]

That is, for a given bid rent function, what is the closest to, (3.4.9) and (3.4.11) or the furthest from, (3.4.10) and (3.4.12), the centre a household associated with that bid rent function can locate? All cases require that the bid rent be successful, that is, that it be no less than the market rent at that location. Solutions to these problems exist because the relevant constraint sets, conditioned on utility, (3.4.9) and (3.4.10), or expenditure, (3.4.11) and (3.4.12), respectively, are convex sets \( \omega_s - p_s \) and \( \omega^\varepsilon_s - p_s \) are concave functions of \( r \) and \( r \) is continuous.

If residential market price is a monotone decreasing continuous convex function of \( r \) then:

- solutions replicate exactly solutions to the relevant utility or expenditure problem;

and

- the solution to the minimum and maximum distances are identical.

If residential market price is a monotone decreasing piecewise smooth convex function of \( r \), then:
the solution to the minimisation problems will be that \( r \) closest to the centre, out of an interval of \( r \), which formed the solution to the relevant utility or expenditure problem; and

the solution to the maximisation problems will be that \( r \) furthest from the centre, out of an interval of \( r \), which formed the solution to the relevant utility or expenditure problem.

This is the case in this thesis because definition 3.3.1 presumes residential space price is a monotone decreasing piecewise smooth convex function of \( r \). This function is supported from below by bid rent curves which are homogeneous for each class of households but completely heterogeneous between classes. Given the level of generality adopted in the thesis household classes are distinguished by their bid rent curves. The residential zone of the city consists of concentric zones of residential land use, where each successively distant zone from the centre becomes shallower.

Substituting the aspatial indirect utility function of (3.3.1) into (3.4.9) and (3.4.10) and the aspatial expenditure function of (3.3.4) into (3.4.11) and (3.4.12) gives the following identities:

\[
R(p_z, w_L, V(p_z(r), p_z, w_L, f(r))) = r \quad ;
\]

(3.4.13)

\[
\underline{R}(p_z, w_L, V(p_z(r), p_z, w_L, f(r))) = r \quad ; \text{and}
\]

(3.4.14)

\[
R^\varepsilon(p_z, w_L, u, E(p_z(r), p_z, w_L, t_c(r), c(r), u)) = r \quad ;
\]

(3.4.15)

\[
\underline{R}^\varepsilon(p_z, w_L, u, E(p_z(r), p_z, w_L, t_c(r), c(r), u)) = r \quad .
\]

(3.4.16)
Taking either (3.4.13) or (3.4.14) dropping the under or over bar and differentiating with respect to \( r \) gives:

\[
\frac{\partial R}{\partial \nu} = \frac{1}{\frac{\partial V}{\partial p_z} \frac{dp_z}{dr} + \frac{\partial V}{\partial I} \frac{dI}{dr}} \leq 0; \tag{3.4.17}
\]

the sign of which depends on the relative values of the two terms of the denominator on the right hand side. These reflect the inherent forces in the trade-off effects of residential location.

Differentiating (3.4.13) or (3.4.14) and with respect to \( p_z \) and \( w_z \) and ignoring under or over bar gives:

\[
\frac{\partial R}{\partial p_z} = -\frac{\partial R \partial V}{\partial \nu \partial p_z}; \quad \text{and} \quad \frac{\partial R}{\partial w_z} = \frac{\partial R \partial V}{\partial \nu \partial w_z}.
\]

The sign of these derivatives could be determined if the common term on the right hand side was known. Thus comparative statics can be made conditional on the relative weights of the net income and residential space price effects of residential location. Establishing these relative weights can be done by recalling the first order condition (3.3.2) for the solution to (3.3.1) and determining whether the relevant spatial problem is one of maximum or minimum distance. If it is minimum distance then (3.3.2) implies that for small perturbations around minimum distance:
\[
\frac{\partial R}{\partial \nu} = \frac{1}{\frac{\partial V}{\partial p_z} \frac{dp_z}{dr} + \frac{\partial V}{\partial I} \frac{dI}{dr}} \geq 0 ; \text{ and }
\]

if it is maximum distance then (3.3.2) implies that for small perturbations around maximum distance

\[
\frac{\partial R}{\partial \nu} = \frac{1}{\frac{\partial V}{\partial p_z} \frac{dp_z}{dr} + \frac{\partial V}{\partial I} \frac{dI}{dr}} \leq 0 .
\]

Thus:

\[
\frac{\partial R}{\partial p_z} = \frac{\partial}{\partial \nu} \frac{\partial R}{\partial p_z} \geq 0 \quad \text{and} \quad \frac{\partial R}{\partial p_z} = \frac{\partial}{\partial \nu} \frac{\partial R}{\partial p_z} \leq 0 ;
\]

\[
\frac{\partial R}{\partial w_L} = \frac{\partial}{\partial \nu} \frac{\partial R}{\partial w_L} \geq 0 \quad \text{and} \quad \frac{\partial R}{\partial w_L} = \frac{\partial}{\partial \nu} \frac{\partial R}{\partial w_L} \leq 0 .
\]

These latter inequalities emphasise the opportunity cost of leisure in \( w_L \) and the effect of an increase in hourly wages on consumption of residential space is ignored. It is not possible to undertake comparative statics on the effect of an increase or decrease in net income at each location through (3.4.13) or (3.4.14). However, the effects of an upward or downward shift of \( I(r) \) can be intuitively derived through utilising the first order properties of the bid rent function, properties 3.2.1 1b:

\[
\frac{\partial \omega_s}{\partial I} (p_z, w_L, I(r), \nu) \geq 0 .
\]

Given the formation of market rents from bid rents an upward shift in net income at all locations increases bids conditioned on a level of utility at all locations suggesting
that minimum distance decreases and maximum distance increases. The converse is true for a downward shift in $I$ at all locations.

The effect of a change in the relationship between net income and radial distance, through a change in marginal transport cost in distance, can be derived from (3.4.13) and (3.4.14) through (3.4.17). Rearranging (3.4.17) gives the condition for optimal $r$ whether it is minimum or maximum radial distance as:

$$\frac{\partial R}{\partial V} \left[ \frac{\partial V}{\partial p_s dr} + \frac{\partial V}{\partial I dr} \right] = 1 .$$

The sign of the first product in the bracket is positive and the second product sign is negative. A decrease in $dI/dr$, say through a reduction in marginal transport cost in radial distance, therefore implies that optimal $r$ increases. That is inner and outer boundaries of the concentric zone move outwards. The converse applies to an increase in $dI/dr$ through an increase in marginal transport.

Taking either (3.4.15) or (3.4.16) dropping the under or over bar and differentiating with respect to $r$ gives:

$$\frac{\partial R^e}{\partial e} = \frac{1}{\frac{\partial E}{\partial p_s dr} + \frac{\partial E}{\partial t_c dr} + \frac{\partial E}{\partial c dr}} > 0; \quad (3.4.18)$$
the sign of which depends on the relative values of the three terms of the denominator on the right hand side. Again these reflect the inherent forces in the trade-off effects of residential location.

Differentiating (3.4.15) or (3.4.16) with respect to \( p_z \), \( w_L \) and \( u \) and ignoring under or over bar gives:

\[
\frac{\partial R^e}{\partial p_z} = \frac{\partial R^e}{\partial e} \frac{\partial E}{\partial p_z}, \quad \frac{\partial R^e}{\partial w_L} = \frac{\partial R^e}{\partial e} \frac{\partial E}{\partial w_L} \quad \text{and} \quad \frac{\partial R^e}{\partial u} = \frac{\partial R^e}{\partial e} \frac{\partial E}{\partial u} .
\]

The sign of these derivatives could be determined if the common term on the right hand side was known. Thus comparative statics can be made conditional on the relative weights of the transport cost effects, both in time and money cost, and residential space price effects of residential location. Establishing these relative weights can be done by recalling the first order condition (3.3.5) for the solution to (3.3.4) and determining whether the relevant spatial problem is one of maximum or minimum distance. If it is minimum distance then (3.3.5) implies that for small perturbations around minimum distance:

\[
\frac{\partial R^e}{\partial e} = \frac{1}{\frac{\partial E}{\partial p_z} \frac{dp_z}{dr} + \frac{\partial E}{\partial t_c} \frac{dt_c}{dr} + \frac{\partial E}{\partial c} \frac{dc}{dr}} \leq 0; \quad \text{and}
\]

if it is maximum distance then (3.3.5) implies that for small perturbations around maximum distance:
\[
\frac{\partial R^e}{\partial e} = \frac{1}{\partial E dp_t + \partial E dt_{c} + \partial E dc} \geq 0 .
\]

Thus:

\[
\frac{\partial R^e}{\partial p_z} = \frac{\partial R^e}{\partial e} \frac{\partial E}{\partial p_z} \geq 0 \quad \text{and} \quad \frac{\partial R^e}{\partial p_z} = \frac{\partial R^e}{\partial e} \frac{\partial E}{\partial p_z} \leq 0 ;
\]

\[
\frac{\partial R^e}{\partial w_L} = \frac{\partial R^e}{\partial e} \frac{\partial E}{\partial w_L} \geq 0 \quad \text{and} \quad \frac{\partial R^e}{\partial w_L} = \frac{\partial R^e}{\partial e} \frac{\partial E}{\partial w_L} \leq 0 ;
\]

\[
\frac{\partial R^e}{\partial u} = \frac{\partial R^e}{\partial e} \frac{\partial E}{\partial u} \geq 0 \quad \text{and} \quad \frac{\partial R^e}{\partial u} = \frac{\partial R^e}{\partial e} \frac{\partial E}{\partial u} \leq 0 .
\]

Again the change in \(w_L\) only refers to the opportunity cost of transport and ignores the income effect on space consumption. It is not possible to undertake comparative statics on the effect of an increase or decrease in the time or pecuniary cost of transport at each location through (3.4.15) or (3.4.16). However the effects of an upward or downward shift of \(t_c(r)\) or \(c(r)\) can be intuitively derived through utilising the first order properties of the bid rent function, properties 3.2.2 1b:

\[
\frac{\partial \omega^e(p_z, w_L, t_c(r), c(r), u, e)}{\partial t_c} < 0 \quad \text{and} \quad \frac{\partial \omega^e(p_z, w_L, t_c(r), c(r), u, e)}{\partial c} < 0 .
\]

Given the formation of market rents from bid rents an upward shift in time and/or pecuniary transport cost at all locations decreases bids conditioned on a level of utility at all locations suggesting that minimum distance increases and maximum distance decreases. The converse is true for a downward shift in \(t_c(r)\) and/or \(c(r)\) at all locations.
The effect of a change in the relationship between $t_e(r)$ and/or $c(r)$ and radial distance, through a change in marginal time and/or pecuniary transport cost in distance, can be derived from (3.4.15) and (3.4.16) through (3.4.18). Rearranging (3.4.18) gives the condition for optimal $r$ whether it is minimum or maximum radial distance as:

$$\frac{\partial R^e}{\partial e} \left[ \frac{\partial E}{\partial p_i} \frac{dp_i}{dr} + \frac{\partial E}{\partial t_e} \frac{dt_e}{dr} + \frac{\partial E}{\partial c} \frac{dc}{dr} \right] = 1 .$$

The sign of the first product in the bracket is negative and for the second and third products sign is positive. A decrease in $dt_e/dr$ or $dc/dr$ implies that optimal $r$ increases. That is inner and outer boundaries of the concentric zone move outwards. The converse is true for an increase in $dt_e/dr$ or $dc/dr$.

A theme in both the Alonso and Muth type residential models is to relate income to residential land or space consumption and location. This theme was set by Muth (1969) and is continuing, see for example Fujita (1989). An examination of the first order conditions (3.3.3), (3.3.6), (3.4.3) and (3.4.7) reveals the trade-off in location between the cost of access (the numerator in all these conditions) and the demand for land or space (the denominator in all these conditions). Thus if increase in the demand for residential space outweighs the increases in the opportunity cost of transport brought about by an increase in the wage rate it is possible to say:
households on high wage rates will generally take up more peripheral locations, and

- generally increasing real wages over time leads to residential suburbanisation.

This issue will not be pursued in depth here. As indicated in chapter one the main aim of the thesis is to examine using duality methods the conditions leading to the lot size hypothesis. Thus discussion of the relationship between location and the demand for residential space will be deferred to section 3.6 with a more thorough examination of the four different demand functions identified in this chapter. At present it would be useful to summarise the different optimal value functions that have been identified as characterising the residential problem.

3.5 Summary of Results.

The utility problem is characterised by five functions. Starting with the aspatial indirect utility function:

\[
V(p_s(r), p_z, w_L, I(r));
\]

(3.5.1)

and maximising over \( r \) gives the spatial indirect utility function:

\[
V_r(p_z, w_L) .
\]

(3.5.2)

Alternatively solving (3.5.1) for space prices gives the aspatial maximum bid rent function for space, conditioned on utility \( v \):
\[ \omega_z(p_z, w_L, I(r), v) \] \hspace{1cm} (3.5.3)

Choosing location to minimise the difference between bid rent and market price, subject to a market rent constrain, results in:

\[ \Omega_n(p_z, w_L, v) \] \hspace{1cm} (3.5.4)

Finally, the direct choice of distance (maximum or minimum) subject to that same market price gives:

\[ R(p_z, w_L, v) \] \hspace{1cm} (3.5.5)

Substituting (3.5.1) into (3.5.5) gives minimum (or maximum) radial distance for arbitrary levels of utility:

\[ R(p_z, w_L, V(p_z, r, p_z, w_L, I(r))) = r \] \hspace{1cm} (3.5.6)

Substituting (3.5.2) into (3.5.4) gives the implicit indirect utility function:

\[ \Omega_z(p_z, w_L, V_r(p_z, w_L)) = 0 \] \hspace{1cm} (3.5.7)

Substituting (3.5.1) into (3.5.3) gives offer price for residential space:

\[ \omega_z(p_z, w_L, I(r), V(p_z(r), p_z, w_L, I(r))) = p_z \] \hspace{1cm} (3.5.8)
Substituting (3.5.5) into (3.5.1) retrieves indirect utility:

\[ V(p_s(R_s(p_z,w_L,v)),p_z,w_L,I(R_s(p_z,w_L,v))) = v. \]  

(3.5.9)

Substituting (3.5.4) into (3.5.3) also retrieves bid rent:

\[ \omega_s(p_z,w_L,I(R_s(p_z,w_L,v)),v) = \omega_s. \]  

(3.5.10)

Finally, substituting (3.5.4) into (3.5.1) gives indirect utility:

\[ V(\Omega_n(p_z,w_L,v),p_z,w_L,I(r)) = v. \]  

(3.5.11)

The Marshallian demand for residential space as a function of distance can be retrieved from (3.5.1). Further, the demand conditioned on utility and abstracting from own price can be retrieved from (3.5.8):

\[ y_s'(p_z,w_L,I(r),v) = - \frac{1}{(\partial V/\partial l)(\partial \omega_s/\partial v)}. \]  

(3.5.12)
Schematically the system is:

\[ V_r(p_z, w_L) \]

\[ V(p_z(r), p_z, w_L, I(r)) \]

\[ y_s(p_z(r), p_z, w_L, I(r), r) \]

\[ y_s'(p_z, w_L, I(r), r) \]

\[ \omega_s(p_z, w_L, I(r), r) \]

\[ \max \]

\[ V(p_z(r), p_z, w_L, I(r)) \]

\[ (3.5.7) \]

\[ (3.5.9) \]

\[ \Omega_{rs}(p_z, w_L, r) \]

\[ (3.5.6) \]

\[ (3.5.11) \]

\[ (3.5.10) \]

The residential choice problem in expenditure is also characterised by five functions.

Starting with the aspatial expenditure function:

\[ E(p_z(r), p_z, w_L, t_c(r), c(r), u) \]  \hspace{1cm} (3.5.13)

and minimising this function over \( r \) (3.5.13) gives the spatial expenditure function:

\[ E_r(p_z, w_L, u) \]  \hspace{1cm} (3.5.14)

The aspatial maximum bid rent conditioned on expenditure is arrived at by solving (3.5.13) for space price:

\[ \omega_s^e(p_z, w_L, t_c(r), c(r), u, e) \]  \hspace{1cm} (3.5.15)

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Choosing location to minimise the difference between bid rent and market price, subject to a market rent constraint, results in:

\[ \Omega^e_{\pi}(p_z, w_L, u, e) . \]  \hspace{1cm} (3.5.16)

Looking at the location problem as a direct choice of distance (minimum or maximum) subject to a market constraint yields:

\[ R^e(p_z, w_L, u, e) . \]  \hspace{1cm} (3.5.17)

Substituting (3.5.13) into (3.5.17), gives minimum or maximum radial distance:

\[ R^e(p_z, w_L, u, E(p_z(r), p_z, w_L, t_c(r), c(r), u)) = r . \]  \hspace{1cm} (3.5.18)

Substituting (3.5.14) into (3.5.16) gives:

\[ \Omega^e_{\pi}(p_z, w_L, u, E_r(p_z, w_L, u)) = 0 . \]  \hspace{1cm} (3.5.19)

Substituting (3.5.13) into (3.5.15) gives market price in distance:

\[ \omega^e_{\pi}(p_z, w_L, t_c(r), c(r), u, E(p_z(r), p_z, w_L, t_c(r), c(r), u)) = p_z . \]  \hspace{1cm} (3.5.20)

Substituting (3.5.17) into (3.5.13) gives expenditure:
\[ E(p, (R^e(p, w_L, u, e)), p, w_L, t_c(R^e(p, w_L, u, e)), c(R^e(p, w_L, u, e)), u) \equiv e . \]

(3.5.21)

Substituting (3.5.17) into (3.5.15) retrieves optimal offer price:

\[ \omega^e_{t_c(p, w_L, t_c(R^e(p, w_L, u, e)), c(R^e(p, w_L, u, e)), u, e) \equiv \omega^e_{t_c} . \]

(3.5.22)

Substituting (3.5.16) into (3.5.13) gives expenditure:

\[ E(\Omega^e_n(p, w_L, u, e), p, w_L, t_c(r), c(r), u) \equiv e . \]

(3.5.23)

The Hicksian compensated demand for residential space as a function of distance can be retrieved from (3.5.13). Alternatively the demand for residential space can be retrieved from (3.5.20). However this is the Hicksian demand conditioned on expenditure and abstracted from own price.

\[ y^e_{t_c}(p, w_L, t_c(r), c(r), u, e) = \frac{1}{\partial \omega^e_{t_c}/\partial e} . \]

(3.5.24)

Figure 3.5.2 illustrates these relations.
3.6 The Demand for Residential Space.

In this chapter two different demand functions for residential space have been identified, stemming from the two familiar optimisation problems. The solution to the utility maximisation problem, (3.3.1), yields the Marshallian demand for living space as a function of radial distance:

\[
y^M_z = y^M_z(p_z(r), p_z, w_L, t_L(r))
\]

Demand function (3.6.1) is given the superscript \(M\) indicating it is a Marshallian demand function.

The solution to the expenditure minimisation problem, (3.3.4) gives the Hicksian (superscript \(H\)) demand function as a function of \(r\):
\[ y_s^{\nu} = y_s^{\nu}(p_s, p_z, w_L, t_c(r), c(r), u) \quad . \quad (3.6.2) \]

A further conditional demand can be retrieved from the bid rent function conditioned on the indirect utility level \( \nu \). This is a conditional Marshallian demand and carries the superscript \( \nu M \):

\[ y_s^{\nu M}(p_s, w_L, I(r), \nu) \quad . \quad (3.6.3) \]

Finally the Hicksian demand conditioned on expenditure level can be retrieved from the bid rent function conditioned on expenditure level and has the superscript \( eH \):

\[ y_s^{eH}(p_s, w_L, t_c(r), c(r), u, e) \quad . \quad (3.6.4) \]

Differentiating (3.6.2) with respect to distance \( r \) gives:

\[
\frac{\partial y_s^{\nu}}{\partial r} = \frac{\partial y_s^{\nu}}{\partial p_s} \frac{dp_s}{dr} + \frac{\partial y_s^{\nu}}{\partial t_c} \frac{dt_c}{dr} + \frac{\partial y_s^{\nu}}{\partial c} \frac{dc}{dr} < 0 \quad . \quad (3.6.5)
\]

The first term on the right hand side of (3.6.5) is non-negative because:

\[ \frac{\partial y_s^{\nu}}{\partial p_s} \leq 0 \quad \text{and} \quad \frac{dp_s}{dr} < 0 \quad . \]

Moreover:

\[ \frac{dt_c}{dr} > 0 \quad \text{and} \quad \frac{dc}{dr} > 0 \quad ; \]

but:
\[
\frac{\partial y_s^H}{\partial t_c} \leq 0 \quad \text{and} \quad \frac{\partial y_s^H}{\partial c} \leq 0.
\]

This leads to:

**Proposition 3.6.1**

\[
\begin{align*}
\frac{\partial y_s^H}{\partial t_c} & > 0 \quad \Rightarrow \quad \frac{\partial y_s^H}{\partial c} > 0 \\
\frac{\partial y_s^H}{\partial t_c} & < 0 \\
\frac{\partial y_s^H}{\partial t_c} & = 0 \quad \Rightarrow \quad \frac{\partial y_s^H}{\partial c} \geq 0
\end{align*}
\]

Proposition 3.6.1 can be further decomposed. Substituting the aspatial expenditure function into (3.6.4) gives the identity:

\[
y_s^H = y_s^{oH}(p_z, w_z, t_c(r), c(r), u, E(p_z(r), p_z, w_z, t_c(r), c(r), u)) \quad .
\]  

(3.6.6)

Differentiating (3.6.6) with respect to \( r \) gives:
Equations (3.6.5) and (3.6.7) imply:

\[
\frac{\partial y_s^H}{\partial t_e} = \frac{\partial y_s^{SH}}{\partial t_e} + \frac{\partial y_s^{EH}}{\partial e} w_L \quad ; \text{and} \quad \frac{\partial y_s^H}{\partial c} = \frac{\partial y_s^{SH}}{\partial c} + \frac{\partial y_s^{EH}}{\partial e} .
\]

These can be substituted into proposition 3.6.1.

Proposition 3.6.1 suggests that it is not possible to be definitive about even the compensated demand for housing and distance. This problem would be exacerbated with uncompensated demand and radial distance, because of the issue of the income and substitution effects of a price change. Fortunately, the proposition stemming from the Marshallian demand is less complex because time and pecuniary transport cost effects are subsumed under the single variable, income net of transport cost.

Differentiating the traditional Marshallian demand, (3.6.1), with respect to \( r \) gives:

\[
\frac{\partial y_s^M}{\partial r} = \frac{\partial y_s^M}{\partial p_s} \frac{dp_s}{dr} + \frac{\partial y_s^M}{\partial I} \frac{dI}{dr} .
\]

Using the fact that utility maximisation requires expenditure minimisation, then the first derivative of the Marshallian demand with respect to price can be replaced with

\[
\frac{\partial y_i^M}{\partial p} = \frac{\partial y_i^H}{\partial p} - \frac{\partial y_i^M}{\partial I} y_i^M.
\]  

(3.6.9)

Substituting (3.6.9) into (3.6.8) and factorising gives:

\[
\frac{\partial y_i^M}{\partial r} = \frac{dp \partial y_i^H}{dr \partial p} + \frac{\partial y_i^M}{\partial I} \left( \frac{dI}{dr} - \frac{dp}{dr} y_i^M \right) \leq 0.
\]  

(3.6.10)

The first term on the right hand side is non-negative and the bracketed term is negative. This then leads to the following proposition:

**Proposition 3.6.2**

\[
\frac{\partial y_i^M}{\partial I} < 0 \iff \frac{\partial y_i^M}{\partial r} < 0 \\
\frac{\partial y_i^M}{\partial I} > 0 \iff \frac{\partial y_i^M}{\partial r} > 0
\]

Proposition 3.6.2 can be further decomposed. Substitute the aspatial indirect utility function into (3.6.3) to give the identity:

\[
y_i^M = y_i^{ind}(p_s, w_L, I(r), V(p_s(r), p_s, w_L, I(r)))
\]  

(3.6.11)

Differentiating (3.6.11) with respect to \( r \) gives:
\[ \frac{\partial y^M_r}{\partial r} = \frac{\partial y^M_s}{\partial I} \frac{dI}{dr} + \frac{\partial y^M_{y^M}}{\partial v} \frac{\partial v}{\partial I} dI + \frac{\partial y^M_{y^M}}{\partial p_s} \frac{dp_s}{dr} . \]  (3.6.12)

Equations (3.6.12) and (3.6.10) imply:

\[ \frac{\partial y^M_s}{\partial I} = \frac{\partial y^M_{y^M}}{\partial I} + \frac{\partial y^M_{y^M}}{\partial v} \frac{\partial v}{\partial I} ; \]

which can be substituted into proposition 3.6.2.

