RADIOASTRONOMICAL INSTRUMENTATION

THE DIAGONAL HORN

by

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PLEASE NOTE

The greatest amount of care has been taken while scanning this thesis,

and the best possible result has been obtained.
DECLARATION

I hereby declare that this thesis does not incorporate any material previously submitted for any other degree award and it does not contain my material previously published by another person except where specifically referenced.
Summary

The horn plays an elemental role in the make up of a radio-telescope. The focus of this research is on one particular type of horn – the diagonal horn.

An analysis of the diagonal horn is made using the Fourier method. The analysis begins from Maxwell’s equations, as the basic building block, and describes the steps involved in developing the radiation pattern.

Based on the theory, a program was written that produces the theoretical graphs referred to throughout the thesis. A diagonal horn was manufactured and the radiation patterns were measured. A comparison of these measured patterns is made against the theoretically generated patterns.

Further research was carried out to demonstrate the effects on the radiation patterns when the horn is fitted with a dielectric plug. This practice may enhance the directivity of the horn at the cost of introducing new losses.
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<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>length of aperture side</td>
</tr>
<tr>
<td>a_r</td>
<td>space radial position vector - x, y, z plane</td>
</tr>
<tr>
<td>a_R</td>
<td>space radial position vector - X, Y, Z plane</td>
</tr>
<tr>
<td>A</td>
<td>constant</td>
</tr>
<tr>
<td>A(p, φ, θ, x, y, z)</td>
<td>vector potential of the current density - electric field</td>
</tr>
<tr>
<td>A(X, Y, Z, x, y, z)</td>
<td>vector potential of the current density - electric field</td>
</tr>
<tr>
<td>b</td>
<td>length of other aperture side (rectangular aperture)</td>
</tr>
<tr>
<td>B</td>
<td>constants</td>
</tr>
<tr>
<td>B</td>
<td>magnetic field density</td>
</tr>
<tr>
<td>C_1 - C_9, C</td>
<td>unknown constants</td>
</tr>
<tr>
<td>d</td>
<td>diameter of the parabolic reflector</td>
</tr>
<tr>
<td>d_a</td>
<td>distance between focus point and actual feed position</td>
</tr>
<tr>
<td>D</td>
<td>electric displacement density</td>
</tr>
<tr>
<td>E</td>
<td>electric charge</td>
</tr>
<tr>
<td>E(x, y, z, t)</td>
<td>electric field</td>
</tr>
<tr>
<td>E_x(x, y)</td>
<td>electric field in the x-direction</td>
</tr>
<tr>
<td>E_y(x, y)</td>
<td>electric field in the y-direction</td>
</tr>
<tr>
<td>E_normal(x, y)</td>
<td>electric field in the x-direction</td>
</tr>
<tr>
<td>E_cross_polarised(x, y)</td>
<td>electric field in the y-direction</td>
</tr>
<tr>
<td>E_0(x, y)</td>
<td>electric field in the z-direction</td>
</tr>
<tr>
<td>E_{0,xy}</td>
<td>constant</td>
</tr>
<tr>
<td>E_F</td>
<td>electric flux density due to magnetic field</td>
</tr>
<tr>
<td>E_{θF}</td>
<td>electric flux density due to magnetic field in θ-direction</td>
</tr>
<tr>
<td>E_{φF}</td>
<td>electric flux density due to magnetic field in φ-direction</td>
</tr>
<tr>
<td>E_A</td>
<td>electric flux density due to electric field</td>
</tr>
<tr>
<td>E_{θA}</td>
<td>electric flux density due to electric field in θ-direction</td>
</tr>
<tr>
<td>E_{φA}</td>
<td>electric flux density due to electric field in φ-direction</td>
</tr>
<tr>
<td>E_θ</td>
<td>total electric flux density in θ-direction</td>
</tr>
<tr>
<td>E_φ</td>
<td>total electric flux density in φ-direction</td>
</tr>
<tr>
<td>f</td>
<td>effective focal length</td>
</tr>
<tr>
<td>F(p, φ, θ, x, y, z)</td>
<td>vector potential of the current density - magnetic field</td>
</tr>
<tr>
<td>F(X, Y, Z, x, y, z)</td>
<td>vector potential of the current density - magnetic field</td>
</tr>
<tr>
<td>H</td>
<td>magnetic field strength</td>
</tr>
<tr>
<td>H(x, y, z, t)</td>
<td>magnetic field</td>
</tr>
<tr>
<td>H_x(x, y)</td>
<td>magnetic field in the x-direction (square horn)</td>
</tr>
<tr>
<td>H_y(x, y)</td>
<td>magnetic field in the y-direction (square horn)</td>
</tr>
<tr>
<td>H_normal(x, y)</td>
<td>magnetic field in the x-direction (diagonal horn)</td>
</tr>
<tr>
<td>H_cross_polarised(x, y)</td>
<td>magnetic field in the y-direction (diagonal horn)</td>
</tr>
<tr>
<td>H_z(x, y)</td>
<td>magnetic field in the z-direction</td>
</tr>
<tr>
<td>J</td>
<td>current density</td>
</tr>
</tbody>
</table>
J(x, y, z, t) electric surface current density
J_x(x, y) electric surface current density in the x-direction
J_y(x, y) electric surface current density in the y-direction
J_z(x, y) electric surface current density in the z-direction
k \frac{2\pi}{\lambda}
K constant depending on the beam width
L flared edge length
L^*(\rho, \phi, \theta) = \int M(x, y) e^{iK(x \sin \theta \cos \phi + y \sin \theta \sin \phi)} dS, part of auxiliary function F(\rho, \phi, \theta, x, y)
L_\theta(\phi, \theta) part of F(\rho, \phi, \theta, x, y) in \theta-direction
L_\phi(\phi, \theta) part of F(\rho, \phi, \theta, x, y) in \phi-direction
L_{\theta\phi}(\theta) part of F(\rho, \phi, \theta, x, y) in \theta-direction with a known \theta
L_{\phi\phi}(\theta) part of F(\rho, \phi, \theta, x, y) in \phi-direction with a known \theta modes
M(x, y, z, t) magnetic surface current density
M_x(x, y) magnetic surface current density in the x-direction
M_y(x, y) magnetic surface current density in the y-direction
M_z(x, y) magnetic surface current density in the z-direction
n number of sides on polygon
n unit normal to the surface, s
N number of sides on polygon
N^*(\rho, \phi, \theta) = \int J(x, y) e^{iK(x \sin \theta \cos \phi + y \sin \theta \sin \phi)} dS, part of auxiliary function A(\rho, \phi, \theta, x, y)
N_\theta(\phi, \theta) part of A(\rho, \phi, \theta, x, y) in \theta-direction
N_\phi(\phi, \theta) part of A(\rho, \phi, \theta, x, y) in \phi-direction
N_{\theta\phi}(\theta) part of A(\rho, \phi, \theta, x, y) in \theta-direction with a known \theta
N_{\phi\phi}(\theta) part of A(\rho, \phi, \theta, x, y) in \phi-direction with a known \theta
\rho_v free charge enclosed by a surface
\rho_p polarisation of dielectric material
P point on reflector
P(\rho, \phi, \theta) point in space
r distance between origin and P and P(\rho, \phi, \theta)
R distance between aperture point and P(\rho, \phi, \theta)
S surface area
t time
U undersampling
x value in x-direction where system has the origin in the centre of aperture plane and propagates in z direction - near field
x \ x'-direction
x' value in x'-direction where system has the origin in the

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corner of aperture plane and propagates in z direction -
nefar field.

\( X \)
value in x-direction where system has the origin in the
centre of aperture plane and propagates in z direction -
far field

\( y \)
value in y-direction where system has the origin in the
centre of aperture plane and propagates in z direction -
nefar field

\( y' \)
y-direction

\( y' \)
value in \( y' \)-direction where system has the origin in the
centre of aperture plane and propagates in z direction -
nefar field.

\( Y \)
value in y-direction where system has the origin in the
centre of aperture plane and propagates in z direction -
far field

\( z \)
value in z-direction where system has the origin in the
centre of aperture plane and propagates in z direction -
nefar field

\( z' \)
z-direction

\( z' \)
value in \( z' \)-direction where system has the origin in the
centre of aperture plane and propagates in z direction -
nefar field.

\( Z \)
value in z-direction where system has the origin in the
centre of aperture plane and propagates in z direction -
far field

\( Z_0 \)
intrinsic impedance

\( \Delta x \)
linear antenna spacing in the focal plane

\( \Delta x_{\text{Nyquist}} \)
Nyquist sampling rate

\( \lambda \)
Wavelength

\( \pi \)
\( \text{Pi } = 3.14159 \)

\( \varepsilon_0 \)
free space permittivity

\( \varepsilon_r \)
relative permittivity

\( \omega \)
Frequency

\( \mu_0 \)
free space permeability

\( \mu_r \)
relative permeability

\( \Psi \)
angle subtended between \( r \) and \( xy \) plane


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ACKNOWLEDGMENTS

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Chapter 1

INTRODUCTION

Cosmic signals originate from molecular ions in gaseous forms and molecular transitions, which travel a great distance through space to reach Earth. Here at the Earth's surface these signals are mixed with other signals from space or with signals that are generated on the Earth itself, such as those from radio, television, microwave and mobile telecommunications, just to name a few. In fact, this is how cosmic signals were discovered. Back in 1932, Karl G. Jansky intended to measure the noise in a radio system across the Atlantic Ocean. In doing so, he discovered several noises in his receiver and concluded that these noises came from thunderstorms and from another constant steady noise source - the centre of our galaxy, where there is a concentration of stars, dust and gas. This was the beginning of the quest to understand our galaxy. The very first steerable radio-telescope was built by Grote Reber to determine from where the radiation was coming. By 1944, he was able to publish some measurements of cosmic radio waves. After World War II, interest in astronomy bloomed and a great deal of development went into radio-telescopes and towards the techniques associated with it. Radio-telescope structures evolved from a simple parabolic type, to Cassegrian ones that contain sub-reflectors for the purpose of reducing the overall structure size. The development of various horns, such as sectorial, rectangular, diagonal, and circular, followed, along with corrugated ones. It was found that more measurements could be taken at the one time by having several horns arranged in the focal plane. This opened up research on mutual coupling, where the signal is reflected back from an adjacent horn. Analogue electronics developed significantly as well, increasing the signal to noise ratio in the receiver by cryogenically cooling them to low temperatures to minimise noise. Increasingly faster digital electronics also form part of the equation, where the captured data is processed by sophisticated software to produce meaningful pictures of the space around us. Each of these developments are interesting and resulted in a very highly developed technological instrument, the radio-telescope, for the purpose of detecting any signal coming from afar.
The aim of this thesis is to focus on the horn, in particular the diagonal horn. In order to acknowledge the significance of the horn in the radio telescope the rest of this chapter is devoted to brief descriptions of the most significant elements of a the radio telescope.

**Radio-telescopes**

Radio-telescope systems can be split into four significant elements. These are, the reflector, the feed, the receiver and the processing software.

**Reflectors**

The reflector's significant role is to capture a constant signal some distance away. Basically, the sensitivity and noise of the radio-telescope will depend on the reflector's size, structure and movability. The reflector's size and structure will influence the feed, so these two factors will be elaborated on.

**Reflector Size**

Consider the situation where a signal somewhere in space can be viewed by our radio-telescope. Now assume that this signal has become stronger making it much easier for the radio-telescope to detect. Applying the reciprocal theorem, by switching roles, assume that this signal generated by a molecular transition is constant and steady, and that some receiver is to detect it some distance away. Assume then that the receiver has become bigger in the last upgrade making the signal easier to be detected. The angular area of the sky that can be detected becomes a reciprocal of the reflector width. The following formula is a popular one found among several optical references\(^{[1][3][5][8]}\).

\[
\theta = \frac{\lambda}{b}
\]

where \(\theta\) is the beam angle in radians, \(\lambda\) the wavelength of the signal and \(b\) is the beamwidth.

Equation 1.1
This concludes that the larger the reflector the smaller the signal that can be detected.

*Reflector Structure*

The structure of the reflector must be such a shape so that all the plane waves coming from the sources are reflected towards a focal point where the feed is located.

![Diagram of parabolic antenna structure](image)

*Figure 1.1 - Parabolic antenna structure.*

Referring to Figure 1.1, assume a plane wave travelling along the path A to P, arrives at the focus, F with a particular phase. Any other plane wave that reached the focal plane can take any other path to reach the focus and still have the same phase. This means any signal from the focal plane, AF, must travel the same distance to reach the focus, F, with the same phase. This is a definition of a parabolic curve.

A ray travelling from F to point O, and then back the focal plane would have travelled a distance equal to 2OF. Therefore, any other path taken must have travelled the same distance. Another path that can be considered in terms of the coordinates of the point P(x, y) is:
\[ 2 \cdot OF = \sqrt{(OF - x)^2 + y^2} + OF - x \]

\[(2 \cdot OF - OF + x)^2 = (OF - x)^2 + y^2\]

\[OF^2 + 2x \cdot OF + x^2 = OF^2 - 2x \cdot OF + x^2 + y^2\]

\[4x \cdot OF = y^2\]

Equation 1.2

This gives the unique equation defining a parabolic shape. Practically, it is very difficult to provide exactly the same distance for every travelling path causing some loss in signal power. This is called Ruze losses.

Within the parabolic structure, the feed is located at the focus along the geometric axes on the focal plane that is blocking the centre of the source, known as Coma distortion. The larger the feed, the greater the distortion.

Another disadvantage is that the feed is pointing to the reflector and behind it is the Earth. It is critical that the beamwidth of the feed is smaller with respect to the parabolic reflector, so that spillover effects are minimised. Signals can reflect off the Earth’s surface causing more unwanted interference. Radiotelescopes can also have folded optics type of structure called ‘Cassegrian’ (Figure 1.2) which consists of the usual parabolic reflector as well as a subreflector. The feed is placed in the centre of the main parabolic reflector pointing to space rather than Earth.
Figure 1.2 - Cassegrain telescope structure.

In the previous structure (Figure 1.1), the horn was placed at the focus, F', of the main parabolic reflector. In the Cassegrain structure another focus, F, is required on the axis near the main reflector where all the rays must reach. This new focus is where the feed will be located in the Cassegrain system. Any ray that reflects off the subreflector requires to be directed at the focus, F. In order to determine the shape of the subreflector a circle with the centre, M passes through both foci, F and F'.

Using geometry, the distance from point P(x,y) to the F' is related as follows:

\[ PF' = SF' + PS' \]

\[ PF'' = (MF'' - MS) + PS' \]

And the distance from point P(x,y) to F is related as:

\[ PF = SF + PS \]

\[ PF'' = (MF'' - MS) + PS' \]

\[ = ( |MF| - |MF'' - MS| \sqrt{1 + D^2}) \]
These two equations (i.e. the distance $PF$ and $PF'$) define the two foci of a hyperbola. So the subreflector has a hyperbola shape.

Due to these properties the focal length is increased to:

$$\text{apparent focal length} = \frac{FF\cdot SF}{SF'}$$

An advantage of such a system is that the horns need to be larger so that they can cover the whole angle of the subreflector. Another advantage is that the structure's apparent length is large compared to the actual size of the structure. So the radio-telescope structure can be smaller in this folded system and just as effective as a very large non-folded structure. The reflector is not required to be larger in order to avoid possible stray rays (unwanted noise), because with the folded system the horns point at the sky (considered to be a black body), and to the source of the signal, which is not noise. This structure allows the receiver to be much more accessible.
Feeds
The feed captures electromagnetic waves and couples them on to transmission lines. The feed consists of a horn or many horns specifically designed for a particular frequency. Since with antennas the reciprocity theory holds then the characteristics of a receiving antenna must be similar to a transmitting antenna. Generally, to determine the horn’s directivity and power pattern, it is assumed that the horn is sourced by a steady signal and when viewing it at some distance away, the pattern that has resulted from the horn transmitting it. Basically, to determine the gain of the antenna, the far field pattern of the antenna is calculated when it is transmitting a unit source. A particular horn is built specifically for a desired directivity and far field pattern.

Figure 1.3 – Beam width at the focus of a parabolic antenna.

The physical dimension of the horn is interlocked with the size of the reflector of the radio-telescope.

Assume a horn with diameter $A_1A_2$. The rays from point $P_1$ to $A_1$ and $A_2$ have a difference in path length of $\lambda$. The diameter $A_1A_2$ of the horn is much smaller than the focal distance $A_0P_0$. This corresponds to Fraunhofer diffraction so Young’s assumptions can be applied meaning the angle $P_1A_0P_0$ is approximately equal to the angle $SA_1A_2$ since both are so small. Using the sin ratio,

$$\sin \angle P_1A_0P_0 = \frac{\lambda}{A_1A_2}$$
The angle of concern is so small, then $\sin \angle P_1A_0P_0 \approx \angle P_1A_0P_0$ giving the phase difference as:

$$\angle P_1A_0P_0 = \frac{P_1P_0}{P_0A_0} = \frac{\lambda}{A_1A_2}$$

The variable in question is the diameter of the horn, so this equation needs to be converted in terms of length $P_1P_0$. As the horn is moved further away, it can be seen that the diameter increases with the length $P_0P_1$, so

$$P_1P_0 = P_0A_0 \frac{\lambda}{A_1A_2}$$

Equation 1.3

Resolution of the two circular apertures is determined by applying the Rayleigh criterion. The apertures are said to be resolved only when the central maximum intensity of one falls on the first dark ring of the other. The first dark ring according to Airy’s disk occurs at $1.220f^{[50]}$ multiplied by the phase difference. Now the formula becomes:

$$P_1P_0 = 1.220P_0A_0 \frac{\lambda}{A_1A_0} \Rightarrow d = 1.220f \frac{\lambda}{a}$$

Equation 1.4

In terms of the radio telescope system parameters:

- $P_1P_0$, $d$ - diameter of the reflector
- $P_0A_0$, $f$ - focal length
- $A_1A_0$, $a$ - beam diameter
- $\lambda$ - operating wavelength
There has been a lot of research in improving the efficiency of the radio-telescope and by placing more horns in the focal plane, several measurements can be taken simultaneously. The array format is driven by the horn that is intended to be used. If the horn has a circular aperture then the hexagonal format is closer packed together than a square format. However, for the diagonal horn, which has a square aperture, the square format is the most compact. By closely packing horns together, coupling (known as mutual coupling) between them affects the far field patterns.

Receivers

Radio signals obtained by the horn are very weak. The receiver is to filter the signal at the particular frequency band from the noise so it can be processed. Noise includes the three degrees Kelvin, known as background noise from space, Earth's atmosphere and the noise produced within the receiver. It is critical for the receiver to filter out the signal from all the noise. Radio-telescope receivers generally have four stages.

1. The low noise pre amplifier, which is the most significant in reducing noise. Receivers are cryogenically cooled to temperatures of 15 K to minimise noise caused by thermal excitation of electrons.

2. The local oscillator is used to transform the signal frequency to a lower intermediate frequency that is easier to process. This is done by combining the local oscillator frequency with the signal frequency and then filtering the unwanted sidebands.

3. The square law detector, which changes the voltage measurement of the signal to a power measurement. This is done because our interest is the power of the signal and power is proportional to voltage squared.

4. The analogue to digital converter, which converts the analogue measured power voltages to digital, so they can be stored and processed by computers.

There are several techniques that can be used to reduce the noise. One is by taking several measurements at slightly different frequencies. The background
noise will then be common to all the frequencies measured and can be accounted for within the processing. Another method is by correlation, where the radio-telescope has several receivers. The voltage signals from the receivers are multiplied together and then are averaged to determine the power voltage. This reduces the noise because it is averaged out amongst the receivers.
Chapter 2

NEAR FIELD AND FAR FIELD

The diagonal horn’s near field will be derived from the basic building blocks, which are Maxwell’s equations. Following that, by applying diagonal horn boundary conditions to the wave equations (Helmholtz Equation), the near field is obtained. These near field equations are further manipulated by the Diffraction Theory to obtain the far field pattern, which shows the unique electromagnetic pattern from which directional characteristics can be analysed.

Near Field (Hoftman Fields)
The diagonal horn near field can be calculated by breaking down the structure into steps. First the diagonal starts as a square waveguide. A sectoral horn is one that has the E-plane sides flared out, while the H-plane sectoral horn has the H-plane sides flared out. A horn with both sides flared out is called a rectangular horn. The diagonal horn has a 45-degree twist along the axis. That is our final diagonal near field.

Rectangular Waveguide
The rectangular waveguide can be analysed from Maxwell’s Equations to determine the electromagnetic field. Boundary conditions are then applied by assuming that the walls of the guide are a perfect conductor in a ground plane.

Maxwell’s equations\cite{34} consist of four expressions. These equations are the basic tools for any electromagnetic problem.

Gauss’s Electric Law: The total electric flux density, \( \varepsilon_0 E \), originating from the closed surface equals to the free charge, \( \rho_v \), enclosed by this surface. Also included in the equation is a constant, \( \rho_p \), for the polarisation of the dielectric material by any external applied field.

\[
\nabla \cdot \varepsilon_0 E = \rho_v + \rho_p
\]
In a linear dielectric material with a relative permittivity, $\varepsilon_r$, it has been found that the polarisation is linearly proportional to the field.

$$\nabla \cdot \varepsilon_r \varepsilon E = \rho_v$$

$$\nabla \cdot D = \rho_v$$

$$\oint D dA = \rho_v$$

Equation 2.1

**Gauss's Magnetic Law**: Unlike electrical charges, magnetic charges do not exist physically in any material. The magnetic field density, $B$, over a closed surface, totals to zero.

$$\nabla \cdot B = 0$$

$$\oint B dA = 0$$

Equation 2.2

Basically, it states that there is no such thing as a magnetic charge to generate magnetic fields and lines representing the magnitude of the flux are closed circuits.

**Faraday's Law**: This law states that a negative time derivative of the magnetic flux through a surface is equal to the induced electric field around the closed loop bounding the surface. In other words, if an electric charge, $E$, is applied, then there will be a change in the magnetic field, $B$. This theory is equated by:

$$\nabla \times E = -\frac{dB}{dt}$$

$$\oint E dl = -\frac{d}{dt} \oint B \cdot dA$$

Equation 2.3

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THE DIAGONAL HORN
Ampere's Law: Any type of changing current with respect to time, whether it is current density or field, is a source of magnetic fields. So, if a magnetic field exists it can be induced either from the current density, \( J \), or the electric field in the medium. In this case the equation states that the changing electric field is equivalent to the total current density including the displacement current with time.

\[
\nabla \times H = J + \frac{dE}{dt}
\]

\[
\oint H \cdot ds = \int J \cdot dA + \frac{d}{dt} \int \varepsilon_0 \varepsilon \cdot dA
\]

Equation 2.4

Electric and magnetic fields from the waveguide are propagating in space, so by using the rectangular coordinates, they must be defined in terms of the directions \( x, y \) and \( z \). The total field can be factorised into components of directional fields that travel in the three directions. The field in the \( z \)-direction can be further split into the aperture field component and propagating one. Not to forget the fourth dimension, time, is also a dependent component. So the fields become:

\[
E(x', y', z', t) = E_x(x', y')E_y(x', y')E_z(x', y')e^{-j\omega z t}
\]

\[
H(x', y', z', t) = H_x(x', y')H_y(x', y')H_z(x', y')e^{-j\omega z t}
\]

where \( \gamma \) is the propagation constant in the \( z \)-direction and \( \omega \) is the frequency.

Substituting into Faraday's Law (Equation 2.3), applying one of the constitutive relations \( B = \mu_0 \mu H \) and considering the definition of the fields as stated above, the equation becomes:

\[
\nabla \times E_x(x', y')E_y(x', y')E_z(x', y')e^{-j\omega z t} = -\frac{d}{dt}(\mu_0 \mu H_x(x', y')H_y(x', y')H_z(x', y')e^{j\omega z t})
\]
Now looking at Ampere’s Law (Equation 2.4), also substituting a constitutive relation \((J = \sigma E)\) and including the fields as they have been described, leads to:

\[
\nabla \cdot H_z(x', y') H_y(x', y') H_x(x', y') e^{-j\omega t + j\Phi} = \sigma \epsilon \nabla \cdot E_z(x', y') E_y(x', y') E_x(x', y') e^{-j\omega t + j\Phi}
\]

\[
\nabla \cdot H_x H_y H_z(x, y) e^{-j\omega t + j\Phi} = \sigma \epsilon \nabla \cdot E_x E_y E_z e^{-j\omega t + j\Phi}
\]

Manipulating, expanding and simplifying gives a summary of these two considerably long equations into the following shorter six:

\[
\frac{dH_z(x', y')}{dy'} + \gamma H_y(x', y') = j \omega \epsilon \epsilon_0 E_z(x', y')
\]

\[
\frac{dH_y(x', y')}{dx'} + \gamma H_z(x', y') = -j \omega \epsilon \epsilon_0 E_y(x', y')
\]

\[
\frac{dH_z(x', y')}{dx'} - \frac{dH_x(x', y')}{dy'} = j \omega \epsilon \epsilon_0 E_z(x', y')
\]

\[
\frac{dE_z(x', y')}{dy'} + \gamma E_y(x', y') = -j \omega \mu_0 \mu H_z(x', y')
\]

\[
\frac{dE_y(x', y')}{dx'} + \gamma E_z(x', y') = j \omega \mu_0 \mu H_y(x', y')
\]

\[
\frac{dE_x(x', y')}{dx'} - \frac{dE_z(x', y')}{dy'} = -j \omega \mu_0 \mu H_x(x', y')
\]

Equation 2.5

Further manipulation by substituting and simplifying results in the popular Helmholtz wave equation in terms of the axial magnetic Hz:

\[
\frac{d^2 H_z(x', y')}{dx'^2} + \frac{d^2 H_z(x', y')}{dy'^2} + \gamma^2 H_z(x', y') = -\omega^2 \mu_0 \mu \epsilon \epsilon_0 H_z(x', y')
\]

A solution can be obtained from the wave equation since it becomes a double differential equation that has been analysed in mathematical and electromagnetic theory texts. The electric field propagating in the z-direction can further be factorised in terms of the x and y direction so that a solution for these directional fields can be obtained. The plane field is then obtained by multiplying the two together, which gives:

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- THE DIAGONAL HORN -
\[ H_z(x', y') = C_1 C_2 \cos Bx' \cos Ay' + C_3 C_4 \cos Bx' \sin Ay' + C_3 C_2 \sin Bx' \cos Ay' + C_4 C_1 \sin Bx' \sin Ay' \]

where \( C_1, C_2, C_3, C_4, A \) and \( B \) are unknown constants.

Before proceeding any further, there are four more important equations that evolve from more mathematics applied to the wave equation. These equations show the relation between the directional fields on the plane in terms of the directional propagating fields. After applying the boundary conditions to simplify \( E_z(x', y') \) or \( H_z(x', y') \) any other field can be calculated. So,

\[
H_z(x', y') = \frac{\gamma}{\gamma^2 + \omega^2 \mu_0 \mu_r \epsilon_r} \frac{dH_z(x', y')}{dx'} + j \frac{\omega \epsilon_r \epsilon_r}{\gamma^2 + \omega^2 \mu_0 \mu_r \epsilon_r} \frac{dE_z(x', y')}{dy'}
\]

\[
H_y(x', y') = -\frac{\gamma}{\gamma^2 + \omega^2 \mu_0 \mu_r \epsilon_r} \frac{dH_y(x', y')}{dy'} - j \frac{\omega \epsilon_r \epsilon_r}{\gamma^2 + \omega^2 \mu_0 \mu_r \epsilon_r} \frac{dE_y(x', y')}{dx'}
\]

\[
E_z(x', y') = -\frac{\gamma}{\gamma^2 + \omega^2 \mu_0 \mu_r \epsilon_r} \frac{dE_z(x', y')}{dx'} - j \frac{\omega \mu_0 \mu_r}{\gamma^2 + \omega^2 \mu_0 \mu_r \epsilon_r} \frac{dH_z(x', y')}{dy'}
\]

\[
E_y(x', y') = -\frac{\gamma}{\gamma^2 + \omega^2 \mu_0 \mu_r \epsilon_r} \frac{dE_y(x', y')}{dy'} + j \frac{\omega \mu_0 \mu_r}{\gamma^2 + \omega^2 \mu_0 \mu_r \epsilon_r} \frac{dH_y(x', y')}{dx'}
\]

Equation 2.6

Boundary conditions at any perfect conductor require that the tangential electric field and the normal magnetic field must be zero at the perfectly conducting walls of the waveguide. The walls of the waveguides are made of conducting material so they are assumed to be a perfect conductor.

Consider in the waveguide only the transverse electric (TE) waves with the condition, \( E_z = 0 \). As a result of the boundary conditions, some conclusion can be made of all the unknown constants.

Assume a waveguide with its aperture in the \( x'y' \)-plane and the origin situated at a corner of the waveguide. The electric field propagates in the \( z \) direction.
Consider the side in the x-direction to be of length a, and that one in the y-direction to have a length of b. That means the boundary conditions are:

\[ E_x(x', y') = 0 \quad \text{when } y' = 0 \text{ and } y' = b \]

\[ E_y(x', y') = 0 \quad \text{when } x' = 0 \text{ and } x' = a \]

Firstly, looking at the electric field in the x-direction of Equation 2.6 and substituting the field in the z-direction, results in:

\[ E_x(x', y') = \frac{\alpha \mu_0 \mu_r}{\gamma^2 + \omega^2 \mu_0 \mu_r \varepsilon_r \varepsilon_r} \left[ C_1 C_3 \cos B x' \cos A y' - C_2 C_4 \cos B x' \cos A y' + C_3 C_5 \sin B x' \sin A y' - C_4 C_6 \sin B x' \cos A y' \right] \]

where the C's are the constants to be determined.