Proposition 3.6.1 is not testable because of unobserved utility. Proposition 3.6.2 appears to be testable, simply by relating space consumption to location. However, a moments reflection suggests that even here empirical testing will be difficult. Units of residential space are assumed homogeneous, whereas empirical tests would have to control for heterogeneous housing capital in a dwelling unit.

Proposition 3.6.2 leads to the Muth version of the Alonso lot size hypothesis:

*Residential space demands will be increasing in distance r if the net income elasticity of demand for living space is less than zero or, alternatively, in the event that the net income elasticity of demand is greater than zero, the effect of income on residential space demand is outweighed by the substitution effect of residential price variation in distance.*
Clearly this is not the lot size hypothesis, which is couched in terms of increasing land consumption per household (residence) in radial distance. It is perfectly feasible for residential space consumption to increase with distance but the quantity of land consumed per residence to diminish if the quantity of land used to produce residential space units diminishes sufficiently in distance. Only if the quantity of land per residential space unit remains unchanged or increases with radial distance can the lot size hypothesis hold. It is essential therefore in deriving a lot size hypothesis in the Muth type model to establish the relationship between radial distance and land input in producing residential space. This is the objective of chapter four where, through the profit function, residential builders input mix will be examined in relation to radial distance from city centre.

3.7 Concluding Remarks.

The Muth model of urban residential choice developed in this chapter, unlike other models within the Muth type framework, utilised the household’s bid rent function and the bidding process in residential price determination. This is usually associated with those models within an Alonso type framework and is very redolent of the Von Thünen agricultural model. The analysis was able to utilise this bidding process because, like residential land, residential space, once it is provided in a dwelling, is fixed at a location. This bidding process and the assumption of a monocentric city places the model within the class of generic Von Thünen models.
Consumer behaviour was analysed within a framework of utility maximisation, through the aspatial indirect utility function, and expenditure minimisation, through the aspatial expenditure function. In both cases, competitive market structures, combined with the maximisation or minimisation objectives of the household, resulted in a household spatial equilibrium characterised by the classic Muth equation.

Even though the analysis is restricted to the general urban structure of monocentricity, it is robust enough to explain different urban structures within that monocentric framework. Differentiated classes of households can be represented in relative location by concentric zones. These zones may be differentiated by phenomena such as household income if the analyst wishes. However, in this chapter household differentiation has been left at a suitably general level, namely household classes are differentiated by the slope of their bid rent functions relative to radial distance.

In section 3.6, propositions 3.6.1 and 3.6.2 laid down the conditions for residential space consumption per dwelling unit to be an increasing function of radial distance, \( r \). Particularly, proposition 3.6.2, by utilising the Slutsky decomposition of an own price change on the demand for residential space, was able to replicate, in residential space terms, the lot size hypothesis associated with urban residential models. In this case the hypothesis is written in terms of increasing dwelling size, measured by residential space units, as opposed to lot size, measured in land units. However, propositions 3.6.1 and 3.6.2 are general enough to outline those conditions which would lead to urban residential spatial structure other than that associated with the traditional lot size hypothesis, but still contained within a monocentric framework. Thus for example proposition 3.6.2 also shows those conditions which would lead to a
decrease in dwelling size with radial distance. This is an important point. The urban economic model based on the trade-off of access against residential space has mostly been developed in North America and has usually been tested within that context. There are exceptions, for example Evans (1973) tests the urban residential model within a British context. The domination of North America in the development and testing of the urban residential model is strong enough almost to suggest that market cities of the monocentric type would only have a residential structure associated with the North American experience. The model should not only be associated with these outcomes. It is a model of long run market equilibrium for a large city region. As such it should be, and is, robust enough to explain patterns of residential structure which do not necessarily replicate the U. S. or European experience, but are nevertheless generated by market forces.

It is all too easy to assert that access against residential space trade-off models cannot explain patterns of urban residential structure experienced in large city regions of say developing countries. This assertion is incorrect. Proposition 3.6.2 for example would explain "inside out" cities, with small, low income, high density housing on the periphery, through the relative configuration of the income and substitution effects of housing price change. The fact that income and own price elasticities for residential space might vary between societies might explain the variation between those societies in urban residential structure. Further this variation in structure would be contained within propositions such as 3.6.2 and points to the strength of the model as a generalisation of urban residential behaviour within large market oriented cities.
As it stands, proposition 3.6.2 cannot replicate the traditional lot size hypothesis stemming from Alonso type models. This is because proposition 3.6.2 refers to housing and not housing land as in the Alonso model. Alonso type models, because owners are builders and purchase residential land directly, can immediately go to the relationship between the demand for land and radial distance. Any propositions on the relationship between residential land and radial distance stemming from such models only reflect the nature of household preferences. No reference can be made to the effect of the technology embodied in a production function in this kind of model.

A feature of the Muth model is that any reference to residential land consumption and distance from the centre must involve the production technology faced by builders. Chapter four therefore examines the behaviour of builders who produce residential space when faced with a regime of residential space prices and a production technology embodied in a typical production function.
Chapter 4: The Residential Builder and the Lot Size Hypothesis.

4.1 Introduction.

Any analysis of urban residential allocation which follows a Muth type model must also examine producer behaviour in residential production if an objective is to form hypotheses about the allocation of urban land. In chapter three residential space was produced out of land and non-land inputs, Muth (1969, p. 48). Muth (1971, p. 244) later refers to the non-land inputs as building structures. In keeping with the earlier definition, this chapter assumes that two inputs are used to produce residential space in a dwelling. One input is specifically land, all other inputs are non-land and this can be treated as homogeneous building capital. Unlike the Alonso model, builders intermediate in the land market by providing residential space product. Therefore, the residential builder’s derived demand for land is an important component of the urban residential model. In the Alonso model any lot size hypothesis is predicated only on the nature of the household’s utility function. Muth type models, by necessity, involve the builder’s production function as well as the household’s utility function in the derivation of any lot size hypothesis.

Carrying over the notation conventions established in previous chapters it is possible to define the builder’s profit problem as:

$$\max_{r \in \mathcal{K}, \omega, \omega_y} \{ p_x(r)y_x - w_y K - w_v(r)V : y_x - f(K, V) \leq 0 \}$$

(4.1.1)
where:

\( p_s(r) \) is the price of residential output, which following from chapter three, is a monotonic decreasing convex function of radial distance.

\( y_s \) is the builder's output in terms of residential space per dwelling unit.

\( f(K,N) \) is the production function. Residential output is an increasing strictly quasi-concave function of building capital, \( K \), (building structures excluding land) and land, \( N \), inputs. Both \( K \) and \( N \) are assumed to be of identical qualities at any location.

\( w \) is input price and the subscript relates price to a particular input. Building capital is ubiquitous so that prices are invariant with radial distance. Land prices are a continuous, but as yet unknown, function of radial distance \( r \).

Problem (4.1.1) envisages that space prices have been set for fixed location residences as in chapter three. Space price is a known function of distance and the variation of space price in distance is an important determinant of the spatial aspects of urban residential land allocation. Further, the inner problem assumes that in producing output the builder faces a well behaved production function and profit maximising behaviour is constrained by this technology. A solution to the outer problem of (4.1.1) requires the establishment of the properties of the function \( w_s(r) \). Before proceeding with this it is necessary to say something about the structure of the residential market.

It is assumed that competitive conditions prevail in the residential space market. There is freedom of entry and exit and producers have no power to set prices at any
location and are presented with the residential space price function $p_s(r)$. The fact that residential builders produce homogeneous units of residential space, varying quantities of which form residences fixed at a location, simplifies the development of the properties of the residential land rent function $w_N(r)$.

Starting with a builder's profit problem, which does not make a priori judgement about land input prices, gives:

$$\max_r \{ \Pi(p_s(r), w_K, w_N) = \max_{K, N, y_s} \{ p_s(r) y_s - w_K K - w_N(r) N : y - f(K, N) \leq 0 \} \}.$$  

(4.1.2)

The aspatial profit function of (4.1.2) is, like any non-location profit function, a non-decreasing convex function of input price and a non-increasing convex function of output price. See for example Chambers (1988, p. 124). Of particular interest in the location analysis of the aspatial profit function will be its level curves in the land price/output price plane. This is illustrated in Figure 4.1.1.

Figure 4.1.1
In this figure the heavy arrow shows the direction of increasing profit and the shading shows the lower contour set, which must be a convex set. If the level curve is a lower contour function then it is convex as illustrated.

Problem (4.1.2) is still a Von Thünen type problem despite the fact that at present the only location variate occurs in revenue and not cost. It is a Von Thünen type problem because location is measured by radial distance from some attractive central focal point. Here the centripetal force is output price which is decreasing in \( r \). In the form of (4.1.2) the solution to the spatial problem is trivial, giving only central solutions. There must be some centrifugal force countering output price, and this will be land input price established in bidding under competitive conditions. Builders’ bids for land at any location \( r \) will reflect their derived demand for land from the market for residential space.

The problem of (4.1.2) does have some interest though and the fact that in that problem the location interest is focused on revenue calls into question the idea of starting out with a profit maximising problem. Recalling Beckmann (1968, p. 11, fig. 2.1) already cited in Chapter one, (4.1.2) would appear to be a problem of maximising revenue through location. However, despite Beckmann’s definition of a spatial revenue maximising problem being characterised by location variance in output prices only, (4.1.2), as it is, cannot be usefully translated into a revenue maximisation problem without further restriction. This is so because of two reasons.

Firstly even though input prices are invariant with location, inputs are a choice and therefore output and total cost will vary with location. Thus simply maximising
revenue will not replicate the outcome to (4.1.2) in the sense that even though it will replicate location solutions, it will not replicate outputs associated with profit maximising solutions. This point is worth clarifying. The simple conversion of (4.1.2) into an unconstrained revenue maximisation problem will result in all residential output collapsing to the centre. Revenue maximisation requires achievement of the highest feasible output price, which is at central locations in this case. At that price residential builders in the happy position of having unconstrained revenue as an objective, will also produce more than a profit maximiser constrained by the fact that production cost has to be covered. A restriction therefore would be to maximise revenue subject to an output or profit constraint. That is for an output or profit level at each location choose \( r \) to maximise revenue. Thus the revenue alternatives to (4.1.2) could take the form:

\[
\max_r \{ \Psi(p_i(r), y) \} \equiv \max_{K, N} \{ p_i(r) f(K, N) : f(K, N) \leq y \} ; \text{ or } (4.1.3)
\]

\[
\max_r \{ \Psi(p_i(r), w_K, w_N) \} \equiv \max_{K, N} \{ p_i(r) f(K, N) : p_i(r) f(K, N) - w_K K - w_N N \geq 0 \} ; \text{ or } (4.1.4)
\]

where \( \Psi \) is the aspatial revenue function conditioned on output, \( y \), in (4.1.3), or on profit in (4.1.4). The latter could be a Baumol (1959) type problem if the constraint is fixed to some profit level exceeding zero.

Both (4.1.3) and (4.1.4) describe some centralising force of revenue and are useful in that they fix revenue as an agglomerative force. Problem (4.1.3) is now moving away from a Von Thünen type problem and is becoming more closely related to a Löschian
type problem of market area. For example, taking the aspatial revenue function of (4.1.3) and solving for $y_i$ gives:

$$y_i = y_i(p_i(r), \psi) . \tag{4.1.5}$$

This is optimal sales conditioned on revenue. Taking the graph of $y_i$ in the radial distance dimension, and assuming identical population distribution and price gradient for each radian, and rotating over $2\pi$ radians, gives the “Löschian cone” (Lösch, 1967, p. 106) familiar to location theorists. The Löschian problem of establishing market area through location is then replicated by solving the problem:

$$\max_r \{ y_i(p_i(r), \psi) \} . \tag{4.1.6}$$

That is, choose location $r$ to maximise sales conditioned on revenue. This only takes us as far as circular markets and requires something further to end up with the hexagonal lattice of market areas that is strongly identified with the spatial market analysis of Lösch, (Lösch, 1967, p. 110).

Whilst this approach might be useful for the analysis of other issues, such as the location of sales oriented firms, it is not useful in analysing the residential land market, as all reference to land price has been concentrated out in defining the initial problem of (4.1.3).

The second objection to the idea that (4.1.2) can be turned into a revenue maximisation problem is that it is the proposition of urban theorists that land prices
established under competitive conditions will not be invariant with location, but will be a continuous function of radial distance \( r \) and will act as a centrifugal force in location. Thus, for example, whilst (4.1.4) might be used to analyse a locational interdependence model \( \textit{à la} \) Hotelling, which leads to some centralisation of sellers, at that point of centralisation competition for central location must result in variation in land prices around that agglomerative location. That is, competition for location will affect land price through the demand for land derived from the demand for central location. Thus land price will naturally act as a decentralising force countering agglomeration. Under these conditions ultimately problem (4.1.4) must address the issue of input cost, through land prices systematically varying with location. It is difficult to justify analysis of the behaviour of a single output firm with spatial variation in input as well as output price through revenue maximisation, unless that firm has a sales revenue maximising objective conditioned on profit or is profit satisficing. Further, where an aspatial revenue function might become particularly useful is analysing the efficient output mix of a multi-product firm (see Fried \textit{et al.}, 1993, p. 9-26). But even then the issue would only be interesting from a location point of view if the efficient mix varies with location due to say the market price for the different outputs varying differently with location.

Finally, utilising a cost dual to (4.1.2) is not useful in location analysis in terms of the current parameters. There is no location variation in cost and until that is introduced the problem would have no location interest. Simply making some \( \textit{a priori} \) judgement about land price and location would suffice to introduce some location variation in cost. However, outcomes would be specific to that \( \textit{a priori} \) judgement. This was not the case in chapter two where the agrarian model was analysed through a
cost dual. There, variation in land price was justified through competitive bidding and competitive bids in distance where determined by variation in transport cost in distance. Whilst transport cost conditions where imposed from outside the model the structure of transport cost imposed had some economic rationale. It is therefore proposed to analyse residential builders' behaviour through traditional profit maximising objectives.

4.2 Builders' Bid Rents and Residential Land Market Rent.

Returning to the profit maximising objective of (4.1.1) it is possible to identify the relationship of land market rent to radial distance through competitive bidding. Inverting the aspatial profit function of (4.1.2) for land rent results in the bid rent function:

$$\omega_N = \omega_N(p_z(r), w_k, \pi) .$$

(4.2.1)

This is a maximum value function because it is derived from a maximum value function and is the maximum bid conditioned on profit level $\pi$. The bid rent curve is the expression for the lower contour function of the aspatial profit function of (4.1.2). Substituting the aspatial profit function of (4.1.2) into the bid rent function (4.2.1) gives the identity:

$$\omega_N(p_z(r), w_k, \Pi(p_z(r), w_k, w_N)) = w_N .$$

(4.2.2)
The first derivative properties of the bid rent function can now be found by differentiating (4.2.2) with respect to its arguments. Differentiating (4.2.2) with respect to $w_N$ and using Hotelling's Lemma gives:

$$\frac{\partial}{\partial \pi} \omega_N(p_s(r), w_K, \pi) \frac{\partial \Pi}{\partial w_N} = 1 \Rightarrow \frac{\partial}{\partial \pi} \omega_N(p_s(r), w_K, \pi) = -\frac{1}{N}.$$ 

The derived demand conditioned on $\pi$ and abstracted from own price can be retrieved by:

$$N^\pi(p_s(r), w_K, \pi) = -\frac{1}{\frac{\partial}{\partial \omega_N} \frac{\partial}{\partial \pi}}.$$ 

Derived demands conditioned on profit will be denoted with the $\pi$ superscript. Differentiating (4.2.2) with respect to $p_s(r)$ and $w_K$ gives the remaining first derivative properties of $\omega_N$. The second order properties are given by a further differentiation.

Properties 4.2.1 - (the properties of $\omega_N$)

1. a) Non-increasing in $\pi$ and $w_K$:

$$\frac{\partial}{\partial \pi} \omega_N(p_s(r), w_K, \pi) = -\frac{1}{N^\pi(p_s(r), w_K, \pi)}$$

$$\frac{\partial}{\partial w_K} \omega_N(p_s(r), w_K, \pi) = -\frac{\partial}{\partial \pi} \omega_N(p_s(r), w_K, \pi) \frac{\partial \Pi}{\partial w_K} = -\frac{K^\pi}{N^\pi(p_s(r), w_K, \pi)} \leq 0$$

b) Non-decreasing in $p_s$:
\[
\frac{\partial}{\partial p_i} \omega_N(p_i(r), w_K, \pi) = -\frac{\partial}{\partial \pi} \omega_N(p_i(r), w_K, \pi) \frac{\partial}{\partial p_i} = \frac{y_i^\pi}{N^\pi} \pi(p_i(r), w_K, \pi) \geq 0
\]

2. The function \(\omega_N\) is convex in \(p_i, w_K\) and \(\pi\).

\[
\begin{align*}
\frac{\partial^2 \omega_N}{\partial p_i^2} &= \frac{\partial}{\partial p_i} \left( \frac{y_i^\pi}{N^\pi} \right) \\
\frac{\partial^2 \omega_N}{\partial w_K^2} &= \frac{\partial}{\partial w_K} \left( -\frac{K_i^\pi}{N^\pi} \right) \\
\frac{\partial^2 \omega_N}{\partial \pi^2} &= \frac{\partial}{\partial \pi} \left( -\frac{1}{N^\pi} \right) \\
\end{align*}
\]

The curvature characteristics of function (4.2.1) can be established through its first and second derivative properties with respect to distance \(r\). By definition:

\[
\frac{dp_i}{dr} < 0 \quad \text{and} \quad \frac{d^2 p_i}{dr^2} \geq 0 \quad \text{and}
\]

recall properties 4.2.1 and Figure 4.1.1:

\[
\frac{\partial \omega_N}{\partial p_i} > 0 .
\]

Finally, recall the properties of the aspatial profit function (4.1.2) and Figure 4.1.1, then:

\[
\frac{\partial^2 \omega_N}{\partial p_i^2 r} \geq 0 .
\]
These relations lead to the following corollaries:

**Corollary 4.2.1**

\[
\frac{\partial \omega_N}{\partial r} = \frac{\partial \omega_N}{\partial p_s} \frac{dp_s}{dr} < 0 .
\]

**Corollary 4.2.2**

\[
\frac{\partial^2 \omega_N}{\partial r^2} = \frac{\partial \omega_N}{\partial p_s} \frac{d^2 p_s}{dr^2} + \frac{\partial^2 \omega_N}{\partial p_s^2} \left( \frac{dp_s}{dr} \right)^2 \geq 0 .
\]

These corollaries make sense, because the profit function is a non-decreasing convex function of \( p_s \) and a non-increasing convex function of \( w_N \). Thus any reduction in profit brought about by a decrease in output price through increasing radial distance, \( r \), must be countered by a decrease in \( w_N \) in order to maintain profit level \( \pi \). Further, Cornes (1992, p. 18) shows that, when a problem is to maximise an objective function, where the parameters occur only in the objective function, and the objective is a convex function of those parameters, the maximum value function is convex in those parameters. The objective function of the inner problem of (4.2.1) is convex in output price and land input price. Further, output price is a convex function of radial distance, so that level curves in the land input price/radial distance plane define the lower contour set of a decreasing convex function in offer price and distance. Finally, because \( \pi \) is a convex increasing function of \( p_s \) and a convex decreasing function of \( w_N \), then the compensatory increase in \( w_N \) to maintain profit with a decrease in \( p_s \), through a change in location, must remain unchanged or increase.

Corollaries (4.2.1) and (4.2.2) lead to the following proposition:
Proposition 4.2.1

Residential builders' bid rent, \( \omega_h \), is a monotone decreasing convex function of radial distance \( r \).

In the agrarian model of chapter two and the residential space model of chapter three equilibrium market price as a function of \( r \) was formed through competitive bidding. In the case of residential builders, competitive conditions result in a complete coincidence of builders' bid rent functions so that in the bidding process residential land market rent as a function of distance will be:

\[
\omega_h(r) = \omega_h = \omega_h(p_t(r), w_K, \pi).
\] (4.2.3)

This is similar to the residential land market outcome of Solow (1973, p. 5), where he assumes complete coincidence of households' bid rent functions in an Alonso type model. Equation (4.2.3) results in:

Proposition 4.2.2

Under competitive conditions the urban residential land market rent is a monotone decreasing convex function of radial distance \( r \).
4.3 The Builder's Equilibrium.

The builder's profit maximising problem can now be stated in terms of an inner and outer problem, but where land market prices are a monotonic decreasing convex function of radial distance $r$. The solution to the problem of maximising the aspatial profit function is the spatial profit function:

$$\Pi_s(w_k) = \max_r \left\{ \Pi(p_s(r), w_k, w_N(r)) \right\} . \quad (4.3.1)$$

A solution to (4.3.1) exists. Recalling (4.2.2), because:

$$\frac{\partial \omega_N}{\partial p_s} = 1 \Rightarrow \frac{dp_s}{dr} = \frac{\partial \omega_N}{\partial r} ;$$

then (4.2.3) implies:

$$\frac{dp_s}{dr} = \frac{\partial \omega_N}{\partial r} = \frac{dw_N}{dr} ;$$

which ensures a solution to the problem of (4.3.1) when land market rent is substituted into the aspatial profit function as a decreasing convex function based on competitive builders' bid rent functions. The first order condition for the solution to (4.3.1) is:

$$\frac{\partial \Pi}{\partial r} = \frac{\partial \Pi}{\partial p_s} \frac{dp_s}{dr} + \frac{\partial \Pi}{\partial w_N} \frac{dw_N}{dr} = 0 . \quad (4.3.2)$$

Using Hotelling's Lemma and substituting into (4.3.2) for $\partial \Pi/\partial p_s$ and $\partial \Pi/\partial w_N$ gives:
\[ \frac{y_s(p_s(r), w_K, w_N(r))}{N(p_s(r), w_K, w_N(r))} = \frac{dw_k}{dp_s} \cdot \frac{dr}{dr} \]

(4.3.3)

At any location \( r \), profit maximisation requires that the ratio of land input to output of residential space per dwelling be fixed so that the tendency for profit to increase through the reduction in land cost equals the tendency for profit to decrease through the reduction in revenue.

Further, recalling that:

\[ \frac{d^2 w_k}{dr^2} \geq 0 ; \]

then equation (4.3.3) implies that the ratio of land input to output is constant or increasing with distance from centre for a given level of profit. In equilibrium all builders make normal profit at all locations so this implication is important. Further discussion on this issue will be deferred to a more detailed examination of the derived demand for land in section 4.5.

Chapters two and three utilised the fact that optimisation problems involving bidding for items which were fixed at a location could be alternatively represented as minimising the difference between bid rent (offer price) and market rent (price) subject to the constraint that in order to capture the fixed location item at least market rent had to be paid at that location. Profit maximising residential builders' behaviours can also be expressed via this alternative problem.
\[ \Omega_{\pi}(w_K, \pi) = \min_r \left\{ \omega_N(p_s(r), w_K, \pi) - w_N(r) : \omega_N(p_s(r), w_K, \pi) - w_N(r) \geq 0 \right\} . \]  

(4.3.4)

The first order conditions characterising the solution to (4.3.4) are:

\[
\frac{\partial \omega_N}{\partial p_s} \frac{dp_s}{dr} = \frac{dw_N}{dr}; \text{ and} \]

\[ \omega_N(p_s(r), w_K, \pi) = w_N(r) . \]

(4.3.5)

Recalling properties 4.2.1 and substituting into the first condition of (4.3.5) gives:

\[ \frac{y^*_s}{N^*_s}(p_s(r), w_K, \pi) = \frac{dw_N/dr}{dp_s/dr} ; \]

(4.3.6)

which replicates (4.3.3). Substituting in the spatial profit function of (4.3.1) into the spatial function \( \Omega_{\pi} \) gives:

\[ \Omega_{\pi}(w_K, \Pi_s(w_K)) = 0 ; \]

(4.3.7)

which is the implicit spatial profit function. Differentiating (4.3.7) with respect to \( w_K \) results in:

\[ \frac{\partial \Omega_{\pi}}{\partial \Pi_s} = -\frac{\partial \Omega_{\pi} \beta}{\partial \Pi_s} \frac{w_K}{\beta} \]

Under competitive conditions because builders will be making normal profit everywhere, the spatial solutions to (4.3.1) and (4.3.4) will not be unique. They will be that interval of radial distances where residential builders' normal profit bid rent
exceeds the bid rent for urban business land use towards the urban centre and agrarian land use toward the periphery. This is illustrated in Figure 4.3.1 where the interval $r_1$ to $r_2$ gives the solution to (4.3.1) and (4.3.4).