Applying the boundary conditions, so when \( y' = 0 \), then \( E_x \) must vanish leading to \( C_4 = 0 \). When \( y' = b \), again \( E_x = 0 \). In this case though, the \( \sin A y' \) term is common. The sinusoidal function will be zero at any factor of \( \pi \). \( A \) depends on the mode of the wave. Thus,

\[ A = \frac{n \pi}{b} \]

Rewriting the equation for the magnetic field in the z direction yields:

\[ H_z(x', y') = C_1 C_3 \cos B x' \cos \frac{n \pi}{b} y' + C_2 C_4 \sin B x' \cos \frac{n \pi}{b} y' \]

Now applying the other condition that governs the electric field in the y-direction of Equation 2.6 will eliminate another unknown constant. The field is:

\[ E_y(x', y') = \frac{\alpha \mu_0 \mu_r}{\gamma^2 + \omega^2 \mu_0 \mu_r \varepsilon_r \varepsilon_r} \left[ - \frac{C_1 C_3}{B} \sin B x' \cos A y' + \frac{C_2 C_3}{B} \cos B x' \cos A y' \right] \]

Another useful boundary condition is when \( x' = 0 \), where the electric field in the y-direction disappears. This means that \( C_2 = 0 \) since \( C_3 \) cannot be equal to zero.
because it is the coefficient of the first term. In a similar fashion as above, B is found when \( x' = a \) and \( E_y = 0 \).

\[
B = \frac{m\pi}{a}
\]

And thus a result has been obtained for the magnetic field and the remainder of the fields can be determined. The equations are simplified with the introduction of \( E_y' = j\frac{\omega \mu_0 \mu_s \pi}{\gamma^2 + \omega^2 \mu_e \epsilon_e \epsilon_s} C \).

\[
H_z(x', y') = C \cos \frac{m\pi}{a} x' \cos \frac{n\pi}{b} y' = -jE_y' \frac{\gamma^2 + \omega^2 \mu_0 \mu_s \epsilon_s \epsilon_e}{\omega \mu_s \mu_e \epsilon_s} \cos \frac{m\pi}{a} x' \cos \frac{n\pi}{b} y'
\]

\[
H_y(x', y') = \frac{\gamma}{\gamma^2 + \omega^2 \mu_e \epsilon_e} C \frac{m\pi}{a} \sin \frac{m\pi}{a} x' \cos \frac{n\pi}{b} y' = -jE_y' \frac{\gamma}{\omega \mu_e \mu_s} \sin \frac{m\pi}{a} x' \cos \frac{n\pi}{b} y'
\]

\[
H_x(x', y') = \frac{\gamma}{\gamma^2 + \omega^2 \mu_e \epsilon_e} C \frac{n\pi}{a} \cos \frac{m\pi}{a} x' \sin \frac{n\pi}{b} y' = -jE_y' \frac{\gamma}{\omega \mu_e \mu_s} \cos \frac{m\pi}{a} x' \sin \frac{n\pi}{b} y'
\]

\[
E_x(x', y') = -j\frac{\omega \mu_0 \mu_s}{\gamma^2 + \omega^2 \mu_e \epsilon_e} C \frac{n\pi}{b} \cos \frac{m\pi}{a} x' \sin \frac{n\pi}{b} y' = -E_y' \frac{\gamma}{\omega \mu_s \mu_e} \cos \frac{m\pi}{a} x' \sin \frac{n\pi}{b} y'
\]

\[
E_y(x', y') = \frac{\omega \mu_0 \mu_s}{\gamma^2 + \omega^2 \mu_e \epsilon_e} C \frac{m\pi}{a} \sin \frac{m\pi}{a} x' \cos \frac{n\pi}{b} y' = -E_y' \frac{\gamma}{\omega \mu_e \mu_s} \sin \frac{m\pi}{a} x' \cos \frac{n\pi}{b} y'
\]

Equation 2.7

Equation 2.7 gives the transverse electric and magnetic field in the various directions. Now consider the transverse magnetic and electric fields where \( H_z = 0 \). Using exactly the same waveguide refers back to the wave equations, Equation 2.5. Again by manipulation, substitution and simplification the wave equation is obtained in terms of the electric field, as follows:

\[
\frac{d^2 E_y(x', y')}{dx'^2} + \frac{d^2 E_y(x', y')}{dy'^2} + \gamma^2 E_y(x', y') = -\omega^2 \mu_0 \mu_s \epsilon_e \epsilon_s E_z(x', y')
\]

Following similar steps as with the TE, mode a solution can be obtained in the TM mode for the axial electric field \( E_z \):

\[
E_z(x', y') = C_5 C_7 \cos B' x' \cos A' y' + C_5 C_9 \cos B' x' \sin A' y' + C_5 C_5 \sin B' x' \cos A' y' + C_5 C_5 \sin B' x' \sin A' y'
\]

where \( C_5, C_6, C_7, C_8, A' \) and \( B \) are unknown constants
The boundary conditions in this case are as follows:

\[ E_z(x',y') = 0 \quad \text{when } y' = 0 \text{ and } y' = b \]
\[ E_z(x',y') = 0 \quad \text{when } x' = 0 \text{ and } x' = a \]

Firstly, looking at the two conditions where there is no axial electric field at the boundaries \( y' = 0 \) and \( x' = 0 \), the result is:

\[ E_z(0,0) = C_6 C_8 \cos B' x' = 0 \]

The unknown constants \( C_6 \) and \( C_7 \) must be therefore be 0. Thus, the electric field equation becomes:

\[ E_z(x', y') = C_6 C_8 \sin B' x' \sin A' y' \]

Now consider the other two boundary conditions where there is no axial electric field at the boundary where \( x' = a \) and \( y' = b \). The result is:

\[ E_z(a, b) = C_6 C_8 \sin B' a \sin A' b = 0 \]

So, just as it is for the TE mode, it is likewise for the TM mode,

\[ A' = A = \frac{n \pi}{b} \quad \text{and} \quad B' = B = \frac{m \pi}{a} \]

Substituting these constants and \( C' = C_6 C_8 \) into Equation 2.6. The equations are simplified with the introduction of \( E_{yz}' = \frac{j \omega \mu_0 \mu_r}{y^2 + \omega^2 \mu_0 \mu_r \varepsilon_0 \varepsilon_r} C' \).
\[ E_z(x', y') = C \frac{m \pi}{a} x' \sin \frac{n \pi}{b} y' = -j E^0_{sy} \frac{\omega \mu_\epsilon_{\lambda} \tau_\gamma}{\omega_\mu_\epsilon_{\lambda} \tau_\gamma} \left( \gamma^2 + \omega^2 \mu_\epsilon_{\lambda} \tau_\gamma \right) \sin \frac{m \pi}{a} x' \sin \frac{n \pi}{b} y' \]

\[ E_x(x', y') = j \frac{\omega \mu_\epsilon_{\lambda} \tau_\gamma}{\gamma^2 + \omega^2 \mu_\epsilon_{\lambda} \tau_\gamma} C \frac{n \pi}{b} \cos \frac{m \pi}{a} x' \sin \frac{n \pi}{b} y' = E^0_{sy} \frac{n \pi}{b} \cos \frac{m \pi}{a} x' \sin \frac{n \pi}{b} y' \]

\[ E_y(x', y') = -j \frac{\omega \mu_\epsilon_{\lambda} \tau_\gamma}{\gamma^2 + \omega^2 \mu_\epsilon_{\lambda} \tau_\gamma} C \frac{m \pi}{a} \cos \frac{m \pi}{a} x' \cos \frac{n \pi}{b} y' = -E^0_{sy} \frac{n \pi}{b} \sin \frac{m \pi}{a} x' \cos \frac{n \pi}{b} y' \]

\[ H_x(x', y') = \frac{\gamma}{\gamma^2 + \omega^2 \mu_\epsilon_{\lambda} \tau_\gamma} C \frac{n \pi}{a} \cos \frac{m \pi}{a} x' \cos \frac{n \pi}{b} y' = -j E^0_{sy} \frac{\gamma}{\omega \mu_\epsilon_{\lambda} \tau_\gamma} \frac{m \pi}{a} \cos \frac{m \pi}{a} x' \sin \frac{n \pi}{b} y' \]

\[ H_y(x', y') = \frac{\gamma}{\gamma^2 + \omega^2 \mu_\epsilon_{\lambda} \tau_\gamma} C \frac{m \pi}{b} \cos \frac{m \pi}{a} x' \sin \frac{n \pi}{b} y' = -j E^0_{sy} \frac{\gamma}{\omega \mu_\epsilon_{\lambda} \tau_\gamma} \frac{m \pi}{a} \cos \frac{m \pi}{a} x' \sin \frac{n \pi}{b} y' \]

Equation 2.8

**E-plane Horn**

As the name implies, the E-plane horn has the sides flared out in the E-direction. The aperture field of the horn is assumed to be the same as the waveguide with an exception of a phase term. Text books are standardised and consider the origin of the xy-plane in the centre of the aperture plane. Up until now, the origin of the waveguide in this thesis was situated at a corner in the xy'-plane. So in order to convert to standards, the electric field with the origin in the centre of the xy-plane becomes:

\[ E_x(x', y') = -E^0_{sy} \frac{m \pi}{a} \sin \left( \frac{\pi m x'}{a} \right) \cos \left( \frac{\pi n y'}{b} \right) \]

\[ E_y(x', y') = -E^0_{sy} \frac{m \pi}{a} \sin \left( \frac{x - a}{2} \right) \cos \left( \frac{y - a}{2} \right) \]

\[ E_z(x', y') = -E^0_{sy} \frac{m \pi}{a} \sin \left( \frac{\pi m x}{a} - \frac{\pi m y}{b} \right) \cos \left( \frac{\pi m y}{b} - \frac{\pi m y}{b} \right) \]

Equation 2.9

Equation 2.9 is not fully correct - the phase term is missing. Propagating waves travel from the apex of the horn outwards and beyond the aperture. These waves act like Huygen sources with a spherical propagation from the source. The aperture plane is flat and fairly close to the source point. However, examining this aperture area, the waves travelling through the centre have a different phase to those travelling along the edges. Assume the sectorial horn
being viewed from the flat sides, as shown in Figure 2.1 with the origin, O, in the centre of the aperture plane. Again the length of the flared side is a on the aperture and the flared edge length, L. The reference phase is taken as that on the aperture plane at z=0, but the actual phase of the field is determined as if a Huygen's source was radiating from the focus F. In Figure 2.1, on the aperture plane at point O, the field at the centre is leading the field along the edge. Geometrical calculations are implicated in obtaining this phase term.

A ray through the wave front is drawn so the phase difference can be calculated from the extra distance travelled by a particular wave as shown in Figure 2.1. The distance between the aperture plane and the wavefront is AW. In triangle PCO,

\[
OC^2 = CP^2 - OP^2
\]

\[
OC^2 = L^2 - \left(\frac{a}{2}\right)^2
\]

![Figure 2.1 - Side view of the E-plane horn with spherical propagating waves.](image)
In triangle ACO,

\[ AC^2 = OC^2 + AO^2 \]
\[ AC^2 = \left( L^2 - \left( \frac{a}{2} \right)^2 \right) + x^2 \]

The extra distance travelled:

\[ AW = WC - AC \]

To simplify the formula use,

\[ WC + AC = WC + (WC - AW) \]
\[ WC + AC = 2WC - AW \]

Since WC is much greater than AW then,

\[ WC + AC \approx 2WC = 2L \]

So,

\[ (WC - AC)(WC + AC) = WC^2 - AC^2 \]
\[ AW \times 2L \approx L^2 - \left( L^2 - \left( \frac{a}{2} \right)^2 + x^2 \right) \]
\[ AW \times 2L \approx \frac{a^2}{4} - x^2 \]
\[ AW \approx \frac{a^2}{8L} - \frac{x^2}{2L} \]

This difference in distance corresponds to a phase difference, which is the difference in distance dividing by the wavelength and multiplying by \(2\pi\). Thus,

\[ \text{phase difference} \approx \frac{2\pi}{\lambda} \left( \frac{a^2}{8L} - \frac{x^2}{2L} \right) \]
Consider the fields for any particular transverse electric mode, **TE**\(_{mn}\) then the electric field is:

\[
E_x(x, y) = E_{xy}^{0} \frac{n}{b} \sin\left(\frac{m \pi x}{a} - \frac{m \pi}{2}\right) \cos\left(\frac{n \pi y}{b} - \frac{n \pi}{2}\right) e^{\frac{j2\pi}{\lambda} \left(\frac{a^2}{8} \frac{x^2}{y^2}\right) + \frac{j2\pi}{\lambda} \left(\frac{a^2}{8} \frac{y^2}{x^2}\right)}
\]

\[
E_y(x, y) = -E_{xy}^{0} \frac{m}{a} \sin\left(\frac{m \pi x}{a} - \frac{m \pi}{2}\right) \cos\left(\frac{n \pi y}{b} - \frac{n \pi}{2}\right) e^{\frac{j2\pi}{\lambda} \left(\frac{a^2}{8} \frac{x^2}{y^2}\right) + \frac{j2\pi}{\lambda} \left(\frac{a^2}{8} \frac{y^2}{x^2}\right)}
\]

\[
H_x(x, y) = -\frac{j \gamma}{\omega \mu_0 \mu_r} E_{xy}^{0} \frac{n}{b} \cos\left(\frac{m \pi x}{a} - \frac{m \pi}{2}\right) \sin\left(\frac{n \pi y}{b} - \frac{n \pi}{2}\right) e^{\frac{j2\pi}{\lambda} \left(\frac{a^2}{8} \frac{x^2}{y^2}\right) + \frac{j2\pi}{\lambda} \left(\frac{a^2}{8} \frac{y^2}{x^2}\right)}
\]

\[
H_y(x, y) = \frac{j \gamma}{\omega \mu_0 \mu_r} E_{xy}^{0} \frac{m}{a} \cos\left(\frac{m \pi x}{a} - \frac{m \pi}{2}\right) \sin\left(\frac{n \pi y}{b} - \frac{n \pi}{2}\right) e^{\frac{j2\pi}{\lambda} \left(\frac{a^2}{8} \frac{x^2}{y^2}\right) + \frac{j2\pi}{\lambda} \left(\frac{a^2}{8} \frac{y^2}{x^2}\right)}
\]

Equation 2.10

And the fields for any particular transverse magnetic mode, **TM**\(_{mn}\) the electric field is:

\[
E_x(x, y) = C \sin\left(\frac{m \pi x}{a} - \frac{m \pi}{2}\right) \sin\left(\frac{n \pi y}{b} - \frac{n \pi}{2}\right) e^{\frac{j2\pi}{\lambda} \left(\frac{a^2}{8} \frac{x^2}{y^2}\right) + \frac{j2\pi}{\lambda} \left(\frac{a^2}{8} \frac{y^2}{x^2}\right)}
\]

\[
E_y(x, y) = E_{xy}^{0} \frac{n}{b} \cos\left(\frac{m \pi x}{a} - \frac{m \pi}{2}\right) \sin\left(\frac{n \pi y}{b} - \frac{n \pi}{2}\right) e^{\frac{j2\pi}{\lambda} \left(\frac{a^2}{8} \frac{x^2}{y^2}\right) + \frac{j2\pi}{\lambda} \left(\frac{a^2}{8} \frac{y^2}{x^2}\right)}
\]

\[
E_y(x, y) = -E_{xy}^{0} \frac{m}{a} \sin\left(\frac{m \pi x}{a} - \frac{m \pi}{2}\right) \cos\left(\frac{n \pi y}{b} - \frac{n \pi}{2}\right) e^{\frac{j2\pi}{\lambda} \left(\frac{a^2}{8} \frac{x^2}{y^2}\right) + \frac{j2\pi}{\lambda} \left(\frac{a^2}{8} \frac{y^2}{x^2}\right)}
\]

\[
H_y(x, y) = -\frac{j \gamma}{\omega \mu_0 \mu_r} E_{xy}^{0} \frac{n}{b} \cos\left(\frac{m \pi x}{a} - \frac{m \pi}{2}\right) \sin\left(\frac{n \pi y}{b} - \frac{n \pi}{2}\right) e^{\frac{j2\pi}{\lambda} \left(\frac{a^2}{8} \frac{x^2}{y^2}\right) + \frac{j2\pi}{\lambda} \left(\frac{a^2}{8} \frac{y^2}{x^2}\right)}
\]

Equation 2.11

---

**H-plane horn**

The H-plane sectoral plane flares out in the H-direction. With respect to our pyramidal horn, this is in the y-direction. Similarly as for the E-plane horn, the
electrical field can be obtained. The transversal electric mode \( \text{TE}_{mn} \) electric fields are:

\[
E_x(x, y) = E_{xy}^0 \frac{m}{b} \sin \left( \frac{n \pi x}{a} - \frac{n \pi}{2} \right) \cos \left( \frac{m \pi y}{b} - \frac{m \pi}{2} \right) e^{\frac{j 2 \pi (b^2 - y^2)}{\lambda (b^2 + y^2)}} \\
E_y(x, y) = -E_{xy}^0 \frac{n}{a} \sin \left( \frac{n \pi x}{a} - \frac{n \pi}{2} \right) \cos \left( \frac{m \pi y}{b} - \frac{m \pi}{2} \right) e^{\frac{j 2 \pi (b^2 - y^2)}{\lambda (b^2 + y^2)}} \\
H_x(x, y) = -j \frac{\gamma}{\omega \mu_0 \mu_r} E_{xy}^0 \frac{n}{a} \sin \left( \frac{n \pi x}{a} - \frac{n \pi}{2} \right) \cos \left( \frac{m \pi y}{b} - \frac{m \pi}{2} \right) e^{\frac{j 2 \pi (b^2 - y^2)}{\lambda (b^2 + y^2)}} \\
H_y(x, y) = -j \frac{\gamma}{\omega \mu_0 \mu_r} E_{xy}^0 \frac{m}{b} \cos \left( \frac{n \pi x}{a} - \frac{n \pi}{2} \right) \sin \left( \frac{m \pi y}{b} - \frac{m \pi}{2} \right) e^{\frac{j 2 \pi (b^2 - y^2)}{\lambda (b^2 + y^2)}}
\]

Equation 2.12

The transversal magnetic mode \( \text{TM}_{mn} \) electric fields are:

\[
E_x(x, y) = C \sin \left( \frac{n \pi x}{a} - \frac{n \pi}{2} \right) \sin \left( \frac{m \pi y}{b} - \frac{m \pi}{2} \right) e^{\frac{j 2 \pi (b^2 - y^2)}{\lambda (b^2 + y^2)}} \\
E_x(x, y) = E_{xy}^0 \frac{m}{b} \cos \left( \frac{n \pi x}{a} - \frac{n \pi}{2} \right) \sin \left( \frac{m \pi y}{b} - \frac{m \pi}{2} \right) e^{\frac{j 2 \pi (b^2 - y^2)}{\lambda (b^2 + y^2)}} \\
E_y(x, y) = -E_{xy}^0 \frac{n}{a} \sin \left( \frac{n \pi x}{a} - \frac{n \pi}{2} \right) \cos \left( \frac{m \pi y}{b} - \frac{m \pi}{2} \right) e^{\frac{j 2 \pi (b^2 - y^2)}{\lambda (b^2 + y^2)}} \\
H_y(x, y) = -jE_{xy}^0 \frac{n}{\omega \mu_0 \mu_r} \sin \left( \frac{n \pi x}{a} - \frac{n \pi}{2} \right) \cos \left( \frac{m \pi y}{b} - \frac{m \pi}{2} \right) e^{\frac{j 2 \pi (b^2 - y^2)}{\lambda (b^2 + y^2)}} \\
H_y(x, y) = -jE_{xy}^0 \frac{m}{\omega \mu_0 \mu_r} \cos \left( \frac{n \pi x}{a} - \frac{n \pi}{2} \right) \sin \left( \frac{m \pi y}{b} - \frac{m \pi}{2} \right) e^{\frac{j 2 \pi (b^2 - y^2)}{\lambda (b^2 + y^2)}}
\]

Equation 2.13

Note, with respect to the diagonal horn shown in Figure 2.2, the two coefficients of the mode appear as if they have been exchanged.

*Rectangular Horn*

Combine the two \( \text{TE} \) mode sectoral horns (that is Equation 2.10 and Equation 2.12) since the diagonal horn is flared in both directions. The extra phase term needs to be included to all the fields in all directions because all of them follow
this Huygen’s source propagation pattern. So the total electric field of a square aperture (i.e. \(a=b\)) for any mode is:

\[
E_x(x, y) = E_{xy}^0 \left[ \frac{n}{a} \cos \left( \frac{m \pi x}{a} - \frac{m \pi}{2} \right) \sin \left( \frac{n \pi y}{a} - \frac{n \pi}{2} \right) + \frac{m}{a} \cos \left( \frac{n \pi x}{a} - \frac{n \pi}{2} \right) \sin \left( \frac{m \pi y}{a} - \frac{m \pi}{2} \right) \right] e^{j \frac{2 \pi}{\lambda} \left( \frac{x^2}{4L} + \frac{y^2}{4L} \right)}
\]

\[
E_y(x, y) = -E_{xy}^0 \left[ \frac{m}{a} \sin \left( \frac{m \pi x}{a} - \frac{m \pi}{2} \right) \cos \left( \frac{n \pi y}{a} - \frac{n \pi}{2} \right) + \frac{n}{a} \sin \left( \frac{n \pi x}{a} - \frac{n \pi}{2} \right) \cos \left( \frac{m \pi y}{a} - \frac{m \pi}{2} \right) \right] e^{j \frac{2 \pi}{\lambda} \left( \frac{x^2}{4L} + \frac{y^2}{4L} \right)}
\]

\[
H_x(x, y) = -j \frac{\gamma}{\omega \mu_0 \mu} E_{xy}^0 \left[ \frac{m}{a} \sin \left( \frac{m \pi x}{a} - \frac{m \pi}{2} \right) \cos \left( \frac{n \pi y}{a} - \frac{n \pi}{2} \right) + \frac{n}{a} \sin \left( \frac{n \pi x}{a} - \frac{n \pi}{2} \right) \cos \left( \frac{m \pi y}{a} - \frac{m \pi}{2} \right) \right] e^{j \frac{2 \pi}{\lambda} \left( \frac{x^2}{4L} + \frac{y^2}{4L} \right)}
\]

\[
H_y(x, y) = -j \frac{\gamma}{\omega \mu_0 \mu} E_{xy}^0 \left[ \frac{n}{a} \cos \left( \frac{m \pi x}{a} - \frac{m \pi}{2} \right) \sin \left( \frac{n \pi y}{a} - \frac{n \pi}{2} \right) + \frac{m}{a} \cos \left( \frac{n \pi x}{a} - \frac{n \pi}{2} \right) \sin \left( \frac{m \pi y}{a} - \frac{m \pi}{2} \right) \right] e^{j \frac{2 \pi}{\lambda} \left( \frac{x^2}{4L} + \frac{y^2}{4L} \right)}
\]

\[
H_z(x, y) = C \left[ \cos \left( \frac{m \pi x}{a} - \frac{m \pi}{2} \right) \cos \left( \frac{n \pi y}{a} - \frac{n \pi}{2} \right) + \cos \left( \frac{n \pi x}{a} - \frac{n \pi}{2} \right) \cos \left( \frac{m \pi y}{a} - \frac{m \pi}{2} \right) \right] e^{j \frac{2 \pi}{\lambda} \left( \frac{x^2}{4L} + \frac{y^2}{4L} \right)}
\]

Equation 2.14

And the fields for transverse magnetic mode, \(TM_{mn}\) the electric field is:

\[
E_x(x, y) = C' \left[ \sin \left( \frac{m \pi x}{a} - \frac{m \pi}{2} \right) \sin \left( \frac{n \pi y}{a} - \frac{n \pi}{2} \right) + \sin \left( \frac{n \pi x}{a} - \frac{n \pi}{2} \right) \sin \left( \frac{m \pi y}{a} - \frac{m \pi}{2} \right) \right] e^{j \frac{2 \pi}{\lambda} \left( \frac{x^2}{4L} + \frac{y^2}{4L} \right)}
\]
\[ E_x(x, y) = E_{xy}^0 \left[ \frac{n \cos \left( \frac{m \pi x}{a} - \frac{m \pi}{2} \right) \sin \left( \frac{n \pi y}{a} - \frac{n \pi}{2} \right)}{a} + \frac{m \sin \left( \frac{n \pi x}{a} - \frac{n \pi}{2} \right) \cos \left( \frac{m \pi y}{a} - \frac{m \pi}{2} \right)}{a} \right] e^{\frac{j 2 \pi \xi^2 \eta^2 \lambda^2}{8L \lambda 2L 8L 2L}} \]

\[ E_y(x, y) = E_{xy}^0 \left[ \frac{m \sin \left( \frac{m \pi x}{a} - \frac{m \pi}{2} \right) \cos \left( \frac{n \pi y}{a} - \frac{n \pi}{2} \right)}{a} + \frac{n \sin \left( \frac{n \pi x}{a} - \frac{n \pi}{2} \right) \cos \left( \frac{m \pi y}{a} - \frac{m \pi}{2} \right)}{a} \right] e^{\frac{j 2 \pi \xi^2 \eta^2 \lambda^2}{8L \lambda 2L 8L 2L}} \]

\[ H_x(x, y) = -jE_{xy}^0 \gamma \frac{\omega \mu_0 \mu_r}{a} \left[ \frac{m \sin \left( \frac{m \pi x}{a} - \frac{m \pi}{2} \right) \cos \left( \frac{n \pi y}{a} - \frac{n \pi}{2} \right)}{a} + \frac{n \sin \left( \frac{n \pi x}{a} - \frac{n \pi}{2} \right) \cos \left( \frac{m \pi y}{a} - \frac{m \pi}{2} \right)}{a} \right] e^{\frac{j 2 \pi \xi^2 \eta^2 \lambda^2}{8L \lambda 2L 8L 2L}} \]

\[ H_y(x, y) = jE_{xy}^0 \gamma \frac{\omega \mu_0 \mu_r}{a} \left[ \frac{n \cos \left( \frac{m \pi x}{a} - \frac{m \pi}{2} \right) \sin \left( \frac{n \pi y}{a} - \frac{n \pi}{2} \right)}{a} + \frac{m \cos \left( \frac{n \pi x}{a} - \frac{n \pi}{2} \right) \sin \left( \frac{m \pi y}{a} - \frac{m \pi}{2} \right)}{a} \right] e^{\frac{j 2 \pi \xi^2 \eta^2 \lambda^2}{8L \lambda 2L 8L 2L}} \]

Equation 2.15

The dominant mode fields, TE_{10} (that is m = 1 and n = 0) become those shown below. These fields are the same as shown by Whyborn's paper:\[33\].

\[ E_x(x, y) = E_{xy}^0 \left[ 0 + \frac{1}{a} \times 1 \times \sin \left( \frac{n \pi y}{a} - \frac{n \pi}{2} \right) \right] e^{\frac{j 2 \pi \xi^2 \eta^2 \lambda^2}{8L \lambda 2L 8L 2L}} \]

\[ = E_{xy}^0 \left[ \frac{1}{a} \sin \left( \frac{n \pi y}{a} - \frac{n \pi}{2} \right) \right] e^{\frac{j 2 \pi \xi^2 \eta^2 \lambda^2}{8L \lambda 2L 8L 2L}} \]
\[ E_x(x,y) = -E_0^0 \left[ \frac{1}{a} \sin \left( \frac{\pi x}{a} - \frac{\pi}{2} \right) \times 1 + 0 \right] e^{j \frac{2\pi x^2}{\lambda a^2} + j \frac{2\pi y^2}{\lambda a^2}} \]

\[ = -E_0^0 \left[ \frac{1}{a} \sin \left( \frac{\pi x}{a} - \frac{\pi}{2} \right) \right] e^{j \frac{2\pi x^2}{\lambda a^2} + j \frac{2\pi y^2}{\lambda a^2}} \]

\[ H_x(x,y) = -j \frac{\gamma}{\omega \mu_0 \mu_r} E_0^0 \left[ \frac{1}{a} \sin \left( \frac{\pi x}{a} - \frac{\pi}{2} \right) \times 1 + 0 \right] e^{j \frac{2\pi x^2}{\lambda a^2} + j \frac{2\pi y^2}{\lambda a^2}} \]

\[ = -j \frac{\gamma}{\omega \mu_0 \mu_r} E_0^0 \left[ \frac{1}{a} \sin \left( \frac{\pi x}{a} - \frac{\pi}{2} \right) \right] e^{j \frac{2\pi x^2}{\lambda a^2} + j \frac{2\pi y^2}{\lambda a^2}} \]

\[ H_y(x,y) = -j \frac{\gamma}{\omega \mu_0 \mu_r} E_0^0 \left[ 0 + \frac{1}{a} \times 1 \times \sin \left( \frac{\pi y}{a} - \frac{\pi}{2} \right) \right] e^{j \frac{2\pi x^2}{\lambda a^2} + j \frac{2\pi y^2}{\lambda a^2}} \]

\[ = -j \frac{\gamma}{\omega \mu_0 \mu_r} E_0^0 \left[ \frac{1}{a} \sin \left( \frac{\pi y}{a} - \frac{\pi}{2} \right) \right] e^{j \frac{2\pi x^2}{\lambda a^2} + j \frac{2\pi y^2}{\lambda a^2}} \]

\[ H_z(x,y) = C \left[ \cos \left( \frac{\pi x}{a} - \frac{\pi}{2} \right) \times 1 + 1 \times \cos \left( \frac{\pi y}{a} - \frac{\pi}{2} \right) \right] e^{j \frac{2\pi x^2}{\lambda a^2} + j \frac{2\pi y^2}{\lambda a^2}} \]

\[ = C \left[ \cos \left( \frac{\pi x}{a} - \frac{\pi}{2} \right) + \cos \left( \frac{\pi y}{a} - \frac{\pi}{2} \right) \right] e^{j \frac{2\pi x^2}{\lambda a^2} + j \frac{2\pi y^2}{\lambda a^2}} \]

Equation 2.16
**Diagonal Horn**

The principal plane of the diagonal horn refers to the plane where the fields are viewed across the diagonals of the aperture plane. Since the diagonal horn has a square open end, these fields are inclined at 45° to those equated in Equation 2.14 and 2.15. So the following fields are of great interest and are said to be in the principal plane.

\[
E_{\text{normal}}(x, y) = \frac{1}{\sqrt{2}} [E_x(x, y) + E_y(x, y)]
\]

\[
E_{\text{cross-polarized}}(x, y) = \frac{1}{\sqrt{2}} [E_x(x, y) - E_y(x, y)]
\]

\[
H_{\text{normal}}(x, y) = \frac{1}{\sqrt{2}} [H_x(x, y) + H_y(x, y)]
\]

\[
H_{\text{cross-polarized}}(x, y) = \frac{1}{\sqrt{2}} [H_x(x, y) - H_y(x, y)]
\]

Equation 2.17

These equations are the near fields of the diagonal horn. These are required by the preceding sections to calculate the radiation pattern.

![Figure 2.2 - The diagonal horn.](image-url)
Far field (Radiation Pattern)

There are various methods known to calculate the radiation pattern. To date they are classified under these titles: - variational method, conformal mapping method, method of curvilinear coordinates, method of polynomial approximation, cross section method, Fourier Transform method and partial join area method. Here, starting with the diffraction theory and finishing with the Fourier Transform, the radiation pattern will be calculated.