![Figure 4.3.1](image)

Points $r_1$ and $r_2$ in Figure 4.3.1 can be identified as the solutions to the following respective radial distance problems:

$$
\underline{R}(w_\Delta, \pi) = \min_r \left\{ r \in \mathbb{R}_+ : \omega_N(p_\Delta(r), w_\Delta, \pi) - w_N(r) \geq 0 \right\} ; \text{ and } (4.3.8)
$$

$$
\overline{R}(w_\Delta, \pi) = \max_r \left\{ r \in \mathbb{R}_+ : \omega_N(p_\Delta(r), w_\Delta, \pi) - w_N(r) \geq 0 \right\}. \text{ (4.3.9)}
$$

These problems are, find the minimum and maximum radial distance respectively that residential builders can operate subject to the payment of at least market rent for land at those locations. Substituting the aspatial profit function of (4.3.1) into (4.3.8) and (4.3.9) respectively gives the identities:
\[
R(w_K, \Pi(p_s(r), w_K, w_N(r))) = r \quad ; \text{and} \quad (4.3.10)
\]

\[
\overline{R}(w_K, \Pi(p_s(r), w_K, w_N(r))) = r . \quad (4.3.11)
\]

Taking either (4.3.10) or (4.3.11) and dropping the under or over bar and differentiating with respect to \(r\) gives:

\[
\frac{\partial R}{\partial \pi} = \frac{1}{\partial \Pi \frac{dp_s}{dr} + \partial \Pi \frac{dw_N}{dr}} \leq 0; \quad (4.3.12)
\]

the sign of which depends on the relative values of the two terms of the denominator on the right hand side. These reflect the inherent forces in the trade-off of residential space prices and land prices in the location of residential building activity.

Differentiating (4.3.10) or (4.3.11) with respect to \(w_K\) gives:

\[
\frac{\partial R}{\partial w_K} = \frac{\partial R}{\partial \pi \frac{\partial \Pi}{\partial w_K}}.
\]

The sign of this could be determined if the sign of \(\partial R \partial \pi\) on the right hand side was known. It is possible to sign this term if the nature of the function \(R\) is identified. If it is the minimum distance, problem (4.3.8), then recalling Figure 4.3.1 for small perturbations around the optimal location.
\[
\frac{\partial \Pi}{\partial p_s} \frac{dp_s}{dr} + \frac{\partial \Pi}{\partial w_h} \frac{dw_h}{dr} \geq 0.
\]

The converse is true for the function \( R \) in the maximisation problem (4.3.9). Thus comparative statics around optimal location \( r \) is not possible unless one is prepared to determine the sign of \( \partial R \partial \pi \). However, if the sign is determined in the way suggested then the effect of a change of non-land input price on the radial extent of residential building can be determined.

\[
\frac{\partial R}{\partial w_K} = \frac{\partial R}{\partial \pi} \frac{\partial \Pi}{\partial w_K} \geq 0 \quad \text{and} \quad \frac{\partial R}{\partial w_K} = \frac{\partial R}{\partial \pi} \frac{\partial \Pi}{\partial w_K} \leq 0
\]

This implies that an increase in non-land input price reduces the extent of the concentric zone of residential building in the city.

It is not possible to undertake comparative statics on the effect of an increase or decrease of output price at each location through (4.3.10) or (4.3.11). However the effects of an upward or downward shift of \( p_s(r) \) can be intuitively derived through utilising the first order properties of the bid rent function, properties 4.2.1 1b,

\[
\frac{\partial \omega_X(p_s(r), w_K, \pi)}{\partial p_s} \geq 0;
\]

means an increase in output price at all locations increases bid rents conditioned on a profit level at all locations. Combining this upward shift with the competitive determination of market rents implies an increase in output price at all locations
increases the extent of the zone of residential building in the city. The converse is true
for a decrease in output price at all locations.

The effect of a change in the relationship of output prices to radial distance can be
derived from (4.3.10) or (4.3.11) through (4.3.12). Rearranging this latter equation
gives the condition for optimal r whether it is minimum or maximum radial distance
as:

\[
\frac{\partial R}{\partial \pi} \left[ \frac{\partial \Pi}{\partial \pi} \frac{dp_s}{dr} + \frac{\partial \Pi}{\partial \pi} \frac{dw_R}{dr} \right] = 1.
\]

The sign of the first product in the bracket is negative and the second product sign is
positive. A decrease in \(dp_s/dr\) therefore implies that optimal \(r\) increases. That is in
the residential building case the inner and outer boundaries of the concentric zone of
residential building activity shifts outwards. The converse applies to an increase in
\(dp_s/dr\).

4.4 Summary of Results

The residential builder’s problem is characterised by the same five functions
established in chapter 2 and reflect their consumption counterparts of chapter 3.
Starting with the aspatial profit function:

\[
\Pi(p_s(r), w_x, w_R(r)) ; \quad (4.4.1)
\]
and maximising the aspatial function over \( r \), results in the spatial profit function:

\[
\Pi_r(w_K) .
\]  
(4.4.2)

The bid rent function is found by inversion of the aspatial profit function:

\[
\omega \beta(p_z(r), w_K, \pi) .
\]  
(4.4.3)

Choosing distance to minimise the difference between bid rent and market rent, subject to a market rent constraint, results in:

\[
\Omega_{n}(w_K, \pi) .
\]  
(4.4.4)

Finally, choosing \( r \) to minimise or maximise \( r \), subject to a market rent condition, establishes the inner or outer boundary of the urban residential land market:

\[
R(w_K, \pi) .
\]  
(4.4.5)

Optimal rent, profit and minimum and maximum distances can be retrieved from these functions by appropriate substitution. Firstly, substituting the aspatial profit function (4.4.1) into the radial distance function of (4.4.5) gives either minimum or maximum radial distance:

\[
R(w_K, \Pi(p_z(r), w_K, w_{\beta}(r))) = r .
\]  
(4.4.6)
Substituting the spatial profit function (4.4.2) into (4.4.4) gives the implicit profit function:

$$\Omega_n(w_K, \Pi_r(w_K)) = 0.$$  \hspace{1cm} (4.4.7)

Substituting the aspatial profit function (4.4.1) into the bid rent function (4.4.3) retrieves land rent:

$$\omega_n(p_s(r), w_K, \Pi(p_s(r), w_K, w_N(r))) \equiv w_N.$$  \hspace{1cm} (4.4.8)

Substituting the radial distance function (4.4.5) into the aspatial profit function (4.4.1) gives profit level:

$$\Pi(p_s(R_r(w_K, \pi), w_K, w_N(R_r(w_K, \pi))) \equiv \pi.$$  \hspace{1cm} (4.4.9)

Substituting the radial distance function (4.4.5) into the bid rent function (4.4.3) retrieves rent:

$$\omega_n(p_s(R_r(w_K, \pi), w_K, \pi) \equiv w_N.$$  \hspace{1cm} (4.4.10)

Finally, substituting (4.4.4) into the aspatial profit function retrieves profit:

$$\Pi(p_s(r), w_K, \Omega_n(w_K, \pi)) \equiv \pi.$$  \hspace{1cm} (4.4.11)
Builders’ derived demand for land can be retrieved from the aspatial profit function (4.4.1) by differentiation with respect to land price. Alternatively the derived demand for land can be obtained by differentiating (4.4.8) with respect to land price. That is:

\[ N^*\left(p_s(r), w_K, \pi \right) = \frac{1}{\partial \omega \beta \pi} \].

(4.4.12)

This is conditioned on profit \( \pi \) and abstracted from own price.

Equations (4.4.1) to (4.4.11) are summarised in Figure 4.4.1.
4.5 The Derived Demand for Land and Radial Distance.

The previous section described a set of equations from which the derived demand for land of residential builders can be retrieved.

Firstly, the unconditional derived demand can be retrieved from the aspatial profit function via Hotelling’s Lemma:

\[ N(p_s(r), w_K, w_N(r)) \]  \hspace{1cm} (4.5.1)

Alternatively the derived demand conditioned on profit level, but abstracted from own price can be retrieved from the bid rent function:

\[ N^a(p_s(r), w_K, \pi) \]  \hspace{1cm} (4.5.2)

The derived demand for land conditioned on output is only available through cost minimisation, which in the case of residential builders requires some \textit{a priori} conditioning on the nature of the relationship between land input price and radial distance.

Differentiating (4.5.1) with respect to radial distance \( r \) gives:

\[ \frac{\partial N}{\partial r} = \frac{\partial N}{\partial p_s} \frac{dp_s}{dr} + \frac{\partial N}{\partial w_N} \frac{dw_N}{dr} \leq 0 \]  \hspace{1cm} (4.5.3)
Recalling that:

\[
\frac{dp_s}{dr} < 0 \quad \text{and} \quad \frac{dw_N}{dr} < 0 \quad ; \quad \text{and;}
\]

if the Hessian matrix of the aspatial profit function is positive semi-definite, that is the aspatial profit function is convex, then:

\[
\frac{\partial N}{\partial w_N} = -\frac{\partial^2 \Pi}{\partial w_N^2} \leq 0 \quad ;
\]

and equation (4.5.2) leads to the proposition:

**Proposition 4.5.1**

\[
\frac{\partial N}{\partial p_s} \leq 0 \Rightarrow \frac{\partial N}{\partial r} \geq 0
\]

\[
\begin{align*}
\frac{\partial N}{\partial p_s} & \geq 0 \quad \Leftrightarrow \quad \frac{\partial N}{\partial p_s} \frac{dp_s}{dr} \leq \frac{\partial N}{\partial w_N} \frac{dw_N}{dr} \\
\frac{\partial N}{\partial r} & \leq 0 \quad \Leftrightarrow \quad \frac{\partial N}{\partial p_s} \frac{dp_s}{dr} \geq \frac{\partial N}{\partial w_N} \frac{dw_N}{dr}
\end{align*}
\]

That is:

the derived demand for land is increasing with radial distance if the derived demand for land is inversely related to the market price for residential space. Otherwise the derived demand for land is increasing with radial distance only if the own price effect on derived demand with distance outweighs the output price effect on the derived demand for land with distance.
However proposition 4.5.1 can be decomposed further by utilising (4.5.2).

Substituting the aspatial profit function into (4.5.2) gives:

\[ N \equiv N^\pi \left( p_i(r), w_k, \Pi(p_i(r), w_k, w_N(r)) \right). \tag{4.5.4} \]

Differentiating (4.5.4) with respect to \( r \) gives:

\[
\frac{\partial N}{\partial r} = \frac{\partial N^\pi}{\partial p_i} \frac{dp_i}{dr} + \frac{\partial N^\pi}{\partial \pi} \frac{dp}{dr} + \frac{\partial N^\pi}{\partial w_N} \frac{dw_N}{dr}.	ag{4.5.6}
\]

Equations (4.5.6) and (4.5.3) imply:

\[
\frac{\partial N}{\partial p_i} = \frac{\partial N^\pi}{\partial p_i} + \frac{\partial N^\pi}{\partial \pi} y_i; 
\]

which could be substituted into proposition 4.5.1. Properties 2.4.1 1.b) imply:

\[
\frac{\partial N^\pi}{\partial p_i} \geq 0; 
\]

so that:

\[
\frac{\partial N}{\partial p_i} \leq 0 \Rightarrow \frac{\partial N^\pi}{\partial \pi} \leq 0. 
\]

Because lot size is a measure per dwelling unit and output in the Muth model by definition is quantity of residential space per dwelling unit, then proposition 4.5.1 is
sufficient to establish the lot size hypothesis that the quantity of land per dwelling is increasing in distance.

4.6 Concluding Remarks.

This chapter continued the Muth type residential model introduced in chapter 3. Like chapter 3, the analysis in this chapter has attempted to follow the general dual approach suggested in section 1.4 of chapter 1 and utilised in the analysis of the basic Von Thünen model in chapter 2. Given the nature of the residential builder’s problem, the dual approach has been more restricted than in previous chapters. Particularly, because it requires a priori judgement about the nature of the relationship between land market rent and distance, the analysis did not utilise cost minimising objectives.

Even though this chapter has been limited to profit maximising behaviour, outcomes have been useful. Proposition 4.5.1 contains the residential land lot size hypothesis traditionally associated with urban residential models. The model developed in chapter three and in this chapter is robust enough to contain alternative lot size, density structures. Combining propositions 4.5.1 and 3.6.1 or 3.6.2 and 3.6.2a of chapter three can provide a fairly rich diet of differing urban structures within the monocentric form. The analysis of a Muth type model within a bidding framework has been successfully completed.
The outcomes to the application of the alternative dual problems to the urban model has been limited but interesting. Comparative statics can be performed around the envelope properties of the function $\Omega$, stemming from the problem of minimising the difference between offer and market rents subject to a market rent constraint. Conditional comparative statics also can be undertaken using the envelope characteristics of the distance function $R$. There are limitations. Both functions are spatial functions, with some interesting parameters being optimised out through optimal radial distance $r$, giving restricted comparative static analysis. This is particularly so in this chapter, where the initial number of parameters determining outcomes was small.

The fact that comparative statics using the function $R$ are conditional might be regarded as a weakness. However, it could be argued that the factors which determined the conditional nature of some of the comparative statics also determine the richness of the model in terms of outcomes.

The propositions of section 3.6, chapter three and 4.5 of this chapter contain the lot size hypothesis. However, it is not possible to test the lot size hypothesis through empirical estimation of the rich set of urban residential models generated in chapter three and this chapter. The empirical thesis of part B, chapters five and six will proceed to test the lot size hypothesis utilising density gradients for the Australian states' capital cities. Recalling chapter one, section 1.5 these tests will be Popperian in the sense that gross density gradients can search for outcomes that are not consistent with the traditional lot size hypothesis. Chapter five will be given over to what might be regarded as methodological issues in devising an appropriate empirical test for the functional form of density gradients within a general Popperian
methodology. Chapter six deals with problems in estimation and outlines results from tests and model estimation.
PART B

EMPIRICAL DEVELOPMENTS
5.1 Introduction.

Testing economic hypotheses can be difficult. Frequently theory provides us with different competing explanations of economic phenomena. Moreover, even though there may be an established economic hypothesis, several alternative models are available because economic theory has not been forthcoming in terms of the exact functional form which relates economic variables. Procedures encompassing these issues have become increasingly well established in econometric theory and practice in the recent past. In particular, setting up a general statistical model nesting alternative models and then testing down to a single parsimonious model is very appealing. This methodology has become well established in dynamic modelling where general models rich in lag structures can nest several competing alternatives. Charemza and Deadman (1992, Chap. 3) correctly attribute the methodology to Hendry. Certainly the Davidson, Hendry, Srba and Yeo (1978) paper is seminal to general to specific modelling and the 1980’s saw a rapid establishment and refinement of this methodology. Papers such as Hendry and Richard (1982) have been influential in this establishment and the methodology is treated in depth in Hendry (1995). Charemza and Deadman (Chap. 4) and Gilbert (1986) are very accessible reviews of general to specific modelling and emphasise the advantages of testing down to specific models through likelihood or least squares criterion based tests on linear or non-linear restrictions in the general model.
General to specific modelling has not been without criticism. Thomas (1996, p. 360) points out that the process could be subject to data mining. This is also a major criticism alluded to by Pagan (1987). The issue here is the distinction between formulating a general model which nests competing alternative models and creating a model so general that one can easily arrive at apparently significant specific economic models from it with little reference to theory. The latter approach would be data mining but it is not a necessary outcome of using general to specific methodology. The methodology was designed to overcome the issues of data mining and if general to specific modelling is used correctly then data mining will not occur as a necessary consequence. The general model is a vehicle for testing down to specific models. If it is to be used within the general framework of Popperian methodology it must contain models which make economic sense from the point of view of perceived theory. Its existence can only be predicated on its containment of alternative competing models stemming from theory. These sentiments are expressed in McAlcer et al. (1985). The requirement for good general to specific modelling in terms of the nesting model formation, transparency in testing down and formal evaluation of the final parsimonious model is succinctly stated in this paper (p. 299). Further, the authors demonstrate a clear and transparent testing down procedure in estimating a dynamic demand for money model (p. 302-304).

Appropriate use of general to specific modelling requires use of a clearly understandable testing down procedure. If this is undertaken, even though the procedure may be subject to inadvertent data mining, data mining can be identified by third party observers of the analysis. This is not the case when one starts with a
specific model, generalises it to produce virtuous properties and subsequently publishes only the virtuous final form. This is quite clearly demonstrated by Hendry and Mizon (1990). They were only able to critically appraise the McAleer et al (1985) findings in terms of the adopted testing down mode to a specific model (the use of the COMFAC algorithm to discriminate between autoregressive error alternatives) precisely because the latter used the general to specific methodology in a transparent fashion. Whilst this is not an example of explicit or implicit data mining, it shows that when general models are formulated clearly and testing down is transparent, peer appraisal of the analysis is facilitated. This is a prerequisite for the application of scientific methodology to economics. If general to specific modelling is to be successful then the path to a specific model through the rejection of alternate competing models must be easily identifiable and verifiable, as this case clearly shows.

Pagan (1987) goes further and suggests that reliance in testing down on well established methods such as likelihood ratio, Wald or Lagrange Multiplier statistics as well as tests based on the Student or Fisher distributions is regrettable and that these may be supplemented by or replaced with innovative alternatives. He recommends alternatives such as the Akaike criterion or at least to use traditional decision rules in a way which is not slavishly mechanical. These are not criticisms of the general to specific methodology. None are dismissive of the methodology. Rather they are concerns about the way that it is used. They are vital in the sense that they point the way to the correct use of general to specific modelling.
Whatever the decision rules used in moving down from a general to a specific model, because they involve decision through rejection, the methodology is Popperian. General to specific modelling emphasises Popperian falsification in economics. Whether falsification can be used in economics is subject to debate. Hands (1992) suggests that economics cannot be developed in a way where hypotheses are easily refutable. He is correct in pointing out that some hypotheses such as utility maximisation are not very testable propositions. Moreover, hypotheses that have been rejected in many situations are still maintained in theory, for example profit maximisation. The inability to set up controlled laboratory experiments in economics certainly mitigates against falsification methods. Nevertheless testing through falsification is an important rationale for empirical economics (Hausman 1988). The inability of theory, through deductive reasoning, to come up with tightly defined hypotheses does not promote verification as a methodology in econometrics. Rather,

"because [the] theory is loosely formulated, such evidence can never do more than to suggest that [the] theory is worth testing in terms of its falsifiable consequences". (Blaug, 1985, p. 702).

This argument is repeated in a reinstatement of Popperian orthodoxy in terms of econometric methodology being founded in falsification rather than verification in Blaug (1992, Chap. 16). Morgan (1988) sees general to specific modelling as an important tool in Popperian falsification in economics. This view is reinforced by Darnell and Evans (1990, Chap. 2) where falsification underpins empiricism as an appeal to the facts. Later, in chapter 6, Darnell and Evans illustrate the method of falsification within econometrics through an exposition of testing down from a
general model through restrictions on nested models and, by rejection of alternatives, arriving at a specific empirical model. General to specific modelling therefore appeals to those who wish to apply Popperian falsification methodology to economic analysis. General to specific modelling through falsification is a Popperian tool but the application of full Popperian methodology requires more. Falsification is not rejection. What number of falsifying experiments are required for the rejection of a hypothesis? This is an extremely vexing question, especially if rejection leads to a theoretical vacuum.

Whilst the use of general to specific methodology is relatively well established in dynamics and time series, it has not been so well utilised in cross sectional studies. This is a consequence of the nature of many issues in economics; they do tend to be dynamic, especially in terms of long run equilibrium and error correction mechanisms adopted around that dynamic equilibrium. However, if the justification for general to specific modelling is the test of falsification, the methodology should be flexible enough to encompass static as well as dynamic modelling. It is possible to use general to specific modelling in static models employing cross sectional data. Competing linear models can be nested within a general cross sectional model and then tested down relatively easily if the specific competing alternatives contain overlapping explanatory variables. In this case, decision rules can be based on standard test procedures whilst moving from a general to a specific model. The process breaks down if competing models do not have overlapping explainers or if the issue is one of alternative functional forms. Competing models with non-overlapping explainers or different functional forms do not nest naturally in a general model.
The problem of competing functional form frequently arises. Theory may not be forthcoming on the functional form of a relationship contained within a model. Alternatively, the functional form adopted in empirical analysis may have serious implications for theory, such as imposing certain arbitrary restrictions on theoretical structures. Where functional form is not indicated by theory choice of empirical form becomes arbitrary. Under these conditions decisions on the nature of empirical functional form may be made with estimation or empirical issues given prominence, as long as the specific functional form adopted is not inconsistent with theory. Models in cross sectional data may start with some arbitrary functional form because of ease in estimation, the virtuous characteristics of the residuals or the tractability of elasticity measures. A popular candidate seems to be the log-linear form. Whilst these issues such as ease of estimation and normality of residuals are important, theoretical considerations underpinning the model should not necessarily take on a secondary role. In terms of urban population density studies as a test of the lot size hypothesis, whilst theory indicates the log-linear form under certain conditions (see, Kau et al., 1986, p. 18-30), ease of estimation of that form also appears to be a consideration, (see Greene and Barnbrook, 1978). The arbitrary choice of say the log-linear form forces the data to produce certain outcomes rather than allowing the data to aid in the choice of functional form and certainly does not constitute a test of functional form. If theory does suggest a functional form, but this form is conditioned on the coincidence of certain economic conditions, the empirical model estimation must nest within it alternatives as a test for that functional form. There is no need to begin empirical analysis with a log-linear specification. It is well known that the Box and Cox (1964, p. 214) transformation model nests within it log-linear and linear in
level forms as well as square root and inverse transformations as special cases. Moreover, the Box Cox power transformation can be extended in several ways (Cook and Weisberg, 1982).

Whilst Box Cox is not the only transformation which nests alternative specifications suitable for testing (for example see Mackinnon and Magee, 1990) in its simple or extended form the Box Cox transformation is popular in use. Tae Hoon Oum (1989) uses the Box Cox transformation to test between linear in levels and log-linear specifications in rail freight transport. Link et al. (1988) uses the transformation in a similar fashion to discriminate functional form in firm size and development spending. De Witte and Cramer (1986) test for the functional form of the Engel relationship with expenditure on foodstuffs. Godfrey and Wickens (1981) deal with the use of Box Cox transformations specifically to test for competing linear and log-linear models as an alternative to non-nested tests based on centred log-likelihood ratios. It is significant that Gourieroux and Monfort (1995, p. 300), include the Box Cox transformation as one of several alternatives to test the naturally non-nested linear and log-linear specifications.

Alternatively, the Box Cox transformation has been used simply to generalise functional form rather then act as a decision rule between, say, log-linear and linear in levels models. It has been used in this way in hedonic price equations for housing. For an example see Coulson and Robbins (1987). In urban economics, Box Cox methods have been used to identify the curvature characteristics of urban density gradients (Kau and Lee, 1976; Lahiri and Numrich, 1983; Kau, Lee and Sirmans, 1986, p. 79-83 and Numrich, 1990). This paper aims to follow the Box Cox form
adopted by these works in testing the urban residential model, by estimating population and household densities for the capital cities of the six Australian states.

Wooldridge (1992) and Berndt et al. (1993) propose an alternative specification derived from Box Cox. They argue that this specification is more interesting in that it focuses attention on the dependent variable, more specifically the conditional mean function, rather than transforms of the dependent variable as in the simple or extended Box Cox models. This is an interesting development and again like the Box Cox transformation nests within it the linear and log-linear forms as special cases. However it now begs the question, if we have a competing alternative to the Box Cox model, how are we going to choose between these competing alternatives? This is an important issue since it is proposed to use the Wooldridge specification and a log transformation of this specification as competing alternatives to Box Cox estimation of urban density gradients.

A method for discriminating between these competing alternatives using a non-nested test is proposed. In particular this is a specification test which is based on the artificial nesting model of Davidson and MacKinnon (1981), which conforms with the general to specific methodology of testing down from a general to a specific equation. The economic model tested is the density gradient model already examined by Kau and Lee (1976) and Numrich (1990). However, the gradients estimated in this thesis utilise population and household density gradients for the six Australian state capitals for 1986 and 1981 census data. Section 2 of this chapter examines the characteristics of the Box Cox and Wooldridge specifications and also suggests another alternative through the log transformation of the Wooldridge specification. This is followed by a
section on non-nested tests using artificial nesting models and the utilisation of such
tests when competing alternatives involve monotonic transformations of the
dependent variable. Chapter 6 deals with issues in estimation and is also given over
to a discussion of test results.