Diffraction theory is applied by using the superposition theorem. This involves firstly, separating the aperture fields into the x-plane, y-plane and the z-plane. The wave fronts of these fields propagate outwards in concentric spheres centred at the source. These fields are perpendicular to the direction of propagation as well as each other. The waves are still transversal so only the tangential parts are necessary. It is far too difficult to develop the radiation pattern directly from the aperture (near) fields. Instead educated estimates (the auxiliary functions) are developed and then proven that they satisfy Maxwell's equations. These auxiliary functions are in terms of current densities so it is necessary prior to the use of the auxiliary functions, that the current densities are determined from the near fields. These current densities are only approximations of the effect of the near field in this area. Firstly, the current densities, \( J(x,y,z,t) \) and \( M(x,y,z,t) \) from both the \( H(x,y,z,t) \) and \( E(x,y,z,t) \) field are calculated. The total far field is the result of vectorially adding the intensity of the far field of each component. Thus the radiation pattern at a particular point is the sum of the contributing fields from all the electric and magnetic sources due to their densities. So the field, \( A(X,Y,Z,x,y,z) \) at the point \( (X, Y, Z) \) due to \( J(x,y,z,t) \) is superimposed with the field \( F(X,Y,Z,x,y,z) \) due to \( M(x,y,z,t) \).

Since the radiation pattern requires many points to be calculated where each point has a field equation containing several integrals, it was an easy option to use Fourier Series and let modern computers do the number crunching.

Current Densities

Due to the aperture fields the current densities, \( J(x,y,z,t) \) and \( M(x,y,z,t) \) will form over the opening of the horn. Practically, it isn’t possible to measure
them. These variables are not zero but an exact formula has not yet been established for the horn situation. If a semi-infinite medium would be placed outside the boundary, one of the current densities could be eliminated and the other can be estimated. This would not give an exact solution because such a system is different to the horn situation. So approximate equivalents of these near fields are used. This method assumes that there are electric or magnetic densities only within the horn opening but zero outside the horn, similar to what was assumed for the boundary conditions.

The approximate densities can be derived by the following equations.

\[
M(x, y, z, t) = -n \times E(x, y, z, t) \\
J(x, y, z, t) = n \times H(x, y, z, t)
\]

Applying this theory to the diagonal horn follows. To avoid the long aperture field equations Equation 2.17 they will be referred to as the electric field, \(E(x, y, z, t)\), and the magnetic field, \(H(x, y, z, t)\).

\[
E(x, y, z, t) = xE_{\text{normal}}(x, y) + yE_{\text{cross polarised}}(x, y)
\]

and the magnetic field, \(H(x, y, z, t)\),

\[
H(x, y, z, t) = xH_{\text{normal}}(x, y) + yH_{\text{cross polarised}}(x, y) + zH_z(x, y).
\]

So now analysing the electric density, one of the above equations can be expressed as a determinant:

\[
M(x, y, z, t) = \begin{vmatrix}
x & y & z \\
E_{\text{normal}}(x, y) & n_x & n_y \\
E_{\text{cross polarised}}(x, y) & 0 & n_z
\end{vmatrix}
= -x(E_{\text{cross polarised}}(x, y) - 0) - y(E_{\text{normal}}(x, y) - 0) + z(E_{\text{normal}}(x, y) - E_{\text{cross polarised}}(x, y))
= xM_x(x, y) + yM_y(x, y) + zM_z(x, y)
\]

Equation 2.18

Similarly analysing the magnetic density, the other equation from above is expressed as a determinant equation:
\[ J(x,y,z,t) = \begin{vmatrix} x & y & z \\ n_x & n_y & n_z \\ H_{\text{norm}}(x,y) & H_{\text{cross-polarized}}(x,y) & H_z(x,y) \end{vmatrix} \]

\[ = x(H_{\text{cross-polarized}}(x,y) - H_z(x,y)) + y(H_z(x,y) - H_{\text{norm}}(x,y)) + z(H_{\text{norm}}(x,y) - H_{\text{cross-polarized}}(x,y)) \]

\[ = xJ_x(x,y) + yJ_y(x,y) + zJ_z(x,y) \]

Equation 2.19

**Auxiliary Functions**

Before looking at the radiating field of large horns, some auxiliary functions are required in the process. First consider the smallest building block, a small Hertzian dipole. Later, all these small dipoles can be placed one after another so the radiation of the entire horn can be determined. Imagine it to be an extremely small dipole so that current is constant throughout the length. The next step is to look at this dipole with length \( dl \), which has a current that produces a current density, \( J(x,y,z,t) \).

In Figure 2.3, the dipole is situated at the origin of the spherical coordinate system. The main problem is to find the field at point, \( P \).

Looking at the situation from point \( P \), it can be seen that the electromagnetic

![Figure 2.3- Dipole in free space causing radiation fields at point P.](image-url)
field depends on the current density distribution by the entire dipole. So,

\[ A(X,Y,Z) = \frac{\mu_0 \mu_r}{4\pi} \int_S J(x,y,z,t) e^{-iKR} \frac{ds}{R} \]

In other words, the vector potential is proportional to the distribution in the volume and inversely proportional to the distance, \( R \). \( R \) refers to the distance between the point and the very small dipole. The point, \( P(X,Y,Z) \) is described by the space coordinate, \( X, Y, Z \) while the dipole uses surface coordinates \( x, y, z \). There is no directional difference between these two coordinates. The lower case coordinates refers to very small increments in near field while the capital letters are greater increments in far field. The distance, \( R \) becomes:

\[ R = \sqrt{(X-x)^2 + (Y-y)^2 + (Z-z)^2} \]

As point \( P \) moves further out less field is expected this is the “radiation” condition. This is proven in many electromagnetic textbooks, using the four Maxwell’s equations (defined in Equations 2.1 to 2.4) and the two definitions relating magnetic and electric field with this vector potential, \( A \). The final relation that is required, relating the potential vector with the potential scalar. All of these result in the wave equation:

\[ \nabla^2 A(X,Y,Z) + k^2 A(X,Y,Z) = -\mu_0 \mu_r J(x,y,z,t) \]

The vector potential, \( A(X,Y,Z) \) is the solution to this wave equation. It is important to note that the potential vector is only an auxiliary function and is used like a tool in this analysis. The electric field is measurable while \( A(X,Y,Z,x,y,z) \) is just a mathematical tool. Thus the vector potential becomes:

\[ A(X,Y,Z,x,y,z) = \frac{\mu_0 \mu_r}{4\pi} \int_S J(x,y,z,t) e^{-iK\sqrt{(X-x)^2 + (Y-y)^2 + (Z-z)^2}} \frac{ds}{\sqrt{(X-x)^2 + (Y-y)^2 + (Z-z)^2}} \]

\[ = \frac{\mu_0 \mu_r}{4\pi} N(X,Y,Z) \]
It was mentioned previously that the far field waves are transversal, meaning the field must be perpendicular to the direction of propagation. In the rectangular coordinate system, this means that there is no field in the z-direction. So the potential vector in the x-direction and the y-direction becomes:

\[
N_x(X, Y, Z) = \int_{\infty}^{\infty} \int_{\infty}^{\infty} J_x(x, y) \frac{e^{-jk\sqrt{(x-x)^2 + (y-y)^2 + (z-z)^2}}}{\sqrt{(X-x)^2 + (Y-y)^2 + (Z-z)^2}} \, dx \, dy
\]

\[
= \sum_{y=-\frac{a}{2}}^{\frac{a}{2}} \sum_{x=-\frac{a}{2}}^{\frac{a}{2}} J_x(x, y) \frac{e^{-jk\sqrt{(x-x)^2 + (y-y)^2 + (z-0)^2}}}{\sqrt{(X-x)^2 + (Y-y)^2 + (Z-0)^2}}
\]

\[
N_y(X, Y, Z) = \int_{\infty}^{\infty} \int_{\infty}^{\infty} J_y(x, y) \frac{e^{-jk\sqrt{(x-x)^2 + (y-y)^2 + (z-z)^2}}}{\sqrt{(X-x)^2 + (Y-y)^2 + (Z-z)^2}} \, dx \, dy
\]

\[
= \sum_{y=-\frac{a}{2}}^{\frac{a}{2}} \sum_{x=-\frac{a}{2}}^{\frac{a}{2}} J_y(x, y) \frac{e^{-jk\sqrt{(x-x)^2 + (y-y)^2 + (z-0)^2}}}{\sqrt{(X-x)^2 + (Y-y)^2 + (Z-0)^2}}
\]

Equation 2.20

In a similar manner, the other auxiliary function for the magnetic field is derived. But in this case, consider a magnetic current source in a homogenous field and again applying Maxwell’s equations and satisfying the conditions J = 0 but M ≠ 0 and \( \nabla \cdot D = 0 \), the result is:

\[
\nabla^2 F(X, Y, Z) + k^2 F(X, Y, Z) = -\varepsilon_e \varepsilon_r M(x, y, z, t)
\]

And the solution to this wave equation similar to before but with different variables is:

\[
F(X, Y, Z) = \frac{\varepsilon_e \varepsilon_r}{4\pi} \int_M(x, y, z, t) \frac{e^{-jk\sqrt{(x-x)^2 + (y-y)^2 + (z-z)^2}}}{\sqrt{(X-x)^2 + (Y-y)^2 + (Z-z)^2}} \, dS = \frac{\varepsilon_e \varepsilon_r}{4\pi} L(X, Y, Z)
\]

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Thus the x-direction and the y-direction becomes:

\[
L_x(X,Y,Z) = \int \int \int M_x(x,y) \frac{e^{-jk \sqrt{(X-x)^2+(Y-y)^2+(Z-z)^2}}}{\sqrt{(X-x)^2+(Y-y)^2+(Z-z)^2}} \, dx \, dy \\
= \sum_{y=-\frac{a}{2}}^{\frac{a}{2}} \sum_{x=-\frac{a}{2}}^{\frac{a}{2}} M_x(x,y) \frac{e^{-jk \sqrt{(X-x)^2+(Y-y)^2+(Z-z)^2}}}{\sqrt{(X-x)^2+(Y-y)^2+(Z-z)^2}}
\]

\[
L_y(X,Y,Z) = \int \int \int M_y(x,y) \frac{e^{-jk \sqrt{(X-x)^2+(Y-y)^2+(Z-z)^2}}}{\sqrt{(X-x)^2+(Y-y)^2+(Z-z)^2}} \, dx \, dy \\
= \sum_{y=-\frac{a}{2}}^{\frac{a}{2}} \sum_{x=-\frac{a}{2}}^{\frac{a}{2}} M_y(x,y) \frac{e^{-jk \sqrt{(X-x)^2+(Y-y)^2+(Z-z)^2}}}{\sqrt{(X-x)^2+(Y-y)^2+(Z-z)^2}}
\]

Equation 2.21

The radiated field from any antenna leaves the antenna as a spherical wave. Thus, these auxiliary functions must be integrated in the Cartesian spherical coordinate system. The above equations are all in the rectangular coordinate system because it is difficult to convert the near fields into spherical coordinate systems and the computation is simplified. These equations are converted to the spherical coordinate system by these equations:

\[
N(\rho, \theta, \phi) = \int \int \int J(x, y, z) \frac{e^{-jkR}}{R} |Jacobian (\rho, \theta, \phi)| \, dz \, dy \, dx
\]

\[
L(\rho, \theta, \phi) = \int \int \int M(x, y, z) \frac{e^{-jkR}}{R} |Jacobian (\rho, \theta, \phi)| \, dz \, dy \, dx
\]

Below are the spherical far fields derived from the rectangular coordinate system.
\[ N_x(\phi, \theta) = \left[ \int_{\infty}^{\infty} J_x(x, y) \cos \theta \cos \phi + J_y(x, y) \cos \theta \sin \phi \right] e^{-jkR} dx \, dy \]
\[ = \left[ \int_{\infty}^{\infty} J_x(x, y) \cos \theta \cos \phi \right] e^{-jkR} dx \, dy + \left[ \int_{\infty}^{\infty} J_y(x, y) \cos \theta \sin \phi \right] e^{-jkR} dx \, dy \]
\[ = \left[ \int_{\infty}^{\infty} J_x(x, y) \frac{e^{-jkR}}{R} dx \, dy \right] \cos \theta \cos \phi + \left[ \int_{\infty}^{\infty} J_y(x, y) \frac{e^{-jkR}}{R} dx \, dy \right] \cos \theta \sin \phi \]
\[ = N_x \cos \theta \cos \phi + N_y \cos \theta \sin \phi \]

\[ N_y(\phi, \theta) = \left[ \int_{\infty}^{\infty} -J_x(x, y) \sin \phi + J_y(x, y) \cos \phi \right] e^{-jkR} dx \, dy \]
\[ = -N_x \sin \phi + N_y \cos \phi \]

\[ L_x(\phi, \theta) = \left[ \int_{\infty}^{\infty} M_x(x, y) \cos \theta \cos \phi + M_y(x, y) \cos \theta \sin \phi \right] e^{-jkR} dx \, dy \]
\[ = L_x \cos \theta \cos \phi + L_y \cos \theta \sin \phi \]

\[ L_y(\phi, \theta) = \left[ \int_{\infty}^{\infty} -M_x(x, y) \sin \phi + M_y(x, y) \cos \phi \right] e^{-jkR} dx \, dy \]
\[ = -L_x \sin \phi + L_y \cos \phi \]

Equation 2.22

The Figure 2.4 shows the difference between the two coordinate systems and how they relate to each other. Note that the same point, P is in a different location in space depending on the coordinate systems. In both systems the origin is found in the centre of the aperture plane. In the rectangular system the field is propagating in the z-direction and this corresponds to the \( \rho \)-direction in the spherical system.

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The distance \( R \), can be further simplified with the assumption that the \( x \) and \( y \) coordinates on the aperture plane are negligible when compared with the \( X \) and \( Y \) plane. Because this assumption was made, the following equations apply to far field radiation. So the simplified distance is:

\[
R = \sqrt{X^2 + Y^2 + (Z - z)^2} = \sqrt{X^2 + Y^2 + Z^2 - 2Zz + z^2} = \sqrt{r^2 - 2rr'\cos \Psi + z^2}
\]

where \( r \) is the distance from the origin to the point and \( \Psi \) is the angle subtended between the plane \( xy \), which contains the origin and the point.

Usually to simplify calculations, only the first term is taken to be \( R \). This is true for magnitude but when looking at the second term for antennas with dimensions of \( r \) greater than one wavelength, the maximum phase error is 22.5°, which is quite significant. Entirely due to this reason, \( R \) has the following values, which are known as the radian approximations.

\[
\text{amplitude} \rightarrow R = r \\
\text{phase} \rightarrow R \approx r - r'\cos \Psi
\]

\( r'\cos \Psi \) is a scalar quantity resulting from the product of two radial position vectors, \( a_R \) and \( a_r \). The space radial position vector, \( a_R \), is preferred in terms of spherical coordinates because waves travel in a spherical fashion. The surface
The radial position vector is in terms of rectangular coordinates since all the near field aperture fields have been derived for the rectangular system. Thus

\[ r \cdot \cos \Psi = a_r \cdot a_R \]

\[ = (x x + y y) \times (x \sin \theta \cos \phi + y \sin \theta \sin \phi + z \cos \theta) \]

\[ = x \sin \theta \cos \phi + y \sin \theta \sin \phi \]

In text books, this is the simplified auxiliary functions that is used:

\[ N_0(\phi, \theta) = \int \int \left[ J_x(x,y) \cos \theta \cos \phi + J_y(x,y) \cos \theta \sin \phi \right] e^{-j kr} \frac{1}{r} \ dx \ dy \]

\[ = \int \int \left[ J_x(x,y) \cos \theta \cos \phi + J_y(x,y) \cos \theta \sin \phi \right] e^{-j kr \cos \Psi} \frac{1}{r} \ dx \ dy \]

\[ = \int \int \left[ J_x(x,y) \cos \theta \cos \phi + J_y(x,y) \cos \theta \sin \phi \right] e^{-j kr \cos \Psi} \ dx \ dy \]

\[ = e^{-j kr} \int \int \left[ J_x(x,y) \cos \theta \cos \phi + J_y(x,y) \cos \theta \sin \phi \right] e^{j kr \cos \Psi} \ dx \ dy \]

\[ N_\phi(\phi, \theta) = \int \int \left[ -J_x(x,y) \sin \phi + J_y(x,y) \cos \phi \right] e^{j kr \sin \theta \cos \phi + y \sin \theta \sin \phi} \ dx \ dy \]

\[ L_0(\phi, \theta) = \int \int \left[ M_x(x,y) \cos \theta \cos \phi + M_y(x,y) \cos \theta \sin \phi \right] e^{j kr \sin \theta \cos \phi + y \sin \theta \sin \phi} \ dx \ dy \]

\[ L_\phi(\phi, \theta) = \int \int \left[ -M_x(x,y) \sin \phi + M_y(x,y) \cos \phi \right] e^{j kr \sin \theta \cos \phi + y \sin \theta \sin \phi} \ dx \ dy \]

Equation 2.23

The advantage of these equations is that only the aperture plane is involved in the integration. The point, P, which is somewhere in the far field, is not part of the integral. So the radiation pattern is totally dependent on the aperture dimensions and aperture fields of the horn.

One very important point to note is that the integrations as they now stand are three-dimensional integrals. But in practical situations this is very difficult to measure and usually only two planes are measured, one in the E plane and the other the H-plane. These are, as quoted in Balanis, "the E-plane pattern is
defined as the plane containing the electric-field vector and the direction of maximum radiation and the H-plane as the plane containing the magnetic field vector and the direction of maximum radiation". Equation 2.23 above can be simplified to two-dimensional integrals either in the E-plane or the H-plane. Both planes are patterns of the electric field in a particular direction.

Generally, only the E-plane and H-plane are of interest. But in this case because the diagonals of the horn play such a great role, two other planes are introduced. These are the D-plane and the G-plane. By substituting the respective angles for the four commonly used planes the integrals are simplified.

The G-plane is where \( \phi = 90^\circ \) cancelling all of the \( \cos \phi \) factors. The simplified equations are:

\[
N_\theta (\theta) = \int \int [J_y (x, y) \cos \theta] e^{j(k(y \sin \theta))} \, dx \, dy
\]

\[
N_\phi (\theta) = \int \int [-J_x (x, y)] e^{j(k(y \sin \theta))} \, dx \, dy
\]

\[
L_\theta (\theta) = \int \int [M_y (x, y) \cos \theta] e^{j(k(y \sin \theta))} \, dx \, dy
\]

\[
L_\phi (\theta) = \int \int [-M_x (x, y) \sin \phi] e^{j(k(y \sin \theta))} \, dx \, dy
\]

Equation 2.24

The D-plane is where \( \phi = 0^\circ \) cancelling all the \( \sin \phi \) factors. The simplified equations are:

\[
N_\theta (\theta) = \int \int [J_x (x, y) \cos \theta] e^{j(k(y \sin \theta))} \, dx \, dy
\]

\[
N_\phi (\theta) = \int \int [J_y (x, y)] e^{j(k(x \sin \theta))} \, dx \, dy
\]

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\[ L_{\theta}(\theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ M_{x}(x,y)\cos\theta \right] e^{jk(x\sin\theta)} \, dx \, dy \]

\[ L_{\phi}(\theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ M_{y}(x,y)\cos\theta \right] e^{jk(x\sin\theta)} \, dx \, dy \]

Equation 2.25

The H-plane is where \( \phi = 45^\circ \), so \( \cos\phi = 1/\sqrt{2} \) and \( \sin\phi = 1/\sqrt{2} \). The simplified equations are:

\[ N_{\theta}(\theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \frac{J_{x}(x,y)\cos\theta}{\sqrt{2}} + \frac{J_{y}(x,y)\cos\theta}{\sqrt{2}} \right] e^{jk(x\sin\theta + y\sin\theta)} \, dx \, dy \]

\[ N_{\phi}(\theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \frac{J_{x}(x,y)}{\sqrt{2}} + \frac{J_{y}(x,y)}{\sqrt{2}} \right] e^{jk(x\sin\theta + y\sin\theta)} \, dx \, dy \]

\[ L_{\theta}(\theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \frac{M_{x}(x,y)\cos\theta}{\sqrt{2}} + \frac{M_{y}(x,y)\cos\theta}{\sqrt{2}} \right] e^{jk(x\sin\theta + y\sin\theta)} \, dx \, dy \]

\[ L_{\phi}(\theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \frac{M_{x}(x,y)}{\sqrt{2}} + \frac{M_{y}(x,y)}{\sqrt{2}} \right] e^{jk(x\sin\theta + y\sin\theta)} \, dx \, dy \]

Equation 2.26

The E-plane is where \( \phi = -45^\circ \), so \( \cos\phi = 1/\sqrt{2} \) and \( \sin\phi = -1/\sqrt{2} \). The simplified equations are:

\[ N_{\theta}(\theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \frac{J_{x}(x,y)\cos\theta}{\sqrt{2}} + \frac{J_{y}(x,y)\cos\theta}{\sqrt{2}} \right] e^{jk(x\sin\theta + y\sin\theta)} \, dx \, dy \]

\[ N_{\phi}(\theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \frac{J_{x}(x,y)}{\sqrt{2}} + \frac{J_{y}(x,y)}{\sqrt{2}} \right] e^{jk(x\sin\theta + y\sin\theta)} \, dx \, dy \]
\[ L_\theta(\theta) = \int \int \left[ \frac{M_x(x,y) \cos \theta - M_y(x,y) \cos \theta}{\sqrt{2}} \right] e^{i \left( \frac{y \sin \theta}{\sqrt{2}} - \frac{x \sin \theta}{\sqrt{2}} \right)} \, dx \, dy \]

\[ L_\phi(\theta) = \int \int \left[ \frac{M_x(x,y)}{\sqrt{2}} + \frac{M_y(x,y)}{\sqrt{2}} \right] e^{i \left( \frac{x \sin \theta}{\sqrt{2}} + \frac{y \sin \theta}{\sqrt{2}} \right)} \, dx \, dy \]

Equation 2.27

When one of the modes is zero, Equation 2.26 and Equation 2.27 reduce to a one dimension integral (see Appendix B).

**Total Field**

Finally, the radiation field is determined by superposition where the fields in the corresponding directions from the sources are added together to give the total field.

Determining E due to \( F(\rho, \phi, \theta, x, y, z) \) is fairly simple and straightforward. Assume the radiating field from a magnetic source in a homogeneous region \( \nabla \cdot \mathbf{D} = 0 \). So,

\[
\mathbf{D} = \nabla \times \frac{1}{4\pi} \int \frac{\mu_0 \mu_r M(x,y,z,t) e^{i k_R}}{R} \, ds = \nabla \times F(\rho, \phi, \theta, x, y, z)
\]

Electric flux density can be expressed in terms of magnetic field.

\[
\mathbf{D} = \varepsilon_r \varepsilon_0 \mathbf{E}_r
\]

\[
\mathbf{E}_r = -\frac{1}{\varepsilon_0 \varepsilon_r} \mathbf{D} = -\frac{1}{\varepsilon_0 \varepsilon_r} \nabla \times F(\rho, \phi, \theta, x, y, z)
\]

\[
= -\frac{1}{\varepsilon_0 \varepsilon_r} \nabla \times \left[ \varepsilon_r \varepsilon_0 e^{-jk_r} \mathbf{L}'(\rho, \phi, \theta) \right]
\]
\[
- \frac{e^{-jr}}{4\pi} \begin{vmatrix}
\rho & \phi & \theta \\
\frac{d}{d\rho} & \frac{d}{d\phi} & \frac{d}{d\theta} \\
0 & L_{\phi}(\phi, \theta) & L_{\theta}(\phi, \theta)
\end{vmatrix}
\]

\[
= -\frac{e^{-jr}}{4\pi} \left[ \rho \left( \frac{d}{d\phi} L_{\phi}(\phi, \theta) - \frac{d}{d\theta} L_{\phi}(\phi, \theta) \right) - \phi \left( \frac{d}{d\rho} L_{\phi}(\phi, \theta) - 0 \right) + \theta \left( \frac{d}{d\rho} L_{\theta}(\phi, \theta) - 0 \right) \right]
\]

Again because the field is transversal then there is no electric flux density in the \( \rho \)-direction. So the simplified electric flux density in the other two directions becomes:

\[
E_{\phi} = -\frac{jke^{-jr}}{4\pi} L_{\phi}(\phi, \theta)
\]

\[
E_{\theta} = \frac{jke^{-jr}}{4\pi} L_{\theta}(\phi, \theta)
\]

Now \( E \) due to \( A(\rho, \phi, \theta, x, y, z) \) needs a bit more detail but follows the same procedure. The electric field is related to the magnetic field and thus it can be determined. So starting from the magnetic flux being expressed as,

\[
B = \nabla \times \frac{1}{4\pi} \sum_{s} \frac{\mu_{0} \mu_{r} J e^{jkr}}{R} ds = \nabla \times A(\rho, \phi, \theta, x, y, z)
\]

Magnetic flux can also be expressed in terms of magnetic field,

\[
B = \mu_{0} \mu_{r} H
\]

Combining the two together,

\[
H = \frac{1}{\mu_{0} \mu_{r}} \nabla \times A(\rho, \phi, \theta, x, y, z)
\]

Substitute in Maxwell’s curl equation,

\[
\nabla \times E_{A} = -j\omega \mu H
\]

\[
\nabla \times E_{A} = -j\omega \mu_{0} \mu_{r} (\nabla \times A(\rho, \phi, \theta, x, y, z))
\]

\[
E_{A} = -j\omega \mu_{0} \mu_{r} A(\rho, \phi, \theta, x, y, z)
\]
\begin{align*}
\mathcal{N} = -j \omega \mu_0 \mu_r \left[ \frac{\mu_0 \mu_r e^{-jkr}}{4\pi r} N^* (\rho, \phi, \theta) \right] \\
= -jkZ_0 \left[ \frac{e^{-jkr}}{4\pi r} N^* (\rho, \phi, \theta) \right]
\end{align*}

Electric field in the separate directions is:

\begin{align*}
E_{\theta} &= -\frac{jke^{-jkr}}{4\pi r} Z_0 N_{\theta} \\
E_{\phi} &= -\frac{jke^{-jkr}}{4\pi r} Z_0 N_{\phi}
\end{align*}

where \( Z_0 \) is the intrinsic impedance

So the total radiating fields are:

\begin{align*}
E_\theta &= E_{r\theta} + E_{A\theta} = -\frac{jke^{-jkr}}{4\pi r} (L_{\phi} + Z_0 N_{\theta}) \\
E_\phi &= E_{r\phi} + E_{A\phi} = \frac{jke^{-jkr}}{4\pi r} (L_{\theta} - Z_0 N_{\phi})
\end{align*}

Equation 2.28

These are the equations that are used in the program to calculate the co-polarised and cross-polarised radiation field.
Chapter 3

RESULTS

The aim of this thesis is to produce a single diagonal horn at 12.75 GHz with the intended use at the Mopra radio-telescope. Using the theory, the ideal dimensions can be determined by using the program predictions of the near field and far field patterns, which can be assessed to ensure the best dimensions for the horn to suit the application. The horn is made from a copper sheet that has been cut to shape and welded together at the joints. The practical radiation patterns were measured using the CSIRO range.

Theoretical

The aim of the theory is to determine the physical lengths of the diagonal horn. A diagonal horn for the Mopra system is designed with the following parameters:

\[ d - \text{diameter of the reflector} - 22\text{m} \]

\[ f - \text{equivalent focal length} - 76.82\text{m} \]

\[ a - \text{beam diameter} - \text{which is to be determined} \]

\[ \lambda - \text{frequency wavelength} - 23\ \text{mm (12.75 GHz)} \]

These parameters are substituted into Equation 1.4

\[ d = 1.220f \frac{\lambda}{a} \]

\[ 22000 = 1.220 \times 76820.5753 \times \frac{23}{\text{horn diameter}} \]

beam width = 10 cm

The diameter of the beam is 10cm. Assuming that the horn aperture must encircle the beamwidth then consider the side of the horn to be equal to the
diameter of the beam, which is 10cm. The length of the flared edge was optimised using the program. The program predictions for a flared edge length of 20cm, 30cm and 40cm are shown on the following pages. The optimised dimension was 35.5cm giving the horn an axis length of 35.1cm. With these dimensions, the horn's flared angle is 16.19 degrees. The horn is connected to a rectangular waveguide. Ideally, the dimensions for a waveguide at this frequency, has the cross sectional dimensions 9x19mm.
*Flared edge length of 20cm*

Figure 3.1 The x-direction aperture electric field in the x-direction. User inputs: 12.75, 100, 200, 1 0, 40, 1, 1, 3, 1.

Figure 3.2 The x-direction aperture electric field in two dimensional view. User inputs: 12.75, 100, 200, 1 0, 40, 1, 5, 3, 1.

Figure 3.3 The y-direction aperture electric field in the x-direction. User inputs: 12.75, 100, 200, 1 0, 40, 2, 1, 3, 1.

Figure 3.4 The y-direction aperture electric field in the two dimensional view. User inputs: 12.75, 100, 200, 1 0, 40, 2, 5, 3, 1.

Figure 3.5 Co-polarisation D-plane radiation pattern. User inputs: 12.75, 100, 200, 1 0, 20, 1, 400, 1, 3, 2.

Figure 3.6 Cross-polarisation D-plane radiation pattern. User inputs: 12.75, 100, 200, 1 0, 20, 1, 400, 1, 3, 2.
Flared edge length of 30cm

Figure 3.7 The x-direction aperture electric field in the x-direction. User inputs: 12.75, 100, 300, 10, 40, 1, 1, 3, 1.

Figure 3.8 The x-direction aperture electric field in two dimensional view. User inputs: 12.75, 100, 300, 10, 40, 1, 5, 3, 1.

Figure 3.9 The y-direction aperture electric field in the x-direction. User inputs: 12.75, 100, 300, 10, 40, 2, 1, 3, 1.

Figure 3.10 The y-direction aperture electric field in the two dimensional view. User inputs: 12.75, 100, 300, 10, 40, 2, 5, 3, 1

Figure 3.11 Co-polarisation D-plane radiation pattern. User inputs: 12.75, 100, 300, 10, 20, 1, 400, 1, 3, 2.

Figure 3.12 Cross-polarisation D-plane radiation pattern. User inputs: 12.75, 100, 300, 10, 20, 1, 400, 1, 3, 2.
Flared edge length of 40cm

Figure 3.13 The x-direction aperture electric field in the x-direction. User inputs: 12.75, 100, 400, 10, 40, 1, 3, 1.