5.2 The Box and Cox Transformation and Alternatives.

The Box Cox model examined will be the simple model where the dependent variable
only is subject to the power transformation. The results also apply to the extended
model where the explanatory variables are also subject to the power transformation.
The arguments for the Wooldridge alternative only apply to the case where the
dependent variable is subject to the transformation. The Box Cox alternative where
the independent variables only are transformed is not relevant to the issues at hand.

Whilst the Box Cox transformation is well documented it is worth while stating some
of the characteristics of the transformation especially in terms of comparison to the
Wooldridge alternative.

The Deterministic Model

It is well known that the general Box Cox model has nested within it linear and log-
linear models dependent on the value taken up by the transformation parameter. It
can also be demonstrated that certain reparameterisations of the Box Cox model will
have the same virtuous nesting characteristics. In the following the Box Cox model is
defined in terms of the power transformation of the response variable only.
In its deterministic form, the model is:

\[ y^{(\lambda)} = \frac{y^\lambda - 1}{\lambda} = x'\beta \quad \{y > 0\} ; \]  

(5.2.1)

where \( x \) is a vector of observations on the set of explanatory variables and \( \beta \) is a vector of parameters. \( y \) is an observation of the response variable.

It is easy to demonstrate that linear and log-linear models are nested in (5.2.1):

\[ \frac{y^\lambda - 1}{\lambda} = x'\beta \quad \left\{\begin{array}{l}
\lambda = 1 \Rightarrow y = x'\beta + 1 \\
\lim_{\lambda \to 0} \ln y = x'\beta
\end{array}\right. ; \]

where the log-linear case is derived by using L'Hôpital's rule directly on the left hand side of (5.2.1).

Alternatively, solving (5.2.1) for \( y \) results in (Wooldridge, 1992 and Berndt et al., 1993):

\[ y = (\lambda x'\beta + 1)^{\frac{1}{\lambda}} . \]  

(5.2.2)

Equation (5.2.2) also contains the special cases:
\[
    y = (\lambda x' \beta + 1)^\lambda \\
    \begin{cases}
    \lambda = 1 \Rightarrow y = x' \beta + 1 \\
    \lim_{\lambda \to 0} \Rightarrow y = \exp(x' \beta) \Rightarrow \ln y = x' \beta
    \end{cases}
\]

Limiting \( \lambda \) to 0 in (5.2.2) is less tractable than (5.2.1). L'Hôpital's rule cannot be applied directly. Taking logs of (5.2.2) results in:

\[
    \ln y = \frac{\ln(\lambda x' \beta + 1)}{\lambda}.
\]  

(5.2.3)

Setting \( \lambda = 0 \) in (5.2.3) results in 0/0 division and therefore is tractable under L'Hôpital's rule. (5.2.3) is interesting in itself because it shows the same special case characteristics as those models it is derived from and therefore could be used as an alternative specification to (5.2.1) and (5.2.2):

\[
    \ln y = \frac{\ln(\lambda x' \beta + 1)}{\lambda} \\
    \begin{cases}
    \lambda = 1 \Rightarrow \ln y = \ln(x' \beta + 1) \Rightarrow y = x' \beta + 1 \\
    \lim_{\lambda \to 0} \Rightarrow \ln y = x' \beta
    \end{cases}
\]

This is all very elementary. (5.2.1), (5.2.2) and (5.2.3) are simply different ways of saying the same thing. The real issue arises when one moves from the deterministic to the stochastic model. The choice of deterministic model for use as a basis for the stochastic model has serious implications for the nature of the disturbances. In the following discussion on the stochastic versions of (5.2.1), (5.2.2) and (5.2.3):
\( \varepsilon \) is a vector of stochastic disturbances, with specification, \( \varepsilon \sim N(0, \sigma^2) \);

\( \mathbf{X} \) is the \( \text{NxK} \) design matrix of conditioning variables;

\( \mathbf{y} \) is an \( \text{Nx1} \) column vector of observations on the response variable;

\( \boldsymbol{\beta} \) is a \( \text{Kx1} \) column vector of parameters;

\( \lambda \) is a scalar parameter; and

\( \mathbf{1} \) is an \( \text{Nx1} \) unit vector.

The Stochastic Model

Taking (5.2.1) and adding normally distributed disturbances results in:

\[
\mathbf{y}^{(\lambda)} = \mathbf{X}\boldsymbol{\beta} + \varepsilon \quad \begin{cases} 
\lambda = 1 \Rightarrow \mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{1} + \varepsilon \\
\lambda \neq 0; \lambda \neq 1 \Rightarrow \mathbf{y} = (\lambda \mathbf{X}\boldsymbol{\beta} + \mathbf{1} + \lambda \varepsilon)^{\lambda} \\
\lim_{\lambda \to 0} \ln \mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \varepsilon \Rightarrow \mathbf{y} = \exp[\mathbf{X}\boldsymbol{\beta} + \varepsilon]
\end{cases};
\tag{5.2.4}
\]

where \( \mathbf{y}^{(\lambda)} \) is an \( \text{Nx1} \) column vector with typical element \( \frac{y_i^{\lambda} - 1}{\lambda} \).

It is easy to identify the special case of \( \lambda = 1 \) and \( \lambda = 0 \) resulting in additive and multiplicative disturbances respectively. This is acceptable if the transformation is being used to discriminate between these two different specifications. However, Berndt et al. convincingly argue that it is important to recognise the complications when Box Cox is used to generalise functional form.

Firstly, there is a problem in assuming normally distributed errors in the Box Cox stochastic model. Because \( \mathbf{y} \) is restricted to strictly positive values the normal
assumption strictly cannot hold. The distribution is at best truncated normal. The
limits of truncation are determined by the sign of \( \lambda \) in the following way (Berndt et al.
1993, p. 70 and Numrich 1990, p. 256):

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>lower limit</th>
<th>upper limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>&gt;0</td>
<td>(-\frac{1}{\lambda} - x_i^j\beta)</td>
<td>(\infty)</td>
</tr>
<tr>
<td>&lt;0</td>
<td>(-\infty)</td>
<td>(-\frac{1}{\lambda} - x_i^j\beta)</td>
</tr>
</tbody>
</table>

Poirier (1978, p. 284) and Poirier and Melino (1978, p. 1207) identify the density for
transformed \( y \) with a truncated normal disturbance. Numrich (1990, p. 257) and
Poirier (1978, p 285) identify the likelihood and the log of the likelihood respectively
to be maximised for correct parameter estimation. In terms of one sided truncation
either from above or below (determined by the sign of \( \lambda \)), the estimation problem is

\[
\max L(\lambda, \beta, \sigma^2) = -N \frac{N}{2} \ln 2\pi - N \frac{N}{2} \ln \sigma^2 \\
+ \sum_{i=1}^{N} \left[ (\lambda - 1) \ln y_i - \frac{(y_i^{(x)} - x_i^j\beta)^2}{2\sigma^2} - \ln \Phi\left(\frac{1/\lambda + x_i^j\beta}{\alpha\sigma}\right) \right];
\]

(5.2.5)

where:

\(\Phi(\cdot)\) is the standard normal distribution function; and

\(\alpha=1\) where \(\alpha>0\) and \(\alpha=-1\) where \(\lambda<0\).
Alternatively, if the degree of truncation is small then it can be argued that normality in approximation is sufficient. This would be the case if:

$$\frac{\sum_{i=1}^{N} x_i \beta}{N};$$

is positive and large relative to its standard error (Gourieroux and Monfort, 1995, p. 300). This seems to be the standard assumption made when Box Cox has been utilised in applied studies. Using maximum likelihood estimation accounting for one sided truncation, Numrich (1990) re-estimates the Kau and Lee (1976 and 1986) urban density gradients and finds that in the majority of cases the estimated transformation parameter is little affected by the *a priori* restriction of normality. A full listing of the transformation parameters estimated under the different specifications can be found in Numrich *op cit.* (p. 258), where there is some slight evidence to suggest that the larger the estimated transformation parameter the greater is the difference between the estimates with and without truncation.

A further well documented complication associated with the use of the Box Cox model with a transformed response variable is that of heteroskedastic disturbances. The stochastic model (5.2.4) assumes identically distributed disturbances. Zarembka (1974, p. 92) demonstrates that the value of $\lambda$ is sensitive to heteroskedasticity since its value functions as a stabiliser of error variance. Under conditions of non-constant error variance, interpreting $\lambda$ solely as some measure of functional form is dangerous. Unfortunately, Box Cox estimation which does not take separate account of heteroskedasticity means that $\lambda$ will reflect error variance stabilisation as well as
functional form. This can result in some ambivalence in approach to the Box Cox model. Contrast Judge et al. (1988, p. 556), which emphasises functional form as a role for the transformation and Judge et al. (1985, p. 840-841), where the error normalising qualities of the transformation are emphasised. Here it is argued that if functional form is an objective of Box Cox estimation, then estimation must account for heteroskedasticity. Again, Spanos (1986, p. 455-456) recommends the Box Cox transformation for its "normalising" characteristics, including its variance stabilising qualities. In the face of heteroskedasticity we cannot sensibly expect to make inferences about functional form from the transformation parameter unless the heteroskedasticity is accounted for in parameter estimation.

Seeks and Layson (1983, p. 162) identify a log-likelihood function which is weighted to account for heteroskedastic errors. They estimate model parameters by using weighted iterated ordinary least squares where $\lambda$ is searched over an interval of -2 to +2. Lahiri and Egy (1981, p. 301) specify a log-likelihood used for parameter estimation which is weighted for heteroskedastic errors:

$$L(\lambda, \beta, \sigma^2, \delta) = -\frac{N}{2} \ln 2\pi - \frac{N}{2} \ln \sigma^2 - \frac{\delta}{2} \sum_{i=1}^{N} \ln z_i - \frac{1}{\sigma^2} (y^{(h)} - X \beta)' \Omega^{-1} (y^{(h)} - X \beta) + (\lambda - 1) \sum_{i=1}^{N} \ln y_i;$$

(5.2.6)

where the omega matrix is a diagonal matrix with elements consisting of observation weights $z_i^6$. Both $\lambda$ and $\delta$ were estimated by grid search on a concentrated log-likelihood function and these estimates were substituted into the derivatives of the likelihood function with respect to $\beta$ and $\sigma^2$, which were then solved for these
parameter estimates. The term δ is a measure of heteroskedasticity, where δ=0 gives the homoskedastic case. This criterion is again used in Lahiri and Numrich (1983). This study is particularly useful in that it re-estimates the Kau and Lee (1976) U.S. urban population density gradients and finds that accounting for heteroskedasticity makes a difference in the value of estimated parameters.

Estimation of the Box Cox model with the power transformation applied to the response variable is not trivial if error truncation or error variance stabilisation is to be undertaken. The Wooldridge alternative to the Box Cox transformation seems to overcome these complications. This is clarified by taking (5.2.2) and adding a normally distributed disturbance term.

\[
y = (\lambda \mathbf{X}\beta + 1)^{\kappa} + \varepsilon \quad \begin{cases} 
\lambda = 1 \Rightarrow y = \mathbf{X}\beta + 1 + \varepsilon \\
\lambda \neq 0; \lambda = 1 \Rightarrow y = (\lambda \mathbf{X}\beta + 1)^{\kappa} + \varepsilon \\
\lim_{\lambda \to 0} \ln y = \mathbf{X}\beta + \varepsilon \Rightarrow y = \exp[\mathbf{X}\beta + \varepsilon]
\end{cases}
\]  

(5.2.7)

Because \( y \) is not restricted to strictly positive values to undertake a power transformation then the error distribution will not be truncated by values taken on by \( \lambda \). Further, when \( \lambda=1 \) and \( \lambda=0 \) the Wooldridge model in stochastic form mimics its Box Cox antecedent. The clear difference is when \( \lambda \) takes on a value other than zero or one. In the Wooldridge model (5.2.7), because \( \lambda \) is separated out from the disturbance, provided heteroskedasticity is accounted for in estimation, the value of \( \hat{\lambda} \) will not be influenced by an attempt to choose a function oriented toward stabilising non-constant variance in the errors. Model (5.2.7) is clearly different from (5.2.4) in the relationship between the estimated value of \( \lambda \) and heteroskedasticity. In (5.2.4),
even if heteroskedasticity is accounted for in estimation, $\hat{\lambda}$ will be determined in part by choice of a functional form which contributes to the elimination of heteroskedasticity.

An issue with the model (5.2.7) is that even though its parameters are estimable using non-linear least squares (NLS), that estimator, as it is for all models, will be inefficient if the disturbances are heteroskedastic. This is explicitly recognised by Wooldridge (1992) and Berndt et al. (1993). There are several well known alternatives for addressing this problem.

One option is to utilise NLS but to estimate standard errors using White’s (1980) robust estimation of the asymptotic covariance matrix. See, for example, Davidson and MacKinnon (1993, p. 552-556) for estimation with non-linear models. An efficient estimator would employ weighted non-linear least squares (WNLS). Wooldridge (1992, p. 940), in using WNLS, suggests several alternatives as weights and Berndt et al. (1993, p. 76) use the squared fitted values from an initial unweighted NLS estimation. Wooldridge points out that the choice of weights increases efficiency even if the weighting function is incorrectly specified and will not harm consistent parameter estimation. Both Wooldridge and Berndt et al. favour the WNLS alternative. Finally, it would be possible to use the log transform to stabilise error variance. This points to the stochastic version of (5.2.3) and this could be used as an adjunct to WNLS estimation when heteroskedasticity is suspected.
Starting with (5.2.3) and adding a normally distributed disturbance results in:

$$\ln y = \frac{\ln(\lambda X\beta + 1)}{\lambda} + \varepsilon \begin{cases} 
\lambda = 1 & \Rightarrow \ln y = \ln(X\beta + 1) + \varepsilon \Rightarrow y = \exp\varepsilon(X\beta + 1) \\
\lambda \neq 0; \lambda 
eq 1 & \Rightarrow y = \exp\varepsilon(\lambda X\beta + 1)^\lambda \\
\lim_{\lambda \to 0} & \Rightarrow \ln y = X\beta + \varepsilon \Rightarrow y = \exp[X\beta + \varepsilon] 
\end{cases}$$

(5.2.8)

where $\exp \varepsilon$ is a diagonal matrix with typical diagonal element $e^{\varepsilon_j}$.

When $\lambda = 0$ the log transformation mimics its Wooldridge and Box Cox antecedents. There are clear differences when $\lambda \neq 0$. The disturbances are multiplicative and $\lambda$ should be estimated with values (in principle) unaffected by heteroskedasticity, conditional on the maintained hypothesis.

There is a problem with using the log transformation. Wooldridge (1992, p. 936) argues that in econometrics the focus of interest should be the untransformed response variable. This is a very important point. If theory couches hypotheses in terms of $y$, econometric tests should, wherever possible, directly address $y$ as the focus of the empirical model. Whilst the expression for the scale invariant point elasticity measures are the same for each model, interpreting the relationship between the model parameters and the $y$ variable is difficult if it has been subject to a power (5.2.1) or a logarithmic (5.2.3) transformation. The argument that estimated parameters should have direct meaning in functional form is very attractive and strongly points us in the direction of (5.2.7). Nevertheless, despite the fact that naive distributional assumptions have to be made when moving from (5.2.4) and (5.2.8) to $y$ through inverse transformation, (5.2.4) and (5.2.8) could be useful alternatives to (5.2.7). For
example if (5.2.7) and (5.2.8) were estimated using NLS and testing indicated (5.2.8) was “preferred” in approximating the process which generated the data, then this would indicate that estimation of (5.2.7) would have to account for some structure similar to multiplicative disturbances. This might suggest that one re-estimates (5.2.7) using appropriate WNLS. In this way the three specifications can be seen as competing alternatives in approximating the data generating process. Following this methodology, it would be appropriate to employ tests to discriminate between the three estimated alternatives. Berndt et al. (1993, p. 78-80) discriminate between the Box Cox transformation and alternative estimates of the Wooldridge type model by using goodness of fit measures. Another way would be to use specification tests based on non-nested procedures. A useful method here is to set up an artificial nesting model (Davidson and Mackinnon 1993, p. 381-384) containing the competing models and test down from this artificial general model to specific models. This is advantageous in the sense that it is in keeping with the framework and rationale of general to specific modelling. Once testing down is complete “selected” models can be tested for linear or log-linear forms by testing restrictions on \( \lambda \). This describes a two step set of tests:

**Step 1** Non-nested tests of the three general specifications; and

**Step 2** Test restrictions within one, two or all of these three specifications, depending upon the outcome of step 1.

These two stage tests follow the methodology of general to specific modelling in the sense that the tests allow for a testing down procedure based on model rejection. This
two step procedure is undertaken when testing for functional form in the urban density gradient model.

The density hypothesis will be tested for the capital cities of the six Australian states, using 1986 and 1981 census data for population and household density gradients, that is 24 different data sets in all. Further details on the data and the 24 data sets can be found in section 1 of chapter six.

As a generalisation we start with the hypothesis

\[ h(y, \alpha) = f(x, \theta), \quad \frac{dy}{dx} < 0. \quad (5.2.9) \]

The three alternative specifications discussed in this section fit into this hypothesis. In this and subsequent sections these three models will be labelled, model 1 Box Cox, model 2 Wooldridge and model 3 logarithmic transform of Wooldridge. The left hand side of (5.2.9) refers to the transformation of the response variable \( y \), which is density measured in persons or households per hectare. On the right hand side, \( x \) is the conditioning variable and is distance in kilometres from urban centre. The unknown parameters are given by \( \alpha \) and \( \theta \).

Prior to model estimation and testing, it would be appropriate to clarify the artificial nesting model that will be used to test for the model specification. This is the objective of the following section.
5.3 The Artificial Nesting Model.

The non-nested tests follow the procedure established by Davidson and Mackinnon (1981) as an alternative to the Cox (1961 and 1962) procedures developed by Pesaran (1974) for linear single equation models and Pesaran and Deaton (1978) for non-linear single equation models. These tests are based on the centred likelihood ratio, Pesaran, \textit{op cit.} (p. 157) and Pesaran and Deaton, \textit{op cit.} (p. 684). Both papers establish the asymptotic properties of the respective linear and non-linear tests and Deaton and Pesaran generalise the test to competing systems of equations. The centred likelihood ratio test was further extended to test logarithmic versus linear alternatives in Aneuryn-Evans and Deaton (1980, p. 278). The Davidson and Mackinnon alternative to these tests is contained within the comprehensive approach of Atkinson (1970). It is computationally less demanding than the centred likelihood ratio type test but Davidson and Mackinnon show that both their J and P tests are asymptotically equal to the negative of the Pesaran and Deaton test statistic (Davidson and Mackinnon, 1981, p. 789-791). They also demonstrate that similar inferences can be derived from their J and P tests compared to the Pesaran and Deaton tests. In their 1981 paper this is done by undertaking pairwise tests on the same five consumption models tested by Pesaran and Deaton. The P test was extended in Mackinnon \textit{et al.} (1983) to test competing models where the response variable has been subject to a monotonic transformation. Whilst the asymptotic properties of these alternative test formats have been established, finite sample characteristics are not so well established. Monte Carlo methods have shown that in small samples these test statistics have a propensity for Type II errors. Godfrey and Pesaran (1983) propose small sample adjustments to
the Cox type test statistic and demonstrate that the adjusted test statistic performs better than the J and F test alternatives in simulation under small samples. Unsure of the finite sample properties, Davidson and Mackinnon in their comparison with Pesaran and Deaton using the consumption model adopt conservative critical regions to avoid Type II errors. A rigorous comparison of the centred likelihood ratio tests, comprehensive tests and the traditional F test can be found in Fisher (1983). Good reviews of non-nested tests are Gourieroux and Monfort (1995b, Chap. 22) and McAleer (1987), which also refers to the Box Cox model as a test to discriminate between the naturally non-nested linear and log-linear specifications. Davidson and Mackinnon (1993, p. 381-388) is a comprehensive account with an emphasis on J and P tests and Pesaran (1990) is a terse but rigorous review of the centred likelihood ratio test.

Davidson and Mackinnon *op cit.* derive several tests of alternative explanations (models) of the phenomenon $y$. Each test is based on a different estimable version of a single compound artificial model.

Using the notation of Davidson and Mackinnon, say there are two alternate models expressed in terms of observation $i$ of the phenomenon $y$:

\begin{align}
  y_i &= f_i(x_i, \beta) + \varepsilon_i \; ; \text{ and} \\
  y_i &= g_i(z_i, \gamma) + \varepsilon_i
\end{align}

(5.3.1) \hspace{1cm} (5.3.2)

where $x$ and $z$ are vectors of explanators and $\beta$ and $\gamma$ are parameter vectors. These models are non-nested if it is not possible to place restrictions on the parameters of
one model so that it equates to the alternative model for arbitrary values of parameters in that alternative model (Davidson and Mackinnon, 1993, p. 381). This is the case for models 1, 2 and 3 as it is only possible to equate these models by placing parameter restrictions on comparable models at the same time.

Defining the null, \( H_0 \), as equation (5.3.1) and \( H_1 \), the alternate model as equation (5.3.2), it is possible to set up the artificial compound model:

\[
H_c: \quad y_i = (1 - \alpha) f_i(x_i, \beta) + \alpha g_i(z_i, \gamma) + \epsilon_i .
\]  
(5.3.3)

In (5.3.3) \( \alpha = 0 \) implies that the null model is the “true” model in the sense that it is a reasonable representation of the data generating process. Alternatively \( \alpha \neq 0 \) implies that \( H_0 \) is not the “true” model. Because all models are different hypotheses around the process which forms \( y_i \), it is possible for the alternate \( H_1 \) to be the “true” model. Rejecting the null in (5.3.3) has implications for the null and the compound model and not the alternate model. Thus the specification test of the artificial nesting model should be reversed forming a new artificial compound model with a pairwise switching of null and alternate. Because it is a specification test, outcomes of pairwise tests can be asymmetric so that in the case of two competing models (5.3.1) and (5.3.2) it is possible to: reject 1; reject 2; reject neither; or finally reject both.

Three tests, known as the J, C and P tests, have been developed from estimable forms of the compound equation (5.3.3). In all three of the resulting artificial nesting models the test is based on the \( t \) ratio of the parameter \( \alpha \) with a null of zero.
The J test is based on an artificial nesting model which involves the joint estimation of $\alpha$ and $\beta$ using NLS:

$$y_i = (1-\alpha)f_i(x_i, \beta) + \alpha \hat{g} + \varepsilon_i \; ;$$  \hspace{1cm} (5.3.4)

where:

$$\hat{g} = g_i(z_i, \hat{\theta}) \; ;$$

is a vector of predicted values generated by estimation of (5.3.2)

The C test involves replacement of the null model with prior estimates. This artificial nesting equation can then be estimated by restricted NLS:

$$y_i = (1-\alpha)\hat{f}_i + \alpha \hat{g} + \varepsilon_i \; ;$$  \hspace{1cm} (5.3.5)

where:

$$\hat{f}_i = f_i(x_i, \hat{\beta}).$$

Whilst this test may be the easiest to perform in terms of computation, the obvious drawback is that the t ratio associated with $\alpha$ is conditioned on $\hat{\beta}$. Because of this the t ratio associated with $\alpha$ has a variance less than unity under the null hypothesis (Davidson and Mackinnon 1981, p. 783) and will be lower than the unconditional ratios. In this sense it can still be utilised as a conservative test in rejection of the null model.
The P test is an alternative to the J test without the conditional drawbacks of the C test. The general artificial compound model $H_c$ is linearised by taking a Taylor series approximation around $\beta = \hat{\beta}$ of the right hand side of $H_c$. For linearised regression of non-linear regression models using first order Taylor approximation see Amemiya (1985, p. 139-140), Greene (1997, p. 452-453).

\[ y_i - \hat{f}_i = \alpha (\hat{g}_i - \hat{f}_i) + \nu^\prime \beta + \epsilon_i ; \]  \hspace{1cm} (5.3.6)

where:

\[ \nu^\prime = \frac{\partial \hat{f}_i}{\partial \beta} . \]

The advantage of this test is that equation (5.3.6) can be estimated using OLS and therefore is computationally less demanding than the J test. This is particularly useful when one or both of the competing models is non-linear, especially if those models are poorly identified locally in certain parameters, resulting in difficulties converging to a well identified global minimum. Problems of convergence procedures stalling at poorly identified local minima would be compounded in the artificial nesting equation (5.3.4). However, method (5.3.6) requires correct expressions for the first derivatives of the null model evaluated at prior parameter estimates.