Figure 3.14 The x-direction aperture electric field in two dimensional view. User inputs: 12.75, 100, 400, 10, 40, 1, 5, 3, 1.

Figure 3.15 The y-direction aperture electric field in the x-direction. User inputs: 12.75, 100, 400, 10, 40, 2, 1, 3, 1.

Figure 3.16 The y-direction aperture electric field in the two dimensional view. User inputs: 12.75, 100, 400, 10, 40, 2, 5, 3, 1.

Figure 3.17 Co-polarisation D-plane radiation pattern. User inputs: 12.75, 100, 400, 10, 20, 1, 400, 1, 3, 2.

Figure 3.18 Cross-polarisation D-plane radiation pattern. User inputs: 12.75, 100, 400, 10, 20, 1, 400, 1, 3, 2.
A great deal of effort was placed in generating a two-dimensional far field radiation pattern but unfortunately the diagram that the software produces does not truly resemble a real radiation pattern. The same algorithms are used to draw the one-dimensional graph, however, with the two-dimensional graph it seems to generate null values, quite abruptly next to the main lobes. See Appendix 6.6 for a greater detail about this null area. Many possible issues were eliminated such as directional additions, the FFT/DFT algorithm, possible error in functions, software displaying in Cartesian coordinates instead of spherical coordinates, aperture fields, far field value is truly far field distance and software bugs. As the diagrams currently stand, the nulls encountered are caused by the L and N results of DFT which, when added together, cancel each other out. This issue is left to be resolved by future work.

Practical
The horn was made out of copper sheet metal welded together and machined to get the ideal joint. This horn was then placed in the far field range at CSIRO where radiation pattern measurements were taken.

The range consists of a large rectangular room with walls covered in pyramidal structures so scattering fields can be absorbed. Another rectangular horn was used for transmitting signals to the diagonal horn on a stand that rotates in any possible direction wished. Firstly it is important to choose a reasonable distance between the two horns so that the magnetic near field radiating from the current on the horn is not considered as part of the far field. The ideal distance has to make sure the horns are in the far field regions. Boundary of the far field region is determined from:

\[
\text{distance} = \frac{4(\text{diameter of horn})^2}{\lambda}
\]

\[
= \frac{4(100)^2}{23.6} = 1.69 \text{m}
\]

This means that the horns must have at least 1.69 meters between them. The maximum limiting factor is if the distance is too large, then the scattering field within the range will affect the radiation field. Between these two limits lies a
zone that is frequently referred to as the quiet zone. So an ideal distance can be estimated. With all the following practical measurement the distance used was 2.72 m.

One more thing before any measurements can take place is to zero the phase across most of the main lobe. This is the ideal, but practically, it must be fairly accurate within the 3dB bandwidth. The phase can be adjusted by choosing a phase centre point. This important point lies along the axis of the diagonal horn. What exactly is a phase centre point? Well, it is the point about which the diagonal horn rotates. Fortunately, this can be changed manually by shifting the horn or can be calculated theoretically by an inbuilt program that makes adjustments as the measurement is taking place. During the experimental stage it was found that the second option was very accurate but much easier to implement. So a brief study of phase centring was completed. Five measurements were taken along different phase centres. These graphs are found in Appendix B. The area under the phase curve within 30 degrees was calculated so there is a comparison between the phase error and phase centre distance along the horn. The graphs are shown on the following page.

It is worthwhile mentioning that generally with narrow waveguide diagonal horns of this type, the phase centre tends to be closer to the opening of the horn. During the measurements it was found that the phase centre was approximately at 24cm away from the vertex of the horn. The phase centre is chosen so that there is a minimum variation in phase within the specific angle (i.e. 30 degrees).
The phase graphs seem to be asymmetrical and lean toward one side. Unfortunately, rectifying this problem was only possible after most of the measurements were taken. This is a slight offset in the alignment of the two horns but this didn't affect the magnitude of the radiation significantly.
Measurements were all automatically performed by the computer. The horns needed adjustments depending on the fields measured:

**E-plane co-polarisation** -
both horns have the coaxial connections in the vertical position.

**E-plane cross-polarisation** -
the diagonal horn has the coaxial connection in the vertical position while the rectangular horn has the coaxial connection in the horizontal position.

**H-plane co-polarisation** -
both horns have the coaxial connections in the horizontal position.

**H-plane cross-polarisation** -
the diagonal horn has the coaxial connection in the horizontal position while the rectangular horn has the coaxial connection in the vertical position.

The computer moved the diagonal horn from -90 degrees to +90 degrees taking measurements of the radiation pattern as it moved. At the end, these are plotted so the darker curve is the magnitude of the radiation pattern and the lighter curve is the phase of the radiation. On the following pages only the magnitude of the radiation pattern were measured.
Theory versus Practical

As can be seen, all the co-polarisations closely match the theoretical expectations. The principal E-plane cross-polarisation has a main lobe similar to the program's output but it is slightly asymmetrical. This is believed to be due to the junction not being manufactured with the required accuracy. It seems that more accuracy is required in producing this horn. A preferred method would be to electroform it where the basic shape can be made to any desired accuracy out of aluminium having copper atoms grown on it and finally etching out the aluminium to the desired horn. Another factor that could be improved is the sides to be made thicker so they are more rigid.

On the following pages, the practical and theoretical radiation pattern is shown for the 12.75 GHz horn, which has a side length 10cm and the flared length edge is 35.5cm.
Figure 3.20 The theoretical radiation pattern in the D-plane co-polarisation

Figure 3.21 The practical radiation pattern in the D-plane co-polarisation
Figure 3.22 The theoretical radiation pattern in the D-plane cross-polarisation

Figure 3.23 The practical radiation pattern in the D-plane cross-polarisation

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Figure 3.24 The theoretical radiation pattern in the E-plane co-polarisation

Figure 3.25 The practical radiation pattern in the E-plane co-polarisation
Figure 3.26 The theoretical radiation pattern in the H-plane co-polarisation

Figure 3.27 The practical radiation pattern in the H-plane co-polarisation
Extended Research – Dielectric Plug

In radioastronomy, signals are received from a long distance away. That means the diagonal horn must have high directivity and gain so it can be useful for this application. Loading the horn with a dielectric plug can alter the radiation pattern. This practice may enhance the directivity of the horn at the cost of introducing new losses. The radiation pattern narrows as the velocity of the wave slows down through a dielectric material.

To predicting how the radiation pattern is altered by the dielectric plug, start by looking back at Maxwell’s equations, in Equation 2.4 the variable J has the dimension of current density. Power per volume can be obtained by multiplying J with E which has the dimension of volts per meter. So Equation 2.4 becomes:

\[
\nabla \times H = J + \frac{d\varepsilon E}{dt} \\
J = \nabla \times H - \varepsilon \frac{dE}{dt} \\
J \cdot E = E \cdot \left( \nabla \times H - \varepsilon \frac{dE}{dt} \right) \\
J \cdot E = [E \cdot \nabla \times H] - \varepsilon E \frac{dE}{dt}
\]

Using the vector identity, \( \nabla \cdot A \times B = B \cdot \nabla \times A - [A \cdot \nabla \times B] \):

\[
J \cdot E = [-\nabla \cdot E \times H + H \cdot \nabla \times E] - \varepsilon E \frac{dE}{dt}
\]

And applying Equation 2.2

\[
J \cdot E = -\nabla \cdot E \times H - H \frac{dB}{dt} - \varepsilon E \frac{dE}{dt} \\
J \cdot E = -\nabla \cdot E \times H - \mu H \frac{dH}{dt} - \varepsilon E \frac{dE}{dt}
\]
Integrating over the volume, \( V \):

\[
\int_V J \cdot \mathbf{E} \, dV = -\int_V \nabla \cdot \mathbf{E} \times \mathbf{H} \, dV - \frac{d}{dt} \int_V \left( \mu \mathbf{H}^2 - \varepsilon \mathbf{E}^2 \right) \, dV
\]

So, the instantaneous power dissipated in the volume is equal to the sum of the rate of flow of energy flowing inwards (due to the negative sign) through the surface of the volume and the stored energy per volume of electric and magnetic density. The last term on the right hand side becomes significant since this is the stored energy that decreases with respect to time.

The power flow, which is the rate of energy flow per unit area of the plane wave, is:

\[
\frac{d}{dt} \left( -\frac{d}{dt} \int_V \left( \mu \mathbf{H}^2 - \varepsilon \mathbf{E}^2 \right) \, dV \right) = \frac{d}{dt} \int_A \frac{1}{2} \left( \varepsilon \mathbf{E} \sqrt{\frac{\mu}{\varepsilon} \mathbf{H} + \mu \mathbf{H} \sqrt{\frac{\varepsilon}{\mu} \mathbf{E}}} \right) \, dA = \frac{d}{dt} \int_A \frac{1}{2} \left( \sqrt{\varepsilon \mu} \mathbf{EH} + \sqrt{\varepsilon \mu} \mathbf{EH} \right) \, dA = \frac{d}{dt} \int_A \sqrt{\varepsilon \mu} \mathbf{EH} \, dA
\]

The power for a wave moving with velocity, \( v \) becomes,

\[
\text{Power} = \sqrt{\varepsilon \mu} \mathbf{EH} \times v = \varepsilon \mathbf{EH} = \frac{E^2}{Z_0}
\]

This case represents the power when a wavefront hits the dielectric surface in such a way that the incident wave travels in a perpendicular direction to it. When the incident wave hits at an angle then an extra factor is added so that the incident power is:

\[
\text{Power} = \frac{E^2}{Z_0} \cos \alpha_1
\]

The conservation of energy theory applies in this case at the surface of the dielectric plug. The dielectric material doesn't consume energy so the energy
from the incident wave must either be transmitted (refracted) into the dielectric plug or reflected back. The power distribution can be written as:

\[
\frac{E_i^2}{Z_1} \cos \alpha_1 = \frac{E_r^2}{Z_1} \cos \alpha_1 + \frac{E_i^2}{Z_2} \cos \alpha_2
\]

where \(\alpha_1\) is the incident angle in air and \(\alpha_2\) is the refracted angle in the dielectric plug.

A ratio between the incident wave and the transmitted wave is obtained, to determine how much of the wave was actually received.

\[
\frac{E_i^2}{Z_1} \cos \alpha_1 - \frac{E_r^2}{Z_1} \cos \alpha_1 = \frac{E_i^2}{Z_2} \cos \alpha_2
\]

\[
\frac{E_i^2}{Z_2} \cos \alpha_2 = \left(\frac{E_i^2}{Z_1} - \frac{E_r^2}{Z_1}\right) \cos \alpha_1
\]

\[
E_i^2 = \left(E_i^2 - E_r^2\right) \frac{Z_2}{Z_1} \frac{\cos \alpha_1}{\cos \alpha_2}
\]

\[
\frac{E_i^2}{E_i^2} = \left(1 - \frac{E_r^2}{E_i^2}\right) \frac{Z_2}{Z_1} \frac{\cos \alpha_1}{\cos \alpha_2}
\]

Throughout this thesis, the diagonal horn was only considered for transverse electric modes so the electric vector is parallel to the boundary plane, thus perpendicularly polarised. Along the surface of the dielectric material, consider the incident electric field strength having a positive x-direction. Assume that the reflected and transmitted field strengths have the same directions. Apply the conditions at the dielectric boundary which result in:

\[
E_i + E_r = E_t
\]

Manipulate this relationship until a ratio between the incident and reflected electric field is obtained.

\[
1 + \frac{E_r}{E_i} = \frac{E_t}{E_i}
\]

\[
\frac{E_r}{E_i} = \frac{E_t}{E_i} - 1
\]
Substitute into the transmission to incident ratio formula:

\[
\frac{E_t^2}{E_i^2} = \left(1 - \frac{E_t}{E_i} - 1\right)^2 \frac{Z_2 \cos \alpha_1}{Z_1 \cos \alpha_2}
\]

\[
\left(\frac{E_t}{E_i}\right)^2 = \left(1 - \frac{E_t}{E_i}ight)^2 + 2 \left(\frac{E_t}{E_i}\right) - 1 \frac{Z_2 \cos \alpha_1}{Z_1 \cos \alpha_2}
\]

\[
\frac{E_t}{E_i} \frac{Z_1 \cos \alpha_2 + \cos \alpha_1}{Z_2} = -2 \frac{Z_2 \cos \alpha_1}{Z_1 \cos \alpha_2 + \cos \alpha_1}
\]

\[
E_t \frac{Z_1 \cos \alpha_2 + \cos \alpha_1}{Z_2} = -2 \cos \alpha_1
\]

\[
E_t = \frac{-2 \cos \alpha_1}{\frac{Z_1 \cos \alpha_2 + \cos \alpha_1}{Z_2}}
\]

\[
E_t = \frac{-2 \cos \alpha_1}{\sqrt{\frac{\varepsilon_2}{\varepsilon_1} \left(1 - \sin^2 \alpha_2\right) + \cos \alpha_1}}
\]

\[
E_t = \frac{-2 \cos \alpha_1}{\sqrt{\frac{\varepsilon_2}{\varepsilon_1} \left(1 - \frac{\varepsilon_1}{\varepsilon_2} \sin^2 \alpha_1\right) + \cos \alpha_1}}
\]

\[
E_t = \frac{-2 \cos \alpha_1}{\sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2 \alpha_1 + \cos \alpha_1}}
\]

This equation states that the transmitted wave field reduces with a material that has a higher dielectric constant. This statement is true when the loaded horn is assumed to receive wave signals. But consider when this loaded horn is to transmit - the wave in the dielectric material will travel into air. Interpreting the equation that would mean that the wave has increased its electric field strength as it travels from the dielectric into air. Thus the loaded horn becomes more efficient (that is a higher gain) than just the horn itself.

The wave’s speed depends on the median through which it travels. That means that there is a different wavelength in the aperture field compared to the one in the far field. In the aperture field equations, there is a phase term, which is a
product of the wavelength. Thus, the wave's phase need to be adjusted along the aperture plane.

![Wavefront Diagram](image)

**Figure 3.28 - Wavefronts along a ray at the boundary of two medians.**

The velocity of the waves along the rays varies depending on the median through which it travels. Since in both medians time is constant then the distance the ray travels must vary. Thus there is a relationship between the ratio of the distances and speeds of the different medians. Consider Figure 3.28, the ratios are:

\[
\frac{AB}{DC} = \frac{v_2}{v_1}
\]

As can be seen from the diagram, the distance, AB travelled by the wavefront in the dielectric material is a wavelength, \(\lambda_2\). Similarly, the distance DC is the wavelength travelled in air, \(\lambda_1\). So the equation becomes:

\[
\frac{\lambda_2}{\lambda_1} = \frac{v_2}{v_1}
\]

Since the velocity in a certain media can be expressed by: 
\[
v = \frac{1}{\sqrt{\mu \varepsilon}}
\]

the ratio of wavelengths can be expressed in terms of the dielectric constants is where \(\varepsilon_r = 1\) in air:
\[
\frac{\lambda_2}{\lambda_1} = \frac{1}{\sqrt{\mu \varepsilon_2}} = \frac{1}{\sqrt{\mu \varepsilon_1}} = \frac{1}{\sqrt{\varepsilon_1 \varepsilon_2}}
\]

So the wavelength in the dielectric plug, \( \lambda_2 \), depends on the wavelength in air and the dielectric constant of the plug material. Given that the same wave in dielectric travels slower than in air.