Whilst these specification tests point to pairwise testing of alternate competing models, it is possible to expand the artificial compound model for M+1 competing non-nested hypotheses such as the joint P test:
\[ y_i - \hat{f}_i = \sum_{j=1}^{M} \alpha_j (\hat{g}_{ji} - \hat{f}_i) + \nu_i^p b + \varepsilon_i. \] (5.3.7)

A limitation of the artificial nesting model thus far is that it is restricted to competing alternate models where the dependent variable of each model is untransformed. Mackinnon, et al. (1983) extended the P test (Pe test) to include non-nested models where one of the competing hypotheses involves a monotonic transformation of the response variable. Thus if model (5.3.1) is the null hypothesis then this can be tested against an alternative model (5.3.8) involving a monotone transformation \( h \) of the response variable:

\[ h_i(y_i) = g_i(z_i, \gamma) + \varepsilon_i. \] (5.3.8)

Combine (5.3.1) and (5.3.8) to form the artificial compound model:

\[ (1 - \alpha) (y_i - f_i(x_i, \beta)) + \alpha (h_i(y_i) - g_i(z_i, \gamma)) = \varepsilon_i. \] (5.3.9)

Manipulating (5.3.9) to some estimable form is not as straightforward as the P test. Linearisation of (5.3.9) results in \( y \) appearing on the right hand side of the estimating equation. This problem is resolved by the expedient of replacing right hand side \( y \) by estimates \( \hat{f} \). This gives the artificial nesting equation:

\[ y_i - \hat{f}_i = \alpha (\hat{g}_i - h_i(\hat{f}_i)) + \nu_i^p b + \varepsilon_i; \] (5.3.10)
which can be estimated by OLS.

Clearly this Pe test equation can be extended to test a null model such as (5.3.1) against M alternative transformation type models:

$$ y_i - \hat{f}_i = \sum_{j=1}^{M} \alpha_j \left( \hat{g}_{j_1} - h_{j_1}(\hat{f}_i) \right) + \nu \hat{b} + \varepsilon_i . $$

(3.11)

Mackinnon et al. (1983) are not forthcoming on the expression for an artificial compound model where the model containing the $y$ transform is the null. Yet given the nature of the specification test it is important that the reverse pairwise test be undertaken.

There appear to be two alternatives open.

Firstly, one can utilise the C test nested equation:

$$ h(y_i) = (1 - \alpha) \hat{g}_i + \alpha h(\hat{f}_i) + \varepsilon_i . $$

(5.3.12)

Given the conditional nature of the estimated test parameter $\alpha$ this test would have to be used conservatively and could only be used with some safety in terms of rejecting the null.

The preferred alternative is to attempt to reverse the derivation of the Pe test to give an artificial nesting model with the transformed model as the null. This is a straightforward procedure. Form the artificial compound model:
(1 - \alpha)(h_i(y_i) - g_i(z_i, \gamma)) + \alpha(y_i - f_i(x_i, \beta)) = \epsilon_i ; \hspace{1cm} (5.3.13)

where the null model is the model containing the transformed response. Then solve (5.3.13) for the transformed response and replace the function \( f \) with its estimate \( \hat{f} \). Linearise using a first order Taylor expansion around \( \alpha = 0 \) and \( \gamma = \hat{\gamma} \). This is still not estimable because on the right hand side \( y \) appears directly and indirectly in the transformation \( h \). Where \( y \) occurs in the transformation \( h \) on the right replace with estimated function \( \hat{g} \). Where \( y \) occurs directly replace with estimates of \( \hat{g} \) and undertake the inverse transformation to get untransformed estimates of \( y \). This is the function \( h^{-1} \) in (5.3.14). The inverse function \( h^{-1} \) exists because the test is restricted to the class of monotone transformations:

\[
h_i(y_i) - \hat{g}_i = \alpha \left( \hat{f}_i - h^{-1}(\hat{g}_i) \right) + \nu_i^* c + \epsilon_i . \hspace{1cm} (5.3.14)
\]

The result (5.3.14) is an artificial nesting model where the estimated transformed residuals are the subject. The parameter \( \alpha \) can now be estimated along with the vector of parameters \( c \) associated with the vector of first derivatives of the right hand side of the null model. Clearly \( \alpha \) can now be used to test the transformed model in exactly the same way as it could be used to test untransformed null models.

Finally the Pe test can be modified to be used with competing models where these models both contain monotonic transformations of the response variable. Say there are two competing models:
\[ h_{it}(y_i) = g_{it}(x_{it}, \beta) + \varepsilon_{it}; \quad \text{and} \]
\[ h_{zt}(y_i) = g_{zt}(z_{it}, \gamma) + \varepsilon_{zt}. \]  

Then defining (5.3.15) as the null model it is possible to create an artificial nesting model estimable by OLS by linearising a non-linear compound model of (5.3.15) and (5.3.16). Unfortunately right hand side variables will contain \( y \) in both transforms. The null transform on the right hand side can be replaced with its estimate \( \hat{g}_t \). The alternate transform on the right hand side can be replaced with the untransformed estimate \( \hat{g}_t \), which is then transformed by \( h_z \):

\[ h_{zt}(y_i) - \hat{g}_t = \alpha \left( \hat{g}_{zt} - h_{zt}(h_{zt}(\hat{g}_t)) \right) + \nu_{ib} + \varepsilon_{zt}. \]  

\[ (5.3.17) \]

5.4 Conclusion.

Artificial nesting regressions (5.3.10), (5.3.14) and (5.3.17) form suitable venues for specification tests for models 1, 2 and 3. This procedure could help discriminate between these alternative specifications, although outcomes will not necessarily be symmetrical. Once these specification tests have been undertaken it would then be possible to move on to the second stage of specification tests within those models that are not rejected in the artificial nesting procedure. These tests would be based around the parameter \( \lambda \) and would test the restriction \( \lambda = 0 \) and \( \lambda = 1 \) where these restrictions test for the log-linear and linear in levels forms associated with all three
specifications. In this way it may be possible to test down from three alternate competing models through a general, albeit artificial, model to possibly one specification and then further test down within that specification to a more specific description of functional form. The data is allowed to chose that functional form which is appropriate rather than start with the a priori restriction of, say, λ=0.

However, stage one is really a series of specification tests and is not a model selection test. Discrimination between models is by rejection. Clearly none of the three models are “true” in that they replicate the data generating process. All are approximations to the data generating process. The specification tests can only discriminate down to that one which is the best out of the three as an approximation to the data generating process. Using pairwise comparisons through using OLS on (5.3.10) for model 2 against models 1 or 3; (5.3.14) for models 1 or 3 against model 2; and (5.3.17) for model 1 against 3 and then 3 against 1, is sufficient to discriminate between models. Running the tests in this way is sufficient to discriminate the best approximation or approximations to the data generating process.

In the two stage test process it will be necessary, unfortunately, to estimate parameters for models 1, 2 and 3 prior to testing within the artificial nesting regressions. Given the non-linear nature of these three alternatives, and the fact that all three models are not locally well identified as the intercept approaches zero, the J test alternatives to (5.3.10), (5.3.14) and (5.3.17) cannot be utilised. However, at this first round of tests it must be stressed that the parameter estimates themselves are not of direct interest but are simply a means for setting up the artificial nesting regressions for the Pe specification tests. It is only when the second stage of the two step test procedure is
undertaken that individual parameter estimates will be of interest. Under these
conditions, therefore, it is important that the first stage tests are made transparent and
that the test procedure is not distorted in any way to favour outcomes according to
existing, possibly favourable, parameter estimates. This *ex ante* knowledge of the
estimated models could be a weakness in the test procedure, especially as the finite
sample properties of the test statistics associated with the Pe test are not well known
apart from a bias toward the null model as sample size is decreased.

This chapter has dealt with a particular method for discriminating between alternate
models which themselves are flexible enough to encompass various functional forms
that density gradients may take. Issues in model estimation have not been dealt with.
This subject forms the first part of chapter six where the various different model
estimation methods are outlined. Issues of estimation and their implications for the
parametric tests used in model discrimination are also dealt with here. The last part of
chapter six is given over to a discussion of the outcomes of model estimation.
Chapter 6. The Data, Model Estimation and Results.

6.1. The Data

The empirical problem is to use density data for the capital cities of the six Australian states to test the lot size hypothesis by estimating population and household density gradients. The methodology is to estimate these gradients using three competing flexible functional form models, which allow the data to determine functional form. Testing between these alternate models is to be undertaken through specification tests based on the artificial regression model of Davidson and Mackinnon (1981 and 1993). The log-linear and linear functional forms are nested within each competing model and testing down within models for these specific functional forms will be undertaken through testing parameter restrictions via likelihood ratio tests.

The data set consists of population and household density data from the 1981 and 1986 Census of Population, Australian Bureau of Statistics (ABS) for the six capital cities. That is, there are two density variables, each partitioned into six subsets, one subset for each city, with each city subset further partitioned into the years 1981 and 1986 subsets. Thus the density data comprises 24 subsets in all. Each city is defined as the Statistical Division (SD) associated with the capital city. For a definition of SD see ABS (1996).

Associated with each density observation measured in persons or households per hectare is a distance observation, which is direct distance in kilometres from the
centre of the Central Business District (CBD) for each city. None of these data appear
directly in the Census of Population, but they have been estimated from census data.

*Density observations:*

The ABS publishes for each Collection District (CD), which is the basic geographical
unit for data collection, population, household, land area in m² and a geographical co-
ordinate in decimal degrees longitude and latitude to six decimal degrees, defining the
location of the CD. Population, household and land area in 1981 and 1986 was
available aggregated to the Census Sub Division (CSD) level. This nomenclature is
now out of date and CSD’s conform to the 1996 definition of a Field Group Area
(FGA). For full definition of CD’s and FGA’s see ABS (1996). In 1981 and 1986
CSD’s and in 1996 their equivalent, the FGA’s, comprise approximately 10-12
geographically contiguous CD’s. At the CSD level persons and households per
hectare were calculated using the CSD population, household and land area data. All
census data aggregated to the CSD level was kindly provided by Census Applications
Pty Ltd of Concorde, NSW. Census data for 1981 was provided on the 1986 CSD
basis.

*Distance observations:*

Each distance observation is direct distance in kilometres between the census co-
ordinate for the CSD and the centre of the CBD for the relevant city. This distance
measure was calculated in the following way.

1. A geographical co-ordinate in decimal degrees longitude and latitude was
calculated by averaging the co-ordinates of the CD’s comprising the CSD.
2. The centre of the CBD geographic co-ordinates for each city were kindly provided by the publishers of the UBD street maps. Alperovich (1982) shows that outcomes of density studies are sensitive to definition of urban centre. In this case the centre was determined by a third party and not estimated within the research.

3. All of these geographical co-ordinates were converted into metric co-ordinates on the Australian Metric Grid. This conversion was undertaken by the State Land Information Centre, Bathurst, N.S.W. Metric outcomes were to a level of significance equal to the geographic co-ordinates.

4. Finally metric distances between each CSD co-ordinate and the CBD co-ordinate were calculated using elementary Pythagoras.

Thus for each CSD in the data set there is a distance observation and four density observations, comprising population density 1986 and 1981 and household density 1986 and 1981.

*Summary statistics:*

Summary statistics for the 24 subsets are provided in Table 6.1.1. In this table columns one two and three refer to the city, the nature of the summary statistics and the size of data set associated with each city. Column four gives summary statistics for the variable distance in kilometres from city centre. Columns five and six provide the same summary statistics for population density, persons per hectare, for 1986 (Y1) and 1981 (Y2) respectively. Finally summary statistics for household density, households per hectare, for 1986 (Y3) and 1981 (Y4) can be found in columns seven and eight of the table.
centre of the Central Business District (CBD) for each city. None of these data appear
directly in the Census of Population, but they have been estimated from census data.

Density observations:
The ABS publishes for each Collection District (CD), which is the basic geographical
unit for data collection, population, household, land area in m² and a geographical co-
ordinate in decimal degrees longitude and latitude to six decimal degrees, defining the
location of the CD. Population, household and land area in 1981 and 1986 was
available aggregated to the Census Sub Division (CSD) level. This nomenclature is
now out of date and CSD’s conform to the 1996 definition of a Field Group Area
(FGA). For full definition of CD’s and FGA’s see ABS (1996). In 1981 and 1986
CSD’s and in 1996 their equivalent, the FGA’s, comprise approximately 10-12
demographically contiguous CD’s. At the CSD level persons and households per
hectare were calculated using the CSD population, household and land area data. All
census data aggregated to the CSD level was kindly provided by Census Applications
Pty Ltd of Concorde, NSW. Census data for 1981 was provided on the 1986 CSD
basis.

Distance observations:
Each distance observation is direct distance in kilometres between the census co-
ordinate for the CSD and the centre of the CBD for the relevant city. This distance
measure was calculated in the following way.

1. A geographical co-ordinate in decimal degrees longitude and latitude was
calculated by averaging the co-ordinates of the CD’s comprising the CSD.
2. The centre of the CBD geographic co-ordinates for each city were kindly provided by the publishers of the UBD street maps. Alperovich (1982) shows that outcomes of density studies are sensitive to definition of urban centre. In this case the centre was determined by a third party and not estimated within the research.

3. All of these geographical co-ordinates were converted into metric co-ordinates on the Australian Metric Grid. This conversion was undertaken by the State Land Information Centre, Bathurst, N.S.W. Metric outcomes were to a level of significance equal to the geographic co-ordinates.

4. Finally metric distances between each CSD co-ordinate and the CBD co-ordinate were calculated using elementary Pythagoras.

Thus for each CSD in the data set there is a distance observation and four density observations, comprising population density 1986 and 1981 and household density 1986 and 1981.

**Summary statistics:**

Summary statistics for the 24 subsets are provided in Table 6.1.1 In this table columns one two and three refer to the city, the nature of the summary statistics and the size of data set associated with each city. Column four gives summary statistics for the variable distance in kilometres from city centre. Columns five and six provide the same summary statistics for population density, persons per hectare, for 1986 (Y1) and 1981 (Y2) respectively. Finally summary statistics for household density, households per hectare, for 1986 (Y3) and 1981 (Y4) can be found in columns seven and eight of the table.
An examination of these summary statistics reveals that the data are similar for the years 1981 and 1986. This is not surprising because significant change in density patterns through say suburbanisation requires a significant time lapse. This opens up possibilities of analysing the data through the pooling of subsets. Pooling of subsets over years is feasible. Further, it could be argued that, as the exercise is to test the density hypothesis as an urban generalisation, the pooling of data across cities would be useful. There is a problem. The summary table indicates considerable variation of

An examination of these summary statistics reveals that the data are similar for the years 1981 and 1986. This is not surprising because significant change in density patterns through say suburbanisation requires a significant time lapse. This opens up possibilities of analysing the data through the pooling of subsets. Pooling of subsets over years is feasible. Further, it could be argued that, as the exercise is to test the density hypothesis as an urban generalisation, the pooling of data across cities would be useful. There is a problem. The summary table indicates considerable variation of
the number of observations between cities. Thus if the population density subsets were pooled the result would be 3208 observations for each density measure, of which 1048 would be contributed by Sydney. It could be argued then that outcomes from simply pooling the data in this way would be dominated by Sydney. Introduction of dummy variables to allow for intercept and slope parameter differences between cities proved difficult. The density models in the Wooldridge and logarithmic alternatives to Box Cox are not locally identified as the intercept term approaches zero. This led to problems of convergence in non-linear estimation, particularly when the number of parameters to be estimated was increased through the use of dummy variables. As a consequence models extended by dummy variables failed to converge and were abandoned. Under these conditions it was decided to undertake analysis for each city for each year. For comparison, analysis was also undertaken at the highest level of pooling (cities and years) but without extending these pooled models by dummy variables.

Further, examination of the summary statistics reveal that the maximum density value for Sydney is considerably higher than that for other cities. In particular three CSD’s in Sydney consistently scored high values for both years and for both density measures. Moreover, these three CSD’s took up very central locations. It could be argued that outcomes, especially in terms of differences between the three models, might be influenced by those extreme values. A scatterplot of population density for the pooled data, Figure 6.1.1, illustrates these extreme values, which comprise the six population density observations contained within the dashed rectangle.
The Sydney analysis and the pooled analysis for population and household density was repeated deleting the three CSD’s associated with the observations indicated by the dashed rectangle in figure 6.1.1. This repeated analysis showed that outcomes for Sydney was not influenced by these observations in terms of choice between the flexible form models and for testing the parameter restrictions $\lambda=0$ and $\lambda=1$. Similarly, the pooled results for population density did not vary in terms of model choice or tests on parameter restrictions when the three CSD’s were removed from the data. This was not the case with household density. Deleting the three cases resulted in rejection of all three models compared to the analysis including the three observations, where testing down between the three competing models led to rejection of models 1 and 3.
6.2 Issues in Estimation.

Schlesselman (1971) shows that Box Cox models estimated with an intercept term result in scale invariance in estimation for the transformation parameter both in value and variance. Unfortunately, this characteristic does not carry over to the slope parameters in joint estimation. Spitzer (1984) demonstrates that, when the transformation parameter $\lambda$ and the slope parameters $\beta$ are jointly estimated, standard hypothesis tests are difficult because the $t$ ratios associated with the $\hat{\beta}$'s are not invariant to scaling of the dependent variable. This has important implications for potential data mining if the choice of scale of the dependent variable can be arbitrarily made by the researcher. This is particularly relevant to density studies where the response variable is a ratio measure which can be completely under the control of the researcher. Realistically, this also applies to a lot of economic data.

There are several ways around this problem. In the Box Cox model it is perfectly feasible to avoid joint estimation of the coefficients. Grid search methods allow estimation of the slope parameters by OLS conditional on a $\lambda$ value, the ultimate choice of $\lambda$ being determined by that which minimises the alternative summed squared residuals. This is the iterated ordinary least squares approach (IOLS). Alternatively, the slope parameters can be concentrated out of the log-likelihood function and the concentrated log-likelihood can be maximised by choosing $\lambda$ over a grid search. Once $\lambda$ has been established the other parameters can be estimated by OLS. The obvious problem with both estimation methods is that the variance of the $\hat{\beta}$'s are conditional on $\lambda$ and will be understated (see Fomby et al., 1984, p. 426-431.
and Spitzer, 1982, p. 311). If Student type tests are to be undertaken then t ratios have to be estimated using a correct estimate of the asymptotic covariance matrix. This can be done in two alternative ways. Firstly, the correct covariance matrix can be arrived at by taking the negative inverse of the Hessian of the log-likelihood function. Alternatively, the correct asymptotic covariance matrix can be estimated directly from first derivatives of the likelihood function with respect to the parameter estimates using the Berndt et al. (1974, p. 657 and p. 664) estimator. Spitzer (1984) favours use of the Hessian, despite the fact that it requires second derivatives of the likelihood function, on the grounds that it is more accurate. Greene (1997, p. 485n) disputes this and favours the Berndt et al. estimator because of the ease of computation. However the maximum likelihood estimate of the covariance matrix is arrived at, it will require extra computation beyond OLS. Additionally, it should be borne in mind that, these are estimates whose asymptotic properties can only be relied upon in suitably sized samples.

Both $\lambda$ and $\beta$ could be estimated jointly after transforming the Box Cox model in a way that results in scale invariant inference about the parameters of the resultant model. Spitzer (1984) favours dividing the dependent variable by the geometric mean. This has the added advantage of ease of computation (Spitzer, 1984, p. 649, after Zarembka, 1968). If the dependent variable is scaled by its geometric mean then the Jacobian term in the log of the likelihood function becomes zero. Whilst the result is scale invariant it is itself a scaling of $y$. Indeed, Spitzer (1984) uses this very scaling to demonstrate the unfortunate properties of non-neutrality to scale in standard Box Cox estimates. Berndt et al. (1993) point out that other scalings of $y$ can produce similar results in invariance, but use the geometric mean scale. If scaling is to be used
then standardisation is imperative in comparisons between models. Estimates of parameters associated with unscaled $y$ can be retrieved by dividing the scaled parameter estimates by the geometric mean of $y$. Spitzer (1984) goes further and scales the independent variables as well by their respective geometric means. The resultant parameter estimates from this model are point elasticity estimates of the unscaled model. Scaling, if transparent, has the advantage of utilising Student type tests with joint parameter estimation. However, it has the disadvantage of removing $y$ directly from the model. This is unfortunate because $y$ is the focus of interest and under standard Box Cox procedures $y$ is already hidden by a power transformation.

Joint estimation of $\lambda$ and $\beta$ can be undertaken without scaling if hypothesis tests use scale invariant statistics. Suitable established scale invariant test statistics would be likelihood based tests. This would leave the dependent variable subject only to a power transformation, which is important if the analysis focuses on $y$. Countering this is the problem of increased computation. With $K$ independent variables in the simple Box Cox model this will lead to a further $K+1$ model estimations. The possibility of failure of convergence in estimation of restricted models is always present. The estimation of confidence regions is well established using the Student distribution and this might mitigate against the use of estimation techniques which limit hypothesis testing to likelihood based statistics. Based on likelihood ratio (Box and Cox, 1964, p. 215-216) it is possible to construct approximate confidence regions around $\hat{\lambda}$ by choosing those $\bar{\lambda}$ which satisfy:

$$2\left[ L(\hat{\beta}, \hat{\lambda}) - L(\hat{\beta}, \bar{\lambda} | \Theta_{\lambda}) \right] \leq \chi^2_{\alpha, \omega}.$$
This gives \((1-\alpha)\times 100\)% confidence intervals for \(\hat{\lambda}\) based on the chi-squared distribution with \(v=1\) degree of freedom, where \(\Theta_\lambda\) is the \(\lambda\) parameter set. This device is well established in the literature for testing the linear and log-linear cases (see for example Kau and Lee, 1976, and Numrich, 1990). Cook and Weisberg (1982, p. 65-66) give the expression for estimating confidence regions around \(y^{(k)}\). The confidence interval around \(\hat{\lambda}\) is approximate because the \(\hat{\beta}\)'s are fixed in the test and are not free to take on other values as in standard likelihood ratio tests. The argument for these approximate intervals is that \(y^{(0)}\) is not defined for \(\lambda=0\). Standard likelihood ratio tests can be used by either:

- Estimating the log of the likelihood for the log-linear model with the appropriate constant and Jacobian; or
- Estimating the log-likelihood with the restriction that \(\lambda\) is some suitably small distance away from 0 from above and below. This will also produce an effective approximate likelihood estimate for likelihood ratio purposes.

The scale invariance issue is not restricted to the Box Cox transformation. Other nonlinear models exhibit the same characteristics (Dagenais and Dufour, 1991). Wooldridge (1992) and Berndt et al. (1993) point out that similar scale non-invariance applies to model 2. The same criticism applies to model 3. This leads to some problems. The dependent variable naturally lends itself to various different scalings because it is a ratio. This issue therefore has to be addressed prior to estimation. Wherever possible it would be useful to avoid re-scaling of the dependent variable. In
two of the three models $\gamma$ has already been subjected to transformation. With this in mind estimation of model 1 was undertaken by maximising the concentrated log-likelihood for $\lambda$ and then employing OLS estimation for the other parameters. Likelihood ratio statistics are reported for $\lambda$ and unconditional $t$ ratios are estimated using Berndt et al. (1974) estimates for the parameters. The conditional $t$ statistics are retained for comparison. Models 2 and 3 were estimated following Berndt et al. (1993) using NLS and it was convenient to undertake WNLS estimates for model 2 at the same time. Likelihood ratio statistics are reported for the slope parameter, as they are for $\lambda$. As indicated previously, failure to converge to an appropriate restricted maximum is a problem when unrestricted parameters are allowed to range. This was the case with the density model with the restrictions $\lambda=0$ and $\lambda=1$ for some data sets. Unfortunately, neither models 2 nor 3 are locally identified around a zero intercept and therefore it is not possible to report likelihood ratio statistics for this term. In this particular case the slope parameter and $\lambda$ are the important coefficients to test and it is argued that failure to undertake hypothesis tests around the intercept term is only a minor issue. Nevertheless, $t$ ratios associated with the slope parameter are reported but these must be used carefully because of lack of scale invariance.

Details of the criterion and weights used in estimation can be found in Appendix 1, which also reports detailed information on test statistics, point elasticity measures and expressions for derivatives used.

As suggested in section three of chapter five, the Pe statistics were estimated from OLS on (5.3.10), (5.13.14) and (5.3.17). Six pairwise artificial regressions were undertaken for each city subset and for the pooled analysis.
6.3 Results

Explanation of tables of results:

Tables 6.3.18 at the end of this chapter summarises all of the results of model estimation and the computation of test statistics associated with estimated parameters. In this table model 2a and 2b refer to NLS and WNLS estimates respectively of model 2. For details of weights adopted in WNLS estimation see Appendix 1. The dependent variables Y1, Y2, Y3, Y4, refer to population density 1986 and 1981 and household density 1986 and 1981 respectively. t ratios are given in brackets below the relevant parameter with the symbol $t_j$, where subscript $j$ refers to the value of the parameter restriction subject to the null hypothesis. The subscript $c$ for t ratios for model 1 in Table 6.3.18 refers to the estimated statistic conditioned on $\lambda$, otherwise estimates are based on the estimated asymptotic covariance matrix and in all cases with degrees of freedom (d. o. f.) = n-3. Where possible chi-square statistics are given. Chi-square statistics have two subscripts. The first indicates the value of the restricted parameter that is the null value and the second gives the degree of freedom, which in all these cases is 1. The 0.95 significance level is adopted (a critical value of 3.84). The ** symbol in Table 6.3.18 indicates failure of the restricted model to converge. Unfortunately, this was the case for restrictions on $\lambda$ for several data sets. For each data set the last row gives two elasticity scores ($\eta_\lambda$). The first score is elasticity at means and the second is elasticity 1 kilometre distance from centre.

The Pe statistics are listed in seven tables, one for each city plus an additional table for the analysis of Sydney with reduced data and are presented in a summary of results by city below. For each city the table is a 3x3 array where rows refer to the
null model and columns refer to the competing model in the artificial regression equation, so that the principal diagonal of this array is blank. In these tests model 2 refers to the unweighted NLS model. Each non-blank element of these arrays contains a column of four Pe statistics, where successive elements of each column refer to data sets Y1, Y2, Y3 and Y4 respectively. The Pe statistics for the pooled analysis is presented in two tables, one for all data and one for analysis where the three extreme CSD’s are deleted. In these tables each element of the 3x3 array contains two scores, the first for population density and the second for household density. In all cases the statistic is distributed t with 1 d. o. f. The critical test is non-rejection of the null hypothesis. Care must be taken here as rejection of the null does not mean acceptance of the alternate competing model in pairwise comparison. It simply means that the artificial regression model cannot be rejected. For all data sets the critical test score adopted is 2 (Davidson and Mackinnon, 1981 conservatively use 2.5). The critical score 6.314, which is the traditional 95% level with 1 d.o.f., is not used because of the unknown finite sample characteristics of the test. Because of evidence of bias to the null, following Davidson and Mackinnon op. cit., this conservative approach adopted is justifiable. It could be argued that this might lead to Type I errors, but this is not so critical, as the tests are repeated reversing null and alternate models.
**Summary of results by city:**

**Adelaide:**

Table 6.3.1

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Y1</td>
<td>-3.4381</td>
<td>-3.7477</td>
</tr>
<tr>
<td></td>
<td>Y2</td>
<td>0.0389*</td>
<td>-3.8322</td>
</tr>
<tr>
<td></td>
<td>Y3</td>
<td>-2.4734</td>
<td>-2.1173</td>
</tr>
<tr>
<td></td>
<td>Y4</td>
<td>-2.9766</td>
<td>-2.0228</td>
</tr>
<tr>
<td>2</td>
<td>Y1</td>
<td>-1.7048</td>
<td>0.5604</td>
</tr>
<tr>
<td></td>
<td>Y2</td>
<td>-0.8297</td>
<td>0.9876</td>
</tr>
<tr>
<td></td>
<td>Y3</td>
<td>-1.1212</td>
<td>1.0785</td>
</tr>
<tr>
<td></td>
<td>Y4</td>
<td>-0.7948</td>
<td>0.9441</td>
</tr>
<tr>
<td>3</td>
<td>Y1</td>
<td>-2.6639</td>
<td>3.6465</td>
</tr>
<tr>
<td></td>
<td>Y2</td>
<td>-1.0721†</td>
<td>3.3899</td>
</tr>
<tr>
<td></td>
<td>Y3</td>
<td>-2.3729</td>
<td>2.0391</td>
</tr>
<tr>
<td></td>
<td>Y4</td>
<td>-2.2503</td>
<td>2.0728</td>
</tr>
</tbody>
</table>

Table 6.3.1 gives the Pe statistics for the four data sets for Adelaide. There is a symmetry in the test statistics. The middle row indicates that when model 2 is the null it cannot be rejected. Alternatively when models 1 or 3 is the null model (rows one and three respectively) these models are rejected consistently when model 2 is the competing model in the artificial regression, with the exception of a null model 1 for Y2 (marked as * in the table). This result can be extended further in the sense that examination of rows 1 and 3 indicate that with the exception already mentioned and also a null model 3 against model 1 for Y2 (marked as † in the table) models 1 and 3 are consistently rejected. Moreover, the non-rejection of model 2 against model 3, which is arguably a heteroskedastic consistent estimation, indicates that the unweighted NLS estimate of model 2 is consistent with the data. Results therefore indicate that an appropriate representation of the data generating process is model 2a, the NLS model, of table 6.3.18. Further testing down between the linear versus the log-linear specification can take place within this flexible functional form. The relevant parameter here is the estimated λ which is reported below in Table 6.3.2.
Likelihood ratio statistics are reported in the third row based on the restrictions $\lambda=0$ and $\lambda=1$ (denoted by the subscript) which is distributed chi-squared with 1 d.o.f.. At the 95% level the critical value is 3.84. The likelihood ratio statistics indicate that, at the 95% level, in all cases the log-linear restriction ($\lambda=0$) cannot be rejected.

Table 6.3.2

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>Y1 (0.474)</th>
<th>Y2 (0.418)</th>
<th>Y3 (0.424)</th>
<th>Y4 (0.410)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi^2$ (3.84)</td>
<td>$(\chi^2_{0.05}, \chi^2_{0.01}, 9.74)$</td>
<td>$(\chi^2_{0.05}, \chi^2_{0.01}, 11.37)$</td>
<td>$(\chi^2_{0.05}, 2.125, \chi^2_{0.01}, 8.04)$</td>
<td>$(\chi^2_{0.05}, 2.325, \chi^2_{0.01}, 9.4)$</td>
</tr>
</tbody>
</table>

Table 6.3.2 is replicated in the second column for Adelaide of the general Table 6.3.18 found at the end of this chapter. In that table relevant t statistics are also reported along with estimates of the slope parameter and intercept. Likelihood ratio tests reported in that table for model 2a indicate that for all data sets the slope parameter is significantly different from zero. Recall that the t statistics reported in Table 6.3.18 are not scale invariant, so should be treated with caution.

Brisbane

Table 6.3.3

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Y1</td>
<td>-3.4381</td>
<td>-0.8301</td>
</tr>
<tr>
<td></td>
<td>Y2</td>
<td>0.0389</td>
<td>-0.9149</td>
</tr>
<tr>
<td></td>
<td>Y3</td>
<td>-2.4734</td>
<td>0.6309</td>
</tr>
<tr>
<td></td>
<td>Y4</td>
<td>-2.9767</td>
<td>2.1715</td>
</tr>
<tr>
<td>2</td>
<td>Y1</td>
<td>4.1483</td>
<td>3.6932</td>
</tr>
<tr>
<td></td>
<td>Y2</td>
<td>3.5790</td>
<td>3.0291</td>
</tr>
<tr>
<td></td>
<td>Y3</td>
<td>1.8347</td>
<td>1.0656</td>
</tr>
<tr>
<td></td>
<td>Y4</td>
<td>1.1974</td>
<td>0.6971</td>
</tr>
<tr>
<td>3</td>
<td>Y1</td>
<td>-6.9346</td>
<td>6.9883</td>
</tr>
<tr>
<td></td>
<td>Y2</td>
<td>-8.3969</td>
<td>8.3464</td>
</tr>
<tr>
<td></td>
<td>Y3</td>
<td>-7.2156</td>
<td>7.2382</td>
</tr>
<tr>
<td></td>
<td>Y4</td>
<td>-8.6451</td>
<td>8.7227</td>
</tr>
</tbody>
</table>
Table 6.3.3 suggests that the Pe test does indicate some discrimination between models. For subsets Y3 and Y4, model 2 cannot be rejected. For subset Y2, model 1 cannot be rejected. For Y1 all models are rejected at least once and it would not be prudent to continue testing down within any of the three models Y1. Finally, for all subsets when model 3 is the null it is consistently rejected, at least suggesting heteroskedasticity is not a problem.

Table 6.3.4

<table>
<thead>
<tr>
<th></th>
<th>Y2 (Model 3)</th>
<th>Y3 (Model 2)</th>
<th>Y4 (Model 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>0.38</td>
<td>0.902</td>
<td>1.039</td>
</tr>
<tr>
<td>$\chi^2 (3.84)$</td>
<td>($\chi^2_{124.2} \chi^2_{172.8}$)</td>
<td>($\chi^2_{6.11} \chi^2_{0.0036}$)</td>
<td>($\chi^{**}$)</td>
</tr>
<tr>
<td>t (1.6506, 1.2847)</td>
<td>(t5.25, t-8.56)</td>
<td>(t0.51, t0.3944)</td>
<td></td>
</tr>
</tbody>
</table>

Table 6.3.4 which is extracted from Table 6.3.18 shows the ability to test restrictions within model 2 for subsets Y3 and Y4 is limited, because the restricted model fails to converge for data set Y4. However, testing down within model 2 for Y3 fails to reject the null $\lambda=1$. For Y4 the t statistic (with parameter restriction indicated by subscript) for estimated $\lambda$ fails to reject the null $\lambda=1$ at the significance levels 0.05 and 0.1, with critical values at the relevant d.o.f. of 1.6506 and 1.2847 respectively. However, recalling that the t statistics for model 2 are not scale invariant, this must be treated with caution. For data set Y2 testing down within model 1 suggests that the restrictions $\lambda=0$ and $\lambda=1$ can both be rejected, where for this model the scale invariant unconditional t statistics are reported as well as the chi-squared statistics. These quite mixed results could be explained by the low $r^2$ reported in table 6.3.18 for all models and for all data subsets for Brisbane.
Table 6.3.5

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(Y_1)</td>
<td>0.8326</td>
<td>0.5217</td>
</tr>
<tr>
<td></td>
<td>(Y_2)</td>
<td>0.9049</td>
<td>0.5773</td>
</tr>
<tr>
<td></td>
<td>(Y_3)</td>
<td>0.9256</td>
<td>0.5439</td>
</tr>
<tr>
<td></td>
<td>(Y_4)</td>
<td>0.9736</td>
<td>0.5762</td>
</tr>
<tr>
<td>2</td>
<td>(Y_1)</td>
<td>0.6748</td>
<td>0.6517</td>
</tr>
<tr>
<td></td>
<td>(Y_2)</td>
<td>0.7591</td>
<td>0.7333</td>
</tr>
<tr>
<td></td>
<td>(Y_3)</td>
<td>0.6117</td>
<td>0.5972</td>
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<td></td>
<td>(Y_4)</td>
<td>-1.3243</td>
<td>1.3280</td>
</tr>
<tr>
<td>3</td>
<td>(Y_1)</td>
<td>-3.7047*</td>
<td>0.9503</td>
</tr>
<tr>
<td></td>
<td>(Y_2)</td>
<td>-3.4198*</td>
<td>1.0582</td>
</tr>
<tr>
<td></td>
<td>(Y_3)</td>
<td>-1.3288</td>
<td>0.9628</td>
</tr>
<tr>
<td></td>
<td>(Y_4)</td>
<td>0.1048</td>
<td>1.0314</td>
</tr>
</tbody>
</table>

Table 6.3.5 indicates that it is not possible to discriminate between any of the models for this city. With the exception of \(Y_1\) and \(Y_2\) when model 3 is the null and model 1 is the competing model (indicated by * in the table) none of the models can be rejected when they are the null model. The results are therefore almost completely asymmetric and indicate the right course of action is non-rejection of all models. However, the Pe test requires large samples. This result could therefore be explained by the small data sets for Hobart (28). As such it is not possible to extract salient results from Table 6.3.18 as in all other cities. However, an examination of Table 6.3.18 suggests that as a generalisation testing for the restrictions \(\lambda=0\) and \(\lambda=1\) produces contradictory results between models. For model 1 the log-linear restriction cannot be rejected and for models 2a, 2b and 3, that is a remaining 12 cases, 6 of them indicate the linear restriction cannot be rejected. Examination of table 6.3.18 for Hobart also indicates consistently low \(r^2\).
Table 6.3.6

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Y1</td>
<td>2.9633</td>
<td>-1.5038*</td>
</tr>
<tr>
<td></td>
<td>Y2</td>
<td>3.2988</td>
<td>-2.6663</td>
</tr>
<tr>
<td></td>
<td>Y3</td>
<td>4.5209</td>
<td>-3.8614</td>
</tr>
<tr>
<td></td>
<td>Y4</td>
<td>4.3899</td>
<td>-4.3988</td>
</tr>
<tr>
<td>2</td>
<td>Y1</td>
<td>0.6313</td>
<td>-0.5414</td>
</tr>
<tr>
<td></td>
<td>Y2</td>
<td>0.3913</td>
<td>-0.2814</td>
</tr>
<tr>
<td></td>
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<td>-1.2098</td>
<td>1.7470</td>
</tr>
<tr>
<td></td>
<td>Y4</td>
<td>-0.8317</td>
<td>1.3298</td>
</tr>
<tr>
<td>3</td>
<td>Y1</td>
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<td>2.8583</td>
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<td></td>
<td>Y2</td>
<td>-2.5212</td>
<td>2.7783</td>
</tr>
<tr>
<td></td>
<td>Y3</td>
<td>-2.9926</td>
<td>-2.9878</td>
</tr>
<tr>
<td></td>
<td>Y4</td>
<td>-3.0059</td>
<td>3.0084</td>
</tr>
</tbody>
</table>

Table 6.3.6 shows outcomes which are remarkably similar to those of Table 6.3.1 for Adelaide. There is a symmetry in the test statistics. The middle row indicates that when model 2 is the null it cannot be rejected. Alternatively when models 1 or 3 is the null model these models are rejected consistently, with the exception of Y1 when model 1 is the null and 3 is the competing model (indicated by * in the table). Again the non-rejection of model 2 against model 3, which is arguably a heteroskedastic consistent estimation, indicates that the NLS estimate of model 2 (model 2a) is consistent with the data. Results therefore indicate that an appropriate representation of the data generating process is model 2a, the NLS model, of table 6.3.18.

Table 6.3.7

<table>
<thead>
<tr>
<th></th>
<th>Y1</th>
<th>Y2</th>
<th>Y3</th>
<th>Y4</th>
</tr>
</thead>
<tbody>
<tr>
<td>λ</td>
<td>-0.2476</td>
<td>0.1806</td>
<td>-0.184</td>
<td>-0.0978</td>
</tr>
<tr>
<td>$\chi^2(3.84)$</td>
<td>$(\chi^2_0,0.973,\chi^2,38.59)$</td>
<td>$(\chi^2_0,0.6696,\chi^2,48.01)$</td>
<td>$(\chi^2_0,0.982,\chi^2,96.28)$</td>
<td>$(\chi^2_0,0.315,\chi^2,70.89)$</td>
</tr>
</tbody>
</table>

In Table 6.3.7 the likelihood ratio statistics indicate that for model 2a, at the 95% level, in all cases the log-linear restriction ($\lambda=0$) cannot be rejected.
Table 6.3.7 is replicated in the second column for Melbourne of the general Table 6.3.18 at the end of this chapter. In this table likelihood ratio tests reported for model 2a indicate that for all data sets the slope parameter is significantly different from zero.

Perth

Table 6.3.8

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Y1</td>
<td>2.0148</td>
<td>-1.0941†</td>
</tr>
<tr>
<td></td>
<td>Y2</td>
<td>2.5868</td>
<td>-3.1669</td>
</tr>
<tr>
<td></td>
<td>Y3</td>
<td>2.8943</td>
<td>-3.8873</td>
</tr>
<tr>
<td></td>
<td>Y4</td>
<td>3.1720</td>
<td>-4.3889</td>
</tr>
<tr>
<td>2</td>
<td>Y1</td>
<td>-1.9801</td>
<td>1.9375</td>
</tr>
<tr>
<td></td>
<td>Y2</td>
<td>-1.4456</td>
<td>0.8802</td>
</tr>
<tr>
<td></td>
<td>Y3</td>
<td>2.1988*</td>
<td>1.0409</td>
</tr>
<tr>
<td></td>
<td>Y4</td>
<td>0.5855</td>
<td>0.7140</td>
</tr>
<tr>
<td>3</td>
<td>Y1</td>
<td>-2.7638</td>
<td>3.1227</td>
</tr>
<tr>
<td></td>
<td>Y2</td>
<td>-2.8746</td>
<td>3.0536</td>
</tr>
<tr>
<td></td>
<td>Y3</td>
<td>-1.8992‡</td>
<td>3.0388</td>
</tr>
<tr>
<td></td>
<td>Y4</td>
<td>-3.0530</td>
<td>2.9889</td>
</tr>
</tbody>
</table>

Table 6.3.8 almost replicates the same level of symmetry found in Tables 6.3.1 and 6.3.6 for Adelaide and Melbourne respectively. With the exception of Y3 when model 1 is the competing model (indicated in the table by * in the table), the middle row of Table 6.3.8 indicates that when model 2 is the null it cannot be rejected. Alternatively when models 1 or 3 are the null model these models are rejected consistently with the exception of Y1 when model 1 is the null and 3 is the competing model (indicated by † in the table) and Y3 when 3 is the null model and 1 is the competing model (indicated by ‡ in the table). It could be argued that for Y3 all three models are rejected and that it is not possible to reject model 2 for Y1, Y2 and Y4. Non-rejection of model 2 in competition with model 3 suggests heteroskedasticity is
not an issue. Notwithstanding the problems for Y3 testing down within model 2a will proceed for all data sets, but the analysis of Y3 must be treated with caution.

Table 6.3.9

<table>
<thead>
<tr>
<th></th>
<th>Y1</th>
<th>Y2</th>
<th>Y3</th>
<th>Y4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>0.439</td>
<td>0.479</td>
<td>0.424</td>
<td>0.372</td>
</tr>
<tr>
<td>$\chi^2$ (3.84)</td>
<td>$(\chi^2_{0.05}, 1.781, \chi^2_{0.1}, 10.67)$</td>
<td>$(\chi^2_{0.05}, 2.478, \chi^2_{0.1}, 13.97)$</td>
<td>$(\chi^2_{0.05}, 1.420, \chi^2_{0.1}, 18.91)$</td>
<td>$(\chi^2_{0.05}, 2.041, \chi^2_{0.1}, 23.27)$</td>
</tr>
</tbody>
</table>

In Table 6.3.9 the likelihood ratio statistics indicate that, at the 95% level, in all cases the log-linear restriction ($\lambda=0$) cannot be rejected.

Sydney:

*All observations (324):*

Table 6.3.10

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Y1</td>
<td>7.6220</td>
<td>-1.1793*</td>
</tr>
<tr>
<td></td>
<td>Y2</td>
<td>7.5131</td>
<td>-2.9614</td>
</tr>
<tr>
<td></td>
<td>Y3</td>
<td>8.7677</td>
<td>9.6750</td>
</tr>
<tr>
<td></td>
<td>Y4</td>
<td>8.3252</td>
<td>9.7624</td>
</tr>
<tr>
<td>2</td>
<td>Y1</td>
<td>0.0829</td>
<td>0.4178</td>
</tr>
<tr>
<td></td>
<td>Y2</td>
<td>0.1865</td>
<td>0.1140</td>
</tr>
<tr>
<td></td>
<td>Y3</td>
<td>-0.6956</td>
<td>1.3472</td>
</tr>
<tr>
<td></td>
<td>Y4</td>
<td>-0.5827</td>
<td>1.1300</td>
</tr>
<tr>
<td>3</td>
<td>Y1</td>
<td>0.7738</td>
<td>0.6355</td>
</tr>
<tr>
<td></td>
<td>Y2</td>
<td>0.0639</td>
<td>0.0289</td>
</tr>
<tr>
<td></td>
<td>Y3</td>
<td>-0.2188</td>
<td>0.0149</td>
</tr>
<tr>
<td></td>
<td>Y4</td>
<td>0.4462</td>
<td>-0.5033</td>
</tr>
</tbody>
</table>

Row one of Table 6.3.10 indicates the rejection of model 1 when it is the null model with the exception of Y1 in competition with model 3 (indicated by * in the table). Rows two and three show that when model 2 or 3 is the null model neither can be rejected. Therefore, it is not possible to discriminate between models 2 and 3. The failure to reject model 3 suggests that heteroskedasticity may be an issue so it seems
appropriate to test the restrictions \( \lambda = 0 \) and \( \lambda = 1 \) within models 2a, 2b and model 3.

This is done in Table 6.3.11.

### Table 6.3.11

<table>
<thead>
<tr>
<th>Model 2a ( \lambda )</th>
<th>( \chi^2 ) (3.84)</th>
<th>Y1</th>
<th>Y2</th>
<th>Y3</th>
<th>Y4</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.8172</td>
<td>( \chi^2, 26.08, \chi^2, 145.1 )</td>
<td>-0.727</td>
<td>-0.6616</td>
<td>-0.5888</td>
<td></td>
</tr>
<tr>
<td>Model 2b ( \lambda )</td>
<td>-0.6803</td>
<td>( \chi^2, 22.66, \chi^2, 155.8 )</td>
<td>-0.4208</td>
<td>-0.6575</td>
<td>-0.5869</td>
</tr>
<tr>
<td>( \chi^2 ) (3.84)</td>
<td>( \chi^2, 994.3, \chi^2, 994.8 )</td>
<td>-0.2417</td>
<td>-0.220</td>
<td>-0.383</td>
<td>-0.335</td>
</tr>
<tr>
<td>Model 3 ( \lambda )</td>
<td>-0.2417</td>
<td>( \chi^2, 7.014, \chi^2, 148.9 )</td>
<td>-0.220</td>
<td>-0.383</td>
<td>-0.335</td>
</tr>
</tbody>
</table>

This table, which comprises the salient features of Table 6.3.18, indicates that in all data sets for all models the restrictions \( \lambda = 0 \) and \( \lambda = 1 \) are rejected. What is interesting is that the estimated \( \lambda \) values for models 2a, 2b and 3 are less than zero, which suggests a non-linear relationship between distance and density which is more extreme than the log-linear relationship. Figure 6.3.1 illustrates this. In this figure the most extreme non-linear result, that is Y1 model 2a, is graphed against the Box Cox estimate and the log-linear form estimate. In the figure curve 1 (heavy) is the plot using model 2a for Y1 where \( \lambda = -0.8172 \). Curve 2 (dashed) is plotted using parameter values estimated with the log-linear restriction placed on the data set. Curve 3 (shaded) is the Box Cox data plot where \( \lambda = 0.32 \).
Reduced observations (521):

Table 6.3.12

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Y1</td>
<td>7.4681</td>
<td>-0.6357*</td>
</tr>
<tr>
<td></td>
<td>Y2</td>
<td>7.2447</td>
<td>-2.4523</td>
</tr>
<tr>
<td></td>
<td>Y3</td>
<td>8.7967</td>
<td>9.7147</td>
</tr>
<tr>
<td></td>
<td>Y4</td>
<td>8.4821</td>
<td>9.6987</td>
</tr>
<tr>
<td>2</td>
<td>Y1</td>
<td>0.1117</td>
<td>0.1240</td>
</tr>
<tr>
<td></td>
<td>Y2</td>
<td>-0.1092</td>
<td>0.3847</td>
</tr>
<tr>
<td></td>
<td>Y3</td>
<td>-1.0378</td>
<td>1.5525</td>
</tr>
<tr>
<td></td>
<td>Y4</td>
<td>-0.9536</td>
<td>1.4230</td>
</tr>
<tr>
<td>3</td>
<td>Y1</td>
<td>0.4809</td>
<td>0.0778</td>
</tr>
<tr>
<td></td>
<td>Y2</td>
<td>-0.2157</td>
<td>-0.3313</td>
</tr>
<tr>
<td></td>
<td>Y3</td>
<td>0.0036</td>
<td>-0.3276</td>
</tr>
<tr>
<td></td>
<td>Y4</td>
<td>0.7329</td>
<td>-0.8536</td>
</tr>
</tbody>
</table>

Table 6.3.12 shows exactly the same outcome as Table 6.3.10, even to the same exception for population density for 1986 for model 3 as the null model indicated by * in the table.
Table 6.3.13

<table>
<thead>
<tr>
<th></th>
<th>Y1</th>
<th>Y2</th>
<th>Y3</th>
<th>Y4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model 2a λ</td>
<td>-0.883</td>
<td>-0.600</td>
<td>-0.610</td>
<td>-0.453</td>
</tr>
<tr>
<td>( \chi^2(3.84) )</td>
<td>(\chi^2, 44.55, \chi^2, 342.5 )</td>
<td>(\chi^2, 26.61, \chi^2, 320.7 )</td>
<td>(\chi^2, 37.23, \chi^2, 422.7 )</td>
<td>(\chi^2, 22.13, \chi^2, 426.9 )</td>
</tr>
<tr>
<td>Model 2b λ</td>
<td>-0.837</td>
<td>-0.850</td>
<td>-0.607</td>
<td>-0.451</td>
</tr>
<tr>
<td>( \chi^2(3.84) )</td>
<td>(\chi^2, 9039, \chi^2, 15360 )</td>
<td>(\chi^2, 4369, \chi^2, 9667 )</td>
<td>(\chi^2, 4308, \chi^2, 11120 )</td>
<td>(\chi^2, 11175, \chi^2, 19545 )</td>
</tr>
<tr>
<td>Model 3 λ</td>
<td>-0.243</td>
<td>-0.234</td>
<td>-0.301</td>
<td>-0.292</td>
</tr>
<tr>
<td>( \chi^2(3.84) )</td>
<td>(\chi^2, 36.4, \chi^2, 439.3 )</td>
<td>(\chi^2, 33.75, \chi^2, 628.0 )</td>
<td>(\chi^2, 63.5, \chi^2, 711 )</td>
<td>(\chi^2, 59.5, \chi^2, 698.5 )</td>
</tr>
</tbody>
</table>

Like Table 6.3.11, table 6.3.13 indicates that the function for Sydney is more extreme than the log-linear restriction.

The results for the reduced number of observations generally follow those for Sydney with all observations and outcomes do not vary with the analysis with all observations. However, even though outcomes indicate a form more extreme than the log-linear form, as a generalisation these outcomes are less extreme than the results for Sydney with all observations.

**Pooled:**

*All observations (3208):*

Table 6.3.14

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Population</td>
<td>9.4291</td>
<td>-2.1616</td>
</tr>
<tr>
<td></td>
<td>Household</td>
<td>11.5289</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Population</td>
<td>0.4025</td>
<td>-0.1737</td>
</tr>
<tr>
<td></td>
<td>Household</td>
<td>-1.0461</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Population</td>
<td>-4.9669</td>
<td>-3.1966</td>
</tr>
<tr>
<td></td>
<td>Household</td>
<td>5.3482</td>
<td></td>
</tr>
</tbody>
</table>

Table 6.3.14 shows that, in pairwise comparison, whenever model 2 is the null it cannot be rejected. Conversely in pairwise comparison models 1 and 3 are consistently rejected as the null model. The results for model 2 are symmetric and
discriminate against models 1 and 3. Outcomes have not been dominated by Sydney, where it was not possible to test down beyond models 2 and 3.

Table 6.3.15

<table>
<thead>
<tr>
<th></th>
<th>Population Density</th>
<th>Household Density</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\lambda)</td>
<td>-0.883</td>
<td>-0.610</td>
</tr>
<tr>
<td>(\chi^2(3.84))</td>
<td>((\chi^2, 44.55, \chi^2, 342.5))</td>
<td>((\chi^2, 37.23, \chi^2, 422.7))</td>
</tr>
</tbody>
</table>

Table 6.3.15 gives the \(\lambda\) estimates for the pooled population and household density sets for model 2a which appears to be the most appropriate representation of the data generating process. Testing the restrictions \(\lambda=0\) and \(\lambda=1\) rejects both restrictions. Outcomes reflect Sydney, pointing to a more extreme functional form than the log-linear form.

Reduced observations (3202):

Table 6.3.16

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Population Household</td>
<td>9.7010</td>
<td>-2.6046</td>
</tr>
<tr>
<td></td>
<td></td>
<td>11.6438</td>
<td>4.8225</td>
</tr>
<tr>
<td>2</td>
<td>Population Household</td>
<td>-0.1004</td>
<td>0.1906</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-1.6934</td>
<td>2.1044</td>
</tr>
<tr>
<td>3</td>
<td>Population Household</td>
<td>-5.3682</td>
<td>-2.4781</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5.6789</td>
<td>5.5387</td>
</tr>
</tbody>
</table>

Table 6.3.16 gives the Pe scores for the pooled models with observations reduced to 3202. Outcomes are the same for population density for the inclusion of all observations. However, for household density scores are now completely symmetric and all models are rejected. Further testing for \(\lambda=0\) and \(\lambda=1\) can only legitimately be undertaken within model 2a for population density.
Table 6.3.17

<table>
<thead>
<tr>
<th></th>
<th>Population Density</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>-0.600</td>
</tr>
<tr>
<td>$\chi^2(3.84)$</td>
<td>$\left(\chi^2_0&lt;26.61, \chi^2, 320.7\right)$</td>
</tr>
</tbody>
</table>

Table 6.3.17 indicates that both $\lambda=0$ and $\lambda=1$ are rejected.

6.4 Conclusion

Non-nested tests are fraught with difficulties stemming from the potential for asymmetric results. Despite this, for three out of the six cities, Adelaide, Melbourne and Perth, the tests were almost entirely symmetric and tested down to model 2. The results indicate that model 2 is a robust alternative to the Box Cox transformation for these data sets. Moreover, where discrimination was possible further testing down indicated the log-linear specification to be a good approximation to the data generating process in density models for these cities. However, for three out of the six cities the log-linear specification did not apply. Sydney is interesting in that whilst it was not possible to discriminate between models 2 and 3 for three of the data sets these models indicate a more extreme non-linear relationship than the log-linear form. Moreover, these results appear not to be influenced by extreme density observations for three central CSD’s. These results, even though more extreme than the log-linear specification, are not inconsistent with the lot size hypothesis and are consistent with a non-linear relationship between density and distance. It might be argued that the small sample problems associated with Hobart makes inference on these data sets difficult. However Brisbane has reasonable sized data sets and the log-linear specification does not seem appropriate for this city. Moreover, even though all
data sets for the cities of Brisbane and Hobart showed a decrease in density with distance from centre, which does not contradict the lot size hypothesis, the relationship between density and distance seems fragile - so much so that one could question whether the traditional lot size hypothesis and the rigid monocentric form applies unambiguously to these two cities.

The pooled analysis displayed symmetry similar to that of Adelaide, Melbourne and Perth and tested down to model 2a for population and household density. However tests for functional form within model 2a indicated rejection of the linear and log-linear forms. This outcome was repeated for the reduced data set for population, but deletion of the extreme central observations affected the Pe test for household density, with completely asymmetrical outcomes, rejecting all models. Thus the case of extreme observations affecting competing models differently cannot be totally ruled out. Further investigation of this is beyond the scope of this study, but indicates an avenue for further work.

The results are dependent on the adoption of a conservative significance level for the Pe test. Had the traditional 0.95 level been adopted, results would have been completely asymmetric. It could be argued that, rather than “stretching and squeezing the data” to fit certain models, the significance level of a test has been “squeezed” resulting in a symmetrical outcome. If these tests are to be used to discriminate between non-linear models with transform of dependent variable, further work is required on the appropriate level of significance to adopt. Until that is undertaken, say through Monte Carlo simulation, tests of rejection or falsification point to the need for a conservative approach.
Table 6.3.18.

<table>
<thead>
<tr>
<th>Model</th>
<th>model 1</th>
<th>model 2a</th>
<th>model 2b</th>
<th>model 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>y=y1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>λ</td>
<td>0.52</td>
<td>0.474</td>
<td>0.464</td>
<td>-0.042</td>
</tr>
<tr>
<td></td>
<td>(χ²=124.9, x₂=47.0)</td>
<td>(χ²=3.62, x₂=9.74)</td>
<td>(χ²=506.4, x₂=22.9)</td>
<td>(χ²=0.5048, x₂=90.22)</td>
</tr>
<tr>
<td></td>
<td>(t₉=9.45 t₈=8.72)</td>
<td>(t₉=5.56 t₈=10.61)</td>
<td>(t₉=3.72 t₈=38.91)</td>
<td>(t₉=0.5740 t₈=-14.08)</td>
</tr>
<tr>
<td>a</td>
<td>7.5125</td>
<td>7.1147</td>
<td>7.0168</td>
<td>3.2668</td>
</tr>
<tr>
<td></td>
<td>(t₉=10.43)</td>
<td>(t₉=10.70)</td>
<td>(t₉=47.60)</td>
<td>(t₉=10.69)</td>
</tr>
<tr>
<td>b</td>
<td>-0.15176</td>
<td>-0.12892</td>
<td>-0.1270</td>
<td>-0.08691</td>
</tr>
<tr>
<td></td>
<td>(t₉=8.04)</td>
<td>(t₉=9.88)</td>
<td>(t₉=7.62)</td>
<td>(t₉=5.63)</td>
</tr>
<tr>
<td>r²</td>
<td>0.50</td>
<td>0.44</td>
<td>0.44</td>
<td>0.58</td>
</tr>
<tr>
<td>ρₙ</td>
<td>-0.670,-0.031</td>
<td>-0.611,-0.030</td>
<td>-0.616,-0.030</td>
<td>-1.510,-0.100</td>
</tr>
<tr>
<td>y=y2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>λ</td>
<td>0.49</td>
<td>0.418</td>
<td>0.274</td>
<td>-0.055</td>
</tr>
<tr>
<td></td>
<td>(χ²=116.2, x₂=57.4)</td>
<td>(χ²=3.20, x₂=11.37)</td>
<td>(χ²=738.3, x₂=262.8)</td>
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<td>(t₉=6.6 t₈=-13.46)</td>
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<td>-0.702,-0.034</td>
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<td>-1.611,-0.109</td>
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<td>(χ²=20.51, x₂=115.3)</td>
<td>(χ²=0.038, x₂=124.9)</td>
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<td>(t₉=2.29 t₈=-4.07)</td>
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<td>-0.07616</td>
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<tr>
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<td>(χ²=2571, x₂=82.29)</td>
<td>(χ²=0.4044 t₈=13.04)</td>
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n=181: critical t, d.o.f. n-3, 0.05=1.6536, 0.1=1.2863.

255
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<th>Model 2b</th>
<th>Model 3</th>
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<td>-0.727, -0.018</td>
<td>-0.734, -0.019</td>
<td>-1.192, -0.049</td>
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<td>$\chi_{124.2, 172.8}$</td>
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<td>$(t_{3.25}, t_{-8.56})$</td>
<td>$(t_{31.01}, t_{-3.14})$</td>
<td>$(t_{18.22}, t_{-2.0253})$</td>
<td>$(t_{0.0470}, t_{-2.0136})$</td>
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<td>$(t_{10.50})$</td>
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<td>-0.854, -0.021</td>
<td>-1.365, -0.057</td>
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<tr>
<th>$\lambda$</th>
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<td>$\chi_{180.2, 216.5}$</td>
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<td>$(t_{5.48}, t_{-10.17})$</td>
<td>$(t_{11.43}, t_{-1.2421})$</td>
<td>$(t_{11.65}, t_{-3.863})$</td>
<td>$(t_{0.023}, t_{-1.8974})$</td>
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<td>$(t_{13.04})$</td>
<td>$(t_{3.314})$</td>
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<td>$(t_{-171.0})$</td>
<td>$(t_{-185.2})$</td>
<td>$(t_{-1.324})$</td>
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<td>-0.904, -0.022</td>
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<thead>
<tr>
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<th>Model 2b</th>
<th>Model 3</th>
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<td>$(t_{10.51}, t_{0.3944})$</td>
<td>$(t_{22.87}, t_{0.6698})$</td>
<td>$(t_{0.0050}, t_{-2.47})$</td>
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n=274: critical t, d.o.f. n-3, 0.05=1.6506, 0.1=1.2847.
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<th>Table 6.3.18. (cont.)</th>
<th>Hobart</th>
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<tr>
<td>((x^0y, 1.462, x_{11} 20.9))</td>
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<td>((t_{y-1.12}))</td>
<td>((t_{1-2.35}))</td>
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<td>((x^0y, 24.52, x_{11} 0.008))</td>
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n=28: critical t, d.o.f. n-3, 0.05=1.7083, 0.1=1.3163.
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<td>( \chi^2 )</td>
<td>( \chi^2 )</td>
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<td>( (t_{x_{0.9730}38.59}) )</td>
<td>( (t_{x_{0.7527}14.75}) )</td>
<td>( (t_{x_{0.1108}87.13}) )</td>
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<td>( (t_{5.25.26}) )</td>
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<table>
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<th>Model 1</th>
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<td>( \chi^2 )</td>
<td>( \chi^2 )</td>
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<td>( (x_{139.9}1715.0) )</td>
<td>( (x_{0.3694}105.10) )</td>
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<td>( (t_{0.5}5.03) )</td>
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<td>-0.757, -0.031</td>
<td>-0.849, -0.055</td>
<td>-0.853, -0.54</td>
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<table>
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<tr>
<th>Model 1</th>
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<tbody>
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<td>( \lambda )</td>
<td>( \chi^2 )</td>
<td>( \chi^2 )</td>
<td>( \chi^2 )</td>
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<td>( (x_{187.1}1256.3) )</td>
<td>( (x_{0.9815}96.28) )</td>
<td>( (x_{258.30}**) )</td>
<td>( (x_{0.4045}104.44) )</td>
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<tr>
<td>( (t_{10.06}15.09) )</td>
<td>( (t_{6.39}14.13) )</td>
<td>( (t_{32.88}221.18) )</td>
<td>( (t_{16.36}11.04) )</td>
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<tr>
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<td>2.3116</td>
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<td>( (t_{14.73}) )</td>
<td>( (t_{40.4}) )</td>
<td>( (t_{164.01}) )</td>
<td>( (t_{50.8}) )</td>
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<tr>
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<td>-0.09553</td>
<td>-0.03791</td>
<td>-0.03857</td>
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<td>( (t_{11.85}) )</td>
<td>( (t_{28.88}) )</td>
<td>( (t_{109.04}) )</td>
<td>( (t_{5.714}) )</td>
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<td>-1.142, -0.079</td>
<td>-0.972, -0.064</td>
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<td>( \chi^2 )</td>
<td>( \chi^2 )</td>
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<td>( (x_{0.200.0}300.0) )</td>
<td>( (x_{0.154}70.89) )</td>
<td>( (x_{23.18}1483.77) )</td>
<td>( (x_{4.25}122.66) )</td>
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<tr>
<td>( (t_{10.37}16.92) )</td>
<td>( (t_{3.67}14.25) )</td>
<td>( (t_{8.79}60.31) )</td>
<td>( (t_{1.692}12.79) )</td>
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<td>2.6331</td>
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<td>( (t_{39.87}) )</td>
<td>( (t_{41.21}) )</td>
<td>( (t_{10.10}) )</td>
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<td>( b )</td>
<td>-0.10019</td>
<td>-0.04989</td>
<td>-0.05097</td>
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<tr>
<td>( (t_{13.28}) )</td>
<td>( (t_{19.86}) )</td>
<td>( (t_{54.62}) )</td>
<td>( (t_{30.43}) )</td>
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<td>( r^2 )</td>
<td>0.47</td>
<td>0.44</td>
<td>0.44</td>
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<tr>
<td>( \eta_{\lambda} )</td>
<td>-0.892, -0.37</td>
<td>-1.082, -0.067</td>
<td>-1.085, -0.066</td>
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n=439: critical t, d.o.f. n-3, 0.05=1.6485, 0.1=1.2835.
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<td>$\lambda$</td>
<td>$\chi^2_{0.05} 14.1, \chi^2_{0.10} 20.1$</td>
<td>$\chi^2_{0.05} 1.7813, \chi^2_{0.10} 10.67$</td>
<td>$\chi^2_{0.05} 188.40, \chi^2_{0.10} 128.05$</td>
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<td>$t_{0.05}$</td>
<td>$t_{0.10} 6.63, t_{0.10} -3.89$</td>
<td>$t_{0.05} 13.14, t_{0.10} -16.79$</td>
<td>$t_{0.05} 12.31, t_{0.10} -28.93$</td>
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<tr>
<td>$a$</td>
<td>8.8917</td>
<td>6.571</td>
<td>5.1774</td>
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<tr>
<td>$t_{0.05}$</td>
<td>$t_{0.10} 5.67$</td>
<td>$t_{0.05} 16.37$</td>
<td>$t_{0.05} 25.09$</td>
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<tr>
<td>$b$</td>
<td>-0.23084</td>
<td>-0.16090</td>
<td>-0.1330</td>
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<td>$t_{0.05}$</td>
<td>$t_{0.10} -4.38$</td>
<td>$t_{0.05} 13.08$</td>
<td>$t_{0.05} 12.73$</td>
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<tr>
<td>$r^2$</td>
<td>0.47</td>
<td>0.46</td>
<td>0.46</td>
</tr>
<tr>
<td>$\eta_{p}$</td>
<td>-0.641, -0.036</td>
<td>-0.710, -0.042</td>
<td>-0.857, -0.053</td>
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<table>
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<td>$\lambda$</td>
<td>$\chi^2_{0.05} 103.4, \chi^2_{0.10} 29.6$</td>
<td>$\chi^2_{0.05} 2.4777, \chi^2_{0.10} 13.97$</td>
<td>$\chi^2_{0.05} 202.85, \chi^2_{0.10} 197.67$</td>
</tr>
<tr>
<td>$t_{0.05}$</td>
<td>$t_{0.10} 6.71, t_{0.10} -5.07$</td>
<td>$t_{0.05} 12.63, t_{0.10} -13.73$</td>
<td>$t_{0.05} 17.24, t_{0.10} -20.01$</td>
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<tr>
<td>$a$</td>
<td>7.9450</td>
<td>7.1569</td>
<td>7.0146</td>
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<tr>
<td>$t_{0.05}$</td>
<td>$t_{0.10} 6.62$</td>
<td>$t_{0.05} 14.11$</td>
<td>$t_{0.05} 22.61$</td>
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<td>$b$</td>
<td>-0.21753</td>
<td>-0.19749</td>
<td>-0.19604</td>
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<tr>
<td>$t_{0.05}$</td>
<td>$t_{0.10} -5.14$</td>
<td>$t_{0.05} 137.5$</td>
<td>$t_{0.05} 144.1$</td>
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<td>0.48</td>
<td>0.47</td>
<td>0.47</td>
</tr>
<tr>
<td>$\eta_{p}$</td>
<td>-0.727, -0.040</td>
<td>-0.808, -0.046</td>
<td>-0.836, -0.047</td>
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<table>
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<tr>
<th>Model 1</th>
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<tr>
<td>$\lambda$</td>
<td>$\chi^2_{0.05} 92.7, \chi^2_{0.10} 47.4$</td>
<td>$\chi^2_{0.05} 1.4202, \chi^2_{0.10} 18.91$</td>
<td>$\chi^2_{0.05} 336.91, \chi^2_{0.10} 272.99$</td>
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<tr>
<td>$t_{0.05}$</td>
<td>$t_{0.10} 5.90, t_{0.10} -6.14$</td>
<td>$t_{0.05} 4.04, t_{0.10} -5.49$</td>
<td>$t_{0.05} 25.52, t_{0.10} -38.29$</td>
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<td>$a$</td>
<td>3.5695</td>
<td>3.6494</td>
<td>3.5829</td>
</tr>
<tr>
<td>$t_{0.05}$</td>
<td>$t_{0.10} 10.13$</td>
<td>$t_{0.05} 7.91$</td>
<td>$t_{0.05} 67.67$</td>
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<tr>
<td>$b$</td>
<td>-0.11931</td>
<td>-0.12830</td>
<td>-0.12999</td>
</tr>
<tr>
<td>$t_{0.05}$</td>
<td>$t_{0.10} -8.18$</td>
<td>$t_{0.05} 148.1$</td>
<td>$t_{0.05} 153.9$</td>
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<tr>
<td>$r^2$</td>
<td>0.49</td>
<td>0.48</td>
<td>0.48</td>
</tr>
<tr>
<td>$\eta_{p}$</td>
<td>-0.785, -0.044</td>
<td>-0.913, -0.051</td>
<td>-0.968, -0.055</td>
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</table>

<table>
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<th>Model 1</th>
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<th>Model 3</th>
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<tbody>
<tr>
<td>$\lambda$</td>
<td>$\chi^2_{0.05} 88.2, \chi^2_{0.10} 55.9$</td>
<td>$\chi^2_{0.05} 2.0412, \chi^2_{0.10} 23.27$</td>
<td>$\chi^2_{0.05} 684.11, \chi^2_{0.10} 510.98$</td>
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<tr>
<td>$t_{0.05}$</td>
<td>$t_{0.10} 6.31, t_{0.10} -7.11$</td>
<td>$t_{0.05} 33.3, t_{0.10} -5.62$</td>
<td>$t_{0.05} 27.43, t_{0.10} -47.02$</td>
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<tr>
<td>$a$</td>
<td>3.37</td>
<td>3.379</td>
<td>3.3620</td>
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<td>$t_{0.05}$</td>
<td>$t_{0.10} 11.75$</td>
<td>$t_{0.05} 7.65$</td>
<td>$t_{0.05} 73.23$</td>
</tr>
<tr>
<td>$b$</td>
<td>-0.1213</td>
<td>-0.12964</td>
<td>-0.12974</td>
</tr>
<tr>
<td>$t_{0.05}$</td>
<td>$t_{0.10} -9.76$</td>
<td>$t_{0.05} 160.2$</td>
<td>$t_{0.05} 159.1$</td>
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<td>0.50</td>
<td>0.50</td>
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<tr>
<td>$\eta_{p}$</td>
<td>-0.862, -0.048</td>
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<td>-1.051, -0.059</td>
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n=158: critical t, d.o.f. n-3, 0.05=1.6549, 0.1=1.2870.
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<td><strong>λ</strong></td>
<td>0.32</td>
<td>-0.8172</td>
<td>-0.6803</td>
<td>-0.2417</td>
</tr>
<tr>
<td>(x_1^2, 236.4, x_9^2, 526.4)</td>
<td>(t_{0.137}, t_{-29.29})</td>
<td>(t_{0.179}, t_{-399.8})</td>
<td>(t_{0.4237}, t_{104.65})</td>
<td>(t_{0.266}, t_{-13.66})</td>
</tr>
<tr>
<td><strong>a</strong></td>
<td>7.4874</td>
<td>1.2009</td>
<td>1.4000</td>
<td>2.6381</td>
</tr>
<tr>
<td>(t_{0.1633}, t_{0.645})</td>
<td>(t_{0.1956})</td>
<td>(t_{0.50483})</td>
<td>(t_{0.679})</td>
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<tr>
<td><strong>b</strong></td>
<td>-0.09891</td>
<td>-0.00362</td>
<td>-0.00580</td>
<td>-0.02779</td>
</tr>
<tr>
<td>(t_{0.12.07}, t_{0.21.79})</td>
<td>(t_{0.1146.2})</td>
<td>(t_{0.53.63})</td>
<td>(t_{0.776.6})</td>
<td>(t_{0.339.4})</td>
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<tr>
<td><strong>r^2</strong></td>
<td>0.48</td>
<td>0.46</td>
<td>0.46</td>
<td>0.47</td>
</tr>
<tr>
<td><strong>η_χ</strong></td>
<td>-0.830, -0.029</td>
<td>-0.946, -0.160</td>
<td>-0.957, -0.112</td>
<td>-1.220, -0.075</td>
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<tr>
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<tbody>
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<td><strong>λ</strong></td>
<td>0.33</td>
<td>-0.727</td>
<td>-0.4208</td>
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<tr>
<td>(x_1^2, 243.1, x_9^2, 510.6)</td>
<td>(t_{0.129}, t_{-26.18})</td>
<td>(t_{0.1149}, t_{-272.9})</td>
<td>(t_{0.2437}, t_{-82.28})</td>
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<tr>
<td><strong>a</strong></td>
<td>7.6952</td>
<td>1.3239</td>
<td>1.9992</td>
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<tr>
<td>(t_{0.1501}, t_{0.5543})</td>
<td>(t_{0.132.3})</td>
<td>(t_{0.38.12})</td>
<td>(t_{0.74.6})</td>
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<tr>
<td><strong>b</strong></td>
<td>-0.10659</td>
<td>-0.00497</td>
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<tr>
<td>(t_{0.11.30}, t_{-22.43})</td>
<td>(t_{0.83.7})</td>
<td>(t_{0.41.20})</td>
<td>(t_{0.70.5})</td>
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<td><strong>r^2</strong></td>
<td>0.49</td>
<td>0.49</td>
<td>0.48</td>
</tr>
<tr>
<td><strong>η_χ</strong></td>
<td>-0.877, -0.03</td>
<td>-0.941, -0.121</td>
<td>-1.095, -0.087</td>
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<table>
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<th></th>
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<td><strong>λ</strong></td>
<td>0.26</td>
<td>-0.6616</td>
<td>-0.6575</td>
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<tr>
<td>(x_1^2, 177.7, x_9^2, 726.5)</td>
<td>(t_{0.1179}, t_{-33.56})</td>
<td>(t_{0.45.56}, t_{-114.4})</td>
<td>(t_{0.60.71}, t_{-153.05})</td>
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<td><strong>a</strong></td>
<td>4.2409</td>
<td>1.3883</td>
<td>1.3954</td>
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<tr>
<td>(t_{0.20.94}, t_{0.44.44})</td>
<td>(t_{0.59.22})</td>
<td>(t_{0.81.47})</td>
<td>(t_{0.89.5})</td>
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<tr>
<td><strong>b</strong></td>
<td>-0.07039</td>
<td>-0.01413</td>
<td>-0.01431</td>
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<td>(t_{0.16.72}, t_{0.21.55})</td>
<td>(t_{0.91.8})</td>
<td>(t_{0.22.9})</td>
<td>(t_{0.92.2})</td>
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<td><strong>r^2</strong></td>
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<td>0.46</td>
<td>0.46</td>
</tr>
<tr>
<td><strong>η_χ</strong></td>
<td>-0.937, -0.034</td>
<td>-1.091, -0.156</td>
<td>-1.096, -0.157</td>
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<td><strong>λ</strong></td>
<td>0.25</td>
<td>-0.5888</td>
<td>-0.5869</td>
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<td>(x_1^2, 177.0, x_9^2, 776.6)</td>
<td>(t_{0.1131}, t_{-33.56})</td>
<td>(t_{0.40.86}, t_{-110.3})</td>
<td>(t_{0.225.23}, t_{-609.17})</td>
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<td><strong>a</strong></td>
<td>4.1681</td>
<td>1.5146</td>
<td>1.5171</td>
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<tr>
<td>(t_{0.22.34}, t_{0.43.57})</td>
<td>(t_{0.56.59})</td>
<td>(t_{0.536.71})</td>
<td>(t_{0.10.97})</td>
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<tr>
<td><strong>b</strong></td>
<td>-0.07315</td>
<td>-0.01790</td>
<td>-0.01794</td>
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<tr>
<td>(t_{0.17.27}, t_{0.22.34})</td>
<td>(t_{0.96.4})</td>
<td>(t_{0.89.0})</td>
<td>(t_{0.257.32})</td>
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<tr>
<td><strong>r^2</strong></td>
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<td>0.46</td>
<td>0.46</td>
</tr>
<tr>
<td><strong>η_χ</strong></td>
<td>-1.010, -0.036</td>
<td>-1.167, -0.151</td>
<td>-1.166, -0.150</td>
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n=524: critical t, d.o.f. n-3, 0.05=1.6479, 0.1=1.2832
Table 6.3.13. (cont.)

Sydney (Reduced observations)

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>Model 1</th>
<th>Model 2a</th>
<th>Model 2b</th>
<th>Model 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\chi_{1,269.5, x_{148.0}}$, $\chi_{339.5, t_{10.6.6, t_{17.3}}}$</td>
<td>$-0.759$</td>
<td>$-0.811$</td>
<td>$-0.214$</td>
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<tr>
<td></td>
<td>($t_{0.55.7}$)</td>
<td>($t_{0.55.7}$)</td>
<td>($t_{0.55.7}$)</td>
<td>($t_{0.55.7}$)</td>
</tr>
<tr>
<td>$a$</td>
<td>8.45</td>
<td>1.26</td>
<td>1.20</td>
<td>2.75</td>
</tr>
<tr>
<td></td>
<td>($t_{0.11.4}$)</td>
<td>($t_{0.59.9}$)</td>
<td>($t_{0.59.9}$)</td>
<td>($t_{0.59.9}$)</td>
</tr>
<tr>
<td>$b$</td>
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<td>-0.0033</td>
<td>-0.0294</td>
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<tr>
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<td>($t_{0.8.8}$)</td>
<td>($t_{0.516.6}$)</td>
<td>($t_{0.516.6}$)</td>
<td>($t_{0.516.6}$)</td>
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<tr>
<td>$r^2$</td>
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<td>0.51</td>
<td>0.51</td>
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<tr>
<td>$\eta_1$</td>
<td>-0.786, -0.027</td>
<td>-0.798, -0.084</td>
<td>-0.855, -0.112</td>
<td>-1.201, -0.070</td>
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</table>

$y=\gamma 2$

<table>
<thead>
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<th>Model 1</th>
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<th>Model 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\chi_{271.6, x_{154.2}}$, $\chi_{355.12, t_{10.2.1, t_{17.4}}}$</td>
<td>$-0.754$</td>
<td>$-0.946$</td>
<td>$-0.200$</td>
</tr>
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<td>($t_{0.909.6}$)</td>
<td>($t_{0.909.6}$)</td>
<td>($t_{0.909.6}$)</td>
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<tr>
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<tr>
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<td>-0.0041</td>
<td>-0.0021</td>
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<td>($t_{0.909.6}$)</td>
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</tr>
<tr>
<td>$r^2$</td>
<td>0.49</td>
<td>0.53</td>
<td>0.53</td>
<td>0.48</td>
</tr>
<tr>
<td>$\eta_1$</td>
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<td>-0.885, -0.108</td>
<td>-0.778, -0.116</td>
<td>-1.293, -0.076</td>
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$y=\gamma 3$

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<th>Model 2b</th>
<th>Model 3</th>
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<tr>
<td></td>
<td>$\chi_{0.207.7, x_{158.9}}$, $\chi_{558.9, t_{9.0.1, t_{20.0}}}$</td>
<td>$-0.572$</td>
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<td>($t_{0.811.3}$)</td>
<td>($t_{0.811.3}$)</td>
</tr>
<tr>
<td>$a$</td>
<td>4.43</td>
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<td>1.56</td>
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<td>0.53</td>
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$y=\gamma 4$

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<td>$\chi_{0.205.8, x_{153.6}}$, $\chi_{536.1, t_{8.8.1, t_{21.6}}}$</td>
<td>$-0.478$</td>
<td>$-0.485$</td>
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<td>($t_{0.341.2}$)</td>
<td>($t_{0.421.2}$)</td>
<td>($t_{0.421.2}$)</td>
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<td>($t_{0.55.6}$)</td>
</tr>
<tr>
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<td>-0.0756</td>
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<td>-0.0211</td>
<td>-0.0391</td>
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<tr>
<td>$r^2$</td>
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<td>0.53</td>
<td>0.54</td>
<td>0.50</td>
</tr>
<tr>
<td>$\eta_1$</td>
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<td>-1.169, -0.111</td>
<td>-1.159, -0.111</td>
<td>-1.423, -0.110</td>
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$n=521$: critical t, d.o.f n-3, 0.05=1. 1.6480, 0.1=1.2832

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### Table 6.3.18 (cont.)

<table>
<thead>
<tr>
<th>Model 1</th>
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<th>Model 2b</th>
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<tr>
<td><strong>Population Density</strong></td>
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<td>λ</td>
<td>0.37</td>
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<td>(χ₁₉₀₉₈, χ₁₅₃₆₀)</td>
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<td>(t₃₉.₅₅, t₄₁₉.₁)</td>
<td>(t₅₂₆.ₐ, t₅₇₆.₆)</td>
<td>(t₅₈.₅₈, t₅₉.₂₄₆)</td>
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<td>(t₉₈.₈)</td>
<td>(t₂₀₉.₈₉)</td>
<td>(t₉₃.₈₈)</td>
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<td>(t₉.₃₈.₇)</td>
<td>(t₅.₉₉)</td>
<td>(t₅.₉₃)</td>
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<tr>
<td>r²</td>
<td>0.32</td>
<td>0.29</td>
<td>0.29</td>
</tr>
<tr>
<td>ηₓ</td>
<td>-0.774, -0.031</td>
<td>-0.761, -0.107</td>
<td>-0.753, -0.096</td>
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</table>

| **Household Density** | | | |
| λ | 0.31 | -0.610 | -0.607 | -0.301 |
| (χ₁₃₄₆₄, χ₁₄₁₁₄) | (χ₁₃₇₂₆, χ₁₄₂₂₇) | (χ₁₃₃₆₈, χ₁₁₁₉₂) | (χ₁₆₃₅, χ₁₇₁₁) |
| (t₃₃.₅, t₇₄.₆) | (t₃₅.₅, t₁₆₇.₅) | (t₃₁₁.₄, t₁₉₂.₂) | (t₃₆.₃, t₇₇.₃) |
| a | 3.61 | 1.41 | 1.41 | 1.93 |
| (t₄₆.₂) | (t₇₉.₅) | (t₁₉₀.₂) | (t₂₁.₄) |
| b | -0.0774 | -0.0179 | -0.0180 | -0.0468 |
| (t₉.₃₄.₁) | (t₉.₃₉.₁) | (t₃₁₅.₄) | (t₁₇) |
| r² | 0.32 | 0.28 | 0.28 | 0.32 |
| ηₓ | -0.87₉⁻⁻⁻⁻₁₃₁ | -0.99₁⁻⁻⁻⁻₀.₁₁₉ | -1.₅₈₄⁻⁻⁻⁻₀.₇₁₀ | -1.₃₂₀⁻⁻⁻⁻₀.₁₀₈ |

n=3208: critical t, d.o.f. n-3, 0.05=1.64₅₅, 0.₁=1.₂₈₁₈

### Pooled (Reduced observations)

<table>
<thead>
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<th>Model 1</th>
<th>Model 2a</th>
<th>Model 2b</th>
<th>Model 3</th>
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<tr>
<td><strong>Population Density</strong></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>λ</td>
<td>0.39</td>
<td>-0.600</td>
<td>-0.8₅₀</td>
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<td>(χ₁₇₂₀₉, χ₁₂₁₉₈)</td>
<td>(χ₁₂₆₆₁, χ₁₃₂₀.₇)</td>
<td>(χ₁₄₃₉₄, χ₁₆₆₇)</td>
<td>(χ₁₃₃₇₅, χ₁₂₇₉₇)</td>
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<td>(t₃₈.₆₅, t₃₆₈.₁)</td>
<td>(t₃₅₂.₉₄, t₅₇₀.₆)</td>
<td>(t₄₆₄, t₂₄.₅)</td>
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<td>1.₅₁</td>
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<td>(t₁₈₃.₉)</td>
<td>(t₁₇₇.₉)</td>
<td>(t₁₃.₇₉)</td>
</tr>
<tr>
<td>b</td>
<td>-0.₁₁₂₀</td>
<td>-0.₀₀₇₇</td>
<td>-0.₀₀₃₇</td>
</tr>
<tr>
<td>(t₂₉.₈)</td>
<td>(t₃₉.₅)</td>
<td>(t₄₈₇)</td>
<td>(t₁₆₄₇)</td>
</tr>
<tr>
<td>r²</td>
<td>0.₃₂</td>
<td>0.₃₁</td>
<td>0.₅₀</td>
</tr>
<tr>
<td>ηₓ</td>
<td>-0.₇₅₃⁻⁻⁻⁻₀.₃ₐ₀</td>
<td>-0.₈₈₈⁻⁻⁻⁻₀.₀₇₈₀</td>
<td>-0.₇₈₃⁻⁻⁻⁻₀.₁₀₈</td>
</tr>
</tbody>
</table>

| **Household Density** | | | |
| λ | 0.₃₃ | -0.₄₅₃ | -0.₄₅₁ | -0.₃₉₂ |
| (χ₁₄₃₆₆, χ₁₃₁₆₃) | (χ₁₂₂₁₃, χ₁₄₂₆₉) | (χ₁₁₁₇₇, χ₁₉₅₄₅) | (χ₁₇₉₅₉₉, χ₁₆₉₈₄) |
| (t₂₆.₀, t₅₂.₇) | (t₄₃.₅, t₁₃₉.₇) | (t₂₈₃.₄, t₉₀₆.₉) | (t₆₁.₆, t₂₆.₇) |
| a | 3.₆₈ | 1.₆₅ | 1.₆₅ | 1.₉₄ |
| (t₄₀.₁) | (t₇₇.₀) | (t₆₀₂.₁) | (t₂₁₃) |
| b | -0.₀₇₅₄ | -0.₀₂₃₂ | -0.₀₂₃₃ | -0.₀₄₇₀ |
| (t₃₀.₈) | (t₃⁹.₁) | (t₃₈₄) | (t₁₇₃) |
| r² | 0.₃₂ | 0.₃₂ | 0.₃₂ | 0.₃₂ |
| ηₓ | -0.₈₅₈⁻⁻⁻⁻₀.₀₃₄ | -₀.₉₉₃⁻⁻⁻⁻₀.₀₈₈ | -0.₉₉₀⁻⁻⁻⁻₀.₀₈₇ | -1.₃₁₃⁻⁻⁻⁻₀.₁₀₅ |

n=3202: critical t, d.o.f. n-3, 0.0₅=1.₆₄₅₅, 0.₁=1.₂₈₁₈.
Chapter 7: Concluding Remarks.

The main thesis has attempted to apply the methods of duality to the analysis of the urban residential model, particularly with a view to replicating the residential lot size hypothesis. This hypothesis was then tested utilising urban density functions for various Australian cities. Even though the theoretical model is specific in the sense that it applies to residences, that residential model itself is an example of a general spatial model attributable to Von Thünen (1826). The thesis argues that the generic Von Thünen model can be analysed utilising dual methods and that such methods are not specific to the analysis of residential behaviour.

Where this paper has differed from other models within the generic spatial Von Thünen model is in the use of aspatial optimal value functions. These are solution functions to optimising problems and are characterised as defining optimal circumstances at every possible location so that the optimising problem is now one of choosing location. However, these aspatial optimal value functions require variable location prices to make the location problem interesting. Location price is the price of those things which are fixed in location and have to be purchased in order to take up a location. In this paper location price therefore refers to either the price of land or built space which has already been produced.

In order to say something sensible about a systematic variation of this price it was necessary to derive another optimal value function. This was the bid rent function. This was defined as that highest offer price at every location that a potential locator
was prepared to make subject to a constraint. This constraint could be expressed either in terms of levels of expenditure or utility in the consumer problem or in terms of cost or profit in the producer problem. The bid rent function is not new and owes its now prolific use in urban economics to Alonso (1960 and 1964). This paper has derived the bid rent function differently, by inverting the optimal value function stemming from the original optimising problem, such as residential utility maximisation, but where no *a priori* judgement has been made about the nature of the relationship between location price and location. Central locations in the generic Von Thünen model are relatively scarce so that competition for scarce location was then used to derive location price either in the form of land or built space price. This price would therefore have some systematic variation in location and would act to allocate scarce location. Clearly the relationship between the market price for land or built space and location was dominated by the relationship between the offer price for land or space and location.

Having established the nature of the price for locationally fixed goods, which is an argument of the aspatial optimal value function such as the profit function in the producer problem, it was possible to:

1. Identify the conditions for optimal location.
2. Retrieve the demand function for the locationally fixed good through Hotelling’s Theorem, Shephard’s Lemma or Roy’s Identity.

Alternatively, it was possible to retrieve the demand function for the locationally fixed good through the bid price function. This was a spatial adaptation of Hotelling’s
Theorem, Shephard's Lemma or Roy's Identity. The demands retrieved in this way would be conditioned on levels in the original optimal value function from which the bid rent function had been derived, and actually abstracted from own price. These restrictions on the retrieved demand apply because the bid rent function defines a maximum offer price in order to achieve a level of utility, expenditure, profit or cost. Even though the demand function retrieved in this way was highly conditioned, it allowed decomposition of those factors influencing variation in the demand for the locationally fixed good over location.

The bid rent function was also used to identify the characteristics of an optimal location. This was done through the problem of minimising the difference between offer and market rent subject to a market rent constraint. The bid rent function was further used to identify minimum or maximum distance located from centre, where the difference between offer rent and market rent constrained the direct choice of these locations. The optimal value function associated with this latter problem was not fully explored simply because the focus of this thesis was on the relationship between the demand for land or built space and location. However, it was noted that this minimum or maximum distance function could be used for comparative statics around the extent of location in radial distance from centre for the particular activity under scrutiny.

The demand for locationally fixed goods, such as land or built space, retrieved from the system of dual functions were general functions of location. This resulted in propositions on the relationship between demand and location which not only
contained the traditional lot size hypothesis, but also encompassed alternative patterns of land or built space utilisation within a monocentric framework.

This level of generality was carried over into the empirical thesis. Various flexible functional form models were used to estimate urban density gradients. This necessitated discrimination of these flexible models via non-nested tests on an artificial regression model encompassing these alternate flexible form models. Discrimination between the alternative flexible form models was surprisingly successful on the adoption of conservative significance levels for the test statistic. This was justified on the grounds that the finite sample properties of the test statistic are unknown.

The parameter estimates of the density gradients were not inconsistent with the traditional lot size hypothesis. In the majority of cases the estimated slope parameter of the density gradient was significant and of the correct sign. However, beyond this, outcomes were mixed. Firstly, doubts must be expressed about the generality of a distance/density relationship for all state capitals. For one of the six cities, Hobart, the slope parameter was not significant for some estimated density gradients. Moreover, for two cities, Brisbane and Hobart, it was not possible to discriminate between the linear and log linear form of the density gradient. For these cities, the density/distance relationship was weak, possibly accounting for this lack of discrimination between specifications. Secondly, for three of the remaining four cities, it was possible to discriminate between the linear and log linear specifications, these cities ultimately testing down to the log linear form. However, for the
remaining city, Sydney, both the linear and log linear forms were rejected, indicating some other non-linear inverse relationship between distance and density.

It could be argued that the monocentric model is too restricted for much further development. Certainly, elements in the urban economics literature now seem to be pointing in the direction of polycentric forms. See, for example, Anas and Kim (1996), McDonald and McMillen (1990) and McDonald and Prather (1994). The analytical method used in this thesis could be developed in this direction. Spatial adaptations of optimal value functions familiar in standard optimisation theory could be a useful tool generally in locational economics and specifically in polycentric urban modelling. Polycentric cities could be examined through aspatial optimal value functions which are themselves functions of northings and eastings on a metric grid laid over a city/region. This would certainly be general enough to encompass polycentric as well monocentric cities. Care must be taken. It appears that essential to any fruitful location analysis is the systematic variation of location price, through land or built space, in the aspatial function. Whilst trend surface analysis has been utilised in forming surfaces across a two dimensional grid in geography, the surfaces themselves tend to be empirical and descriptive. Any kind of analysis about the forces forming the surface tends to take place outside of the model used to estimate the surface. That is, they tend to be functions only of eastings and northings and could form the basis upon which to induce theories to explain the surface. Similar pitfalls could await the use of such methods in economics. It could be argued that, in order to produce reasonable aspatial optimal value functions which are surfaces in northings and eastings, a priori judgements would have to be made about the way that location prices vary across the two dimensional grid. In other words, location prices
might be treated exogenously when in fact they should be formed within the model. If this was the case then the result might simply be a complex description of a set of circumstances which have been predetermined by a priori reasoning.

The theoretical thesis has demonstrated that it is possible, through the utilisation of duality theory, to generalise models of location where economic actors are, in choice of location, subject to centripetal and centrifugal economic forces. The thesis has shown that the full application of duality method allows the development of novel optimal value functions stemming from several dual location problems. These optimal value functions, through their envelope properties, lead to useful comparative statics in location. Finally, in terms of empirical tests of locational hypotheses, the empirical thesis has demonstrated a method which allows the data freedom to choose function form within alternate flexible functional form models. Further, it has been shown that it is possible to use a specification test based on semi-parametric methods to successfully discriminate between alternative flexible functional form models.

A.1 Estimation.

Model I:
Criterion:
Maximum Likelihood
Log of Likelihood Function

\[ L(\lambda, \beta, \sigma^2) = -\frac{N}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \left( y^{(\lambda)} - X\beta \right) \left( y^{(\lambda)} - X\beta \right) + \ln J \]

where the log of the Jacobian is:

\[ \ln J = \left( \lambda - 1 \sum_{i=1}^{N} \ln y_i \right) \]

Estimation:
QMLE, maximise the concentrated log-likelihood function \( L^* \) with respect to \( \lambda \).
Then use OLS to estimate remaining parameters \( \beta \). Where:

\[ L^*(\lambda) = \left( \lambda - 1 \sum_{i=1}^{N} \ln y_i \right) - \frac{N}{2} \left( \ln(2\pi) + 1 \right) - \frac{N}{2} \ln \hat{\sigma}^2 \]
**Models 2 & 3:**

Criterion:

NLS

Minimise the objective function:

\[ S(\beta) = (y - f(X, \beta))' (y - f(X, \beta)) \]

The equivalent to the Box Cox transformation parameter \( \lambda \) is contained within the parameter set \( \beta \) of the function \( f \).

**Model 2b:**

Criterion:

WNLS

Minimise the objective function:

\[ S(\beta) = \sum_{i=1}^{N} w_i [y_i - f(x_i, \beta)]^2 \]

where:

\[ w_i = \frac{N\omega_i}{\sum_{j=1}^{N} \omega_j} \]

and \( \omega_i \) is the reciprocal of the squared error of the unweighted regression
A.2 $t$ Statistics.

Model 1:

The estimated covariance matrix is after Berndt, Hall, Hall and Hausman (1974):

$$(MLE) \Sigma_{b, \hat{\lambda}, \hat{\sigma}^2} = [\hat{W}^T \hat{W}]^{-1}$$

with typical element being the derivative of the log-likelihood with respect to the unknown parameters evaluated by the parameter estimates, that is (Greene, 1997, p. 484)

$$\frac{\partial L_1}{\partial \beta} = \frac{y_i^{(\lambda)} - x_i \beta}{\sigma^2} x_i$$

$$\frac{\partial L_1}{\partial \lambda} = \ln y_i - \frac{y_i^{(\lambda)} - x_i \beta}{\sigma^2} \left[ \frac{1}{\lambda} \left( y^\lambda \ln y - y^{(\lambda)} \right) \right]$$

$$\frac{\partial L_1}{\partial \sigma^2} = \frac{1}{2\sigma^2} \left( \frac{y_i^{(\lambda)} - x_i \beta}{\sigma^2} \right)^2 - 1$$

Models 2 and 3:

The estimated covariance matrix is:

$$\hat{V}_s(b, \hat{\lambda}) = \hat{\sigma}^2 [\hat{G}^T \hat{G}]$$
A typical element of $\hat{G}$ is the derivative of the least squares criterion from numeric approximation. For 2a and 3 these were checked against

$$\hat{V}(b, \lambda) = \hat{\sigma}^2 [\hat{Z}'\hat{Z}]$$

With typical element of $\hat{Z}$ being the analytic derivative of the least squares criterion evaluated at the parameter estimates.

A.3 Derivatives and Elasticity.

Model 1:

For derivatives refer to section A2 of this appendix.

Model 2:

(Wooldridge, 1992, p. 939)

$$y_i = (\lambda x_i^\beta + 1)^X$$

$$\frac{\partial y_i}{\partial \beta} = \frac{1}{\lambda} (\lambda x_i^\beta + 1)^{(X-1)} x_i$$

$$\frac{\partial y_i}{\partial \lambda} = (\lambda x_i^\beta + 1)^X \frac{\lambda x_i^\beta - (\lambda x_i^\beta + 1) \ln(\lambda x_i^\beta + 1)}{\lambda^2 (\lambda x_i^\beta + 1)}$$
Model 3:

\[ \ln y_i = \frac{1}{\lambda} \ln(\lambda x^\beta + 1) \]

\[ \frac{\partial \ln y_i}{\partial \beta} = \frac{x_i}{\lambda x^\beta + 1} \]

\[ \frac{\partial \ln y_i}{\partial \lambda} = \frac{1}{\lambda} \left[ \frac{x_i^\beta}{\lambda x^\beta + 1} - \frac{1}{\lambda} \ln(\lambda x^\beta + 1) \right] \]

For all models elasticity is:

\[ \eta_{ij} = \beta_j x_j \frac{x_j}{\lambda x^\beta + 1} \left\{ \beta_j \mid \beta ; x_j \mid x \right\} \]
References


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