*Geometric Optical Analysis*

Geometric optical analysis means that the waves are assumed to be straight rays where at the intersection between two medians Snell’s Law is applied. This type of analysis can be used where the geometry of the system is many wavelengths in dimensions. This isn’t the case in this situation since the travelling wave is assumed to have a wavefront of a concentric nature travelling in more directions unlike a straight ray. So it is incorrect to analyse the junction of the medians this way.

*Wave Theory*

Analysing this plug using wave theory means that the effects on the radiation pattern are analysed by looking at what happens to the wavefront as it propagates from the aperture plane of the horn to the far field. The aperture fields (Equation 2.17) have been derived from Maxwell’s equations that are based on simple charges. The dielectric plug is placed within the horn thus changing the aperture field because the wavelength has been altered. The radiation pattern is derived from the aperture fields. The wavefronts of these fields propagate outwards in concentric spheres. So the total radiation (Equation 2.28) at a particular point is the sum of all the contributing aperture fields (See Chapter 2 regarding Diffraction Theory). The total radiation pattern consists of Equations 2.21, which are Fourier Transforms of the current
densities (Equation 2.18 and 2.19). This means the relationship between the aperture fields and the radiation pattern is of a Fourier Transform nature. Properties of the Fourier transform can be applied to estimate the effects on the radiation pattern.

As mentioned earlier in this chapter the wave slows down in the dielectric plug. By inserting a plug the time taken for wave to travel through it is scaled by a dielectric constant factor so the time scaling property can be applied.

In the non loaded horn the wavefront will be travelling at a velocity, \( v_1 \), taking time, \( t \). So for this aperture field, \( E_{\text{aperture}} \) the radiation pattern, \( E_{\text{radiation}} \) will be:

\[
E_{\text{aperture}}(t) \leftrightarrow E_{\text{radiation}}(\omega)
\]

But for a loaded horn where the wavefront is slowed down to a velocity of \( \frac{1}{\sqrt{\varepsilon_2}} v_1 \), it takes longer time to travel the same distance. The wave front would take \( \sqrt{\varepsilon_2} t \). So the effect on the radiation pattern is:

\[
E_{\text{aperture}}(\sqrt{\varepsilon_2} t) \leftrightarrow \frac{1}{\sqrt{\varepsilon_2}} E_{\text{radiation}}\left(\frac{\omega}{\sqrt{\varepsilon_2}}\right)
\]

The comparison between these two equations shows that if the wavefront slows down the radiation pattern narrows. Basically, “time expansion implies frequency compression. Conversely, time compression implies frequency expansion.”[64]
Theory versus Practical

A plug was manufactured out of polystyrene, which has a dielectric constant of 2.56. It fits securely within the diagonal horn. So the dimensions of the plug are similar to the diagonal horn. It was much easier for the plug to be produced with a flat aperture surface. However, the theory developed in this section uses simplified equations that are accurate when the wave leaves the surface at 90 degrees. The difference in length along the axis of the plug if the surface was curved, is 4mm. Comparing this to the actual size of the axial length of 35.1 centimetres the difference between the curved or flat surface is quite insignificant but the ease of manufacturing the flat apertured plug was a definite advantage. Also this difference is fairly insignificant when comparing it to wavelength of the frequency of interest, which is 3.6 cm. So in this section to keep the analysis simple, it is assumed that the wave front exits the dielectric surface at 90 degrees implying the surface is curved.

The following pages are comprised of the practical measurements taken with the dielectric plug. Comparing these with the measurements of an unloaded horn (Figures 3.21, 3.23, 3.25, 3.27), it can be seen that the radiation pattern has narrowed slightly - for instance, taking the beamwidth at -3db gain, Figure 3.21 shows a beamwidth of 15 degrees, while Figure 3.29 shows a beamwidth of 14 degrees. So the wavefront theory is accurate for this application.
Figure 3.29: D-plane, co-polarisation radiation pattern of the 12.75 GHz diagonal horn. Compare with non-loaded Figure 3.21.

Figure 3.30: E-plane, co-polarisation radiation pattern of the 12.75 GHz diagonal horn. Compare with non-loaded Figure 3.25.
Figure 3.31 - H-plane, co-polarisation radiation pattern of the 12.75 GHz diagonal horn. Compare with non-loaded Figure 3.27.
CONCLUSION

In determining optimal dimensions for the horn that is suited for its intended application, theoretical building blocks were taken and used to construct a program. The program is able to predict the near field and far field patterns, which can then be assessed to ascertain the best dimensions for the horn.

The 1-dimensional diagrams are fundamentally accurate when comparing the theoretical predictions to the practical measurements. In most practical cases, the side bands consist of introduced noise and as such the theoretical outcomes become slightly higher in value than those in the measured graphs. One discrepancy between cross-polarised theory and practical is that, for the former, a null point is expected at the centre of the horn, while, for the latter, the measurement is off-centre and the null point is not visible. One possibility as to why this occurred may be whilst taking the practical measurements the horn may not have been centred accurately enough. However, for the purposes of this thesis, the precision of the horn and test environment were adequate enough to demonstrate the desired outcomes, albeit slightly off-centre.

However, the predicted 2-dimensional graphs were not as accurate, in comparison. Although a great deal of time was dedicated towards trying to determine the cause of the error, this was unfortunately, not fully achieved. A detailed discussion of the 2-dimensional values, as calculated by the program, is presented in Appendix E and Appendix F, however, further research and investigation is required and is beyond the scope of this thesis.
Appendix

Appendix A - Practical Graphs
Appendix B - Dominant Mode
Appendix C - Program Details
Appendix D - Modes
Appendix E - Two dimensional Radiation
Appendix F - The radiation Pattern value.
Appendix A

This appendix incorporates all the measurements that were taken of the different fields with varying offsets of the phase centre. These were measured using the 12.75GHz diagonal horn with 10 cm aperture side length and a 35.5 cm flared edge length.

D-plane Co-Polarisation Radiation Pattern

Figure A.1 - Phase centre offsetted 0.20 meters from vertex of horn.
Figure A.2 - Phase centre offset 0.22 meters from vertex of horn.

Figure 4.3 - Phase centre offset 0.24 meters from vertex of horn.
Figure A.4 - Phase centre offset 0.26 meters from vertex of horn.

Figure A.5 - Phase centre offset 0.28 meters from vertex of horn.
Figure A.6 - Phase centre offsetted 0.20 meters from vertex of horn.
Figure A.7 - Phase centre offsetted 0.22 meters from vertex of horn.

Figure A.8 - Phase centre offsetted 0.24 meters from vertex of horn.
Figure A.9 - Phase centre offset 0.26 meters from vertex of horn.

Figure A.10 - Phase centre offset 0.28 meters from vertex of horn.
H-plane Co Polarisation Radiation Pattern

Figure A.11 - Phase centre offsetted 0.20 meters from vertex of horn.
Figure A.12 - Phase centre offsetted 0.22 meters from vertex of horn.

Figure A.13 - Phase centre offsetted 0.24 meters from vertex of horn.
Figure A.14 - Phase centre offset 0.26 meters from vertex of horn.

Figure A.15 - Phase centre offset 0.28 meters from vertex of horn.
Appendix B - Special Case modes

The integral equations reduce to one-dimensional integrals when one of the modes is zero. This simplifies the integrals greatly. In most papers and textbooks the dominant mode is used which fits in this class. Actually, only one of the modes has to be zero and the other may be any integer.

Starting from the aperture plane, Equation 2.12 where the assumption that $m=1$ and $n=0$ was made. Then the aperture plane fields become:

$$E_x(x, y) = E_{xy}^0 \left[ \frac{1}{a} \sin \left( \frac{\pi y}{a} - \frac{\pi}{2} \right) \right] e^{j \frac{2\pi}{L} \left[ \frac{a^2 - x^2 - y^2}{2L} \right]}$$

$$E_y(x, y) = -E_{xy}^0 \left[ \frac{1}{a} \cos \left( \frac{\pi x}{a} \right) \right] e^{j \frac{2\pi}{L} \left[ \frac{a^2 - x^2 - y^2}{2L} \right]}$$

$$H_z(x, y) = -j \frac{\gamma}{\omega \mu} E_{xy}^0 \left[ \frac{1}{a} \sin \left( \frac{\pi x}{a} - \frac{\pi}{2} \right) \right] e^{j \frac{2\pi}{L} \left[ \frac{a^2 - x^2 - y^2}{2L} \right]}$$
\[ H_y(x,y) = -j \frac{\gamma}{\omega \mu} E_{xy}^0 \left[ \frac{1}{a} \sin \left( \frac{\pi y}{a} - \frac{\pi}{2} \right) \right] e^{j \frac{2\pi}{\lambda} \left( \frac{a^2 - x^2 - y^2}{2L} \right)} \]

\[ = -j \frac{\gamma}{\omega \mu} E_{xy}^0 \left[ \frac{1}{a} \cos \frac{\pi y}{a} \right] e^{j \frac{2\pi}{\lambda} \left( \frac{a^2 - x^2 - y^2}{2L} \right)} \]

The aperture equations are the same as Equation 2.13 except that the special case equations are to be substituted. The current density equations, Equation 2.18 and 2.19, also remain unchanged. The only other change that is necessary is the simplification of the auxiliary function integrals, Equation 2.22.

So the principle D-plane is where \( \phi = 90 \) cancelling all the cos\( \phi \) factors. The simplified equations become:

\[ N_\phi(\theta) = \int \int J_y(x,y) \cos \theta e^{jk(y \sin \theta)} \, dx \, dy \]

\[ = \int \int [-H_y(x,y) \cos \theta] e^{jk(y \sin \theta)} \, dx \, dy \]

\[ = \int \int \left[ -j \frac{\gamma}{\omega \mu} E_{xy}^0 \frac{m}{a} \cos \frac{m \pi x}{a} e^{j \frac{2\pi}{\lambda} \left( \frac{a^2 - x^2 - y^2}{2L} \right)} \cos \theta \right] e^{jk(y \sin \theta)} \, dx \, dy \]

\[ = -j \frac{\gamma}{\omega \mu} E_{xy}^0 \frac{m}{a} e^{j \frac{2\pi a}{\lambda} \left( \frac{a^2}{4L} \right)} \int \int \left[ \cos \frac{m \pi x}{a} e^{j \frac{2\pi}{\lambda} \left( \frac{a^2 - x^2 - y^2}{2L} \right)} \cos \theta \right] e^{jk(y \sin \theta)} \, dx \, dy \]

\[ = -j \frac{\gamma}{\omega \mu} E_{xy}^0 \frac{m}{a} \int \int \cos \theta e^{jk(y \sin \theta)} \, dy \int \cos \frac{m \pi x}{a} e^{j \frac{m^2}{4L}} \, dx \]

\[ N_\phi(\theta) = \int \int [-J_y(x,y)] e^{jk(y \sin \theta)} \, dx \, dy \]

\[ = \int \int [-H_y(x,y)] e^{jk(y \sin \theta)} \, dx \, dy \]

RADIOASTRONOMY INSTRUMENTATION - THE DIAGONAL HORN -
\[ L_\theta (\theta) = \int_\infty^\infty \int_\infty^\infty \left[ E_y(x, y) \cos \theta \right] e^{jk(y \sin \theta)} \, dx \, dy \]

\[ = \int_\infty^\infty \int_\infty^\infty \left[ \cos \frac{m \pi y}{a} \cdot e^{-j \frac{\rho^2}{\lambda c}a} \right] e^{jk(y \sin \theta)} \, dy \, \int_\infty^\infty e^{-j \frac{\rho^2}{\lambda c} \cos \theta} \, dx \]

\[ L_\phi (\theta) = \int_\infty^\infty \int_\infty^\infty \left[ -M_x(x, y) \right] e^{jk(y \sin \theta)} \, dx \, dy \]

\[ = \int_\infty^\infty \int_\infty^\infty \left[ \cos \frac{m \pi x}{a} \cdot e^{-j \frac{\rho^2}{\lambda c} \cos \theta} \right] \, dy \int_\infty^\infty e^{-j \frac{\rho^2}{\lambda c} \cos \theta} \, dx \]

Equation B.1

The total radiation field in the E-plane becomes (i.e. Substitute into Equation 2.28):

\[ E_\theta = -\frac{je^{-jk r}}{4\pi r} \left( L_\phi (\theta) + Z_0 N_\phi (\theta) \right) \]

\[ = -\frac{je^{-jk r}}{4\pi r} \left( -E_{xy} m e^{j \frac{2 \pi x}{\lambda c} a} \int_\infty^\infty e^{j \frac{\rho^2}{\lambda c} \sin \theta} \, dy \int_\infty^\infty \cos \frac{m \pi x}{a} e^{-j \frac{\rho^2}{\lambda c} \cos \theta} \, dx + \right) \]

\[ Z_0 \left( -j E_{xy} m e^{j \frac{2 \pi x}{\lambda c} a} \int_\infty^\infty e^{-j \frac{\rho^2}{\lambda c} \sin \theta} \, dy \int_\infty^\infty \cos \frac{m \pi x}{a} e^{-j \frac{\rho^2}{\lambda c} \cos \theta} e^{-j \frac{\rho^2}{\lambda c} \cos \theta} \, dx \right) \]

\[ = \frac{je^{-jk r}}{4\pi r} E_{xy} m e^{j \frac{2 \pi x}{\lambda c} a} \int_\infty^\infty e^{-j \frac{\rho^2}{\lambda c} \sin \theta} \, dy \left( \int_\infty^\infty \cos \frac{m \pi x}{a} e^{-j \frac{\rho^2}{\lambda c} \cos \theta} \, dx + j \int_\infty^\infty \cos \frac{m \pi x}{a} e^{-j \frac{\rho^2}{\lambda c} \cos \theta} e^{-j \frac{\rho^2}{\lambda c} \cos \theta} \, dx \right) \]

\[ E_\phi = -\frac{je^{-jk r}}{4\pi r} \left( L_\phi (\theta) - Z_0 N_\phi (\theta) \right) \]

RADIOASTRONOMY INSTRUMENTATION
- THE DIAGONAL HORN -
\[
\frac{-jke^{-j\phi}}{4\pi} \left( E^0_{xy} \frac{m}{a} e^{\frac{2\pi i}{\lambda} \frac{a}{4L}} \int_{-\infty}^{\infty} \cos \frac{m \pi y}{a} e^{-\frac{m^2 \pi^2}{\lambda^2}} \cos \theta e^{j\phi \sin \theta} dy \int_{-\infty}^{\infty} e^{-\frac{\pi x^2}{\lambda^2}} dx - \right) \\
Z_0 \left( jE^0_{xy} \frac{m}{\omega \mu} e^{\frac{2\pi i}{\lambda} \frac{a}{4L}} \int_{-\infty}^{\infty} \cos \frac{m \pi y}{a} e^{-\frac{m^2 \pi^2}{\lambda^2}} e^{j\phi \sin \theta} dy \int_{-\infty}^{\infty} e^{-\frac{\pi x^2}{\lambda^2}} dx \right)
\]

\[
\frac{jke^{-j\phi}}{4\pi} E^0_{xy} \frac{m}{a} e^{\frac{2\pi i}{\lambda} \frac{a}{4L}} \int_{-\infty}^{\infty} e^{-\frac{\pi y^2}{\lambda^2}} dy \int_{-\infty}^{\infty} \cos \frac{m \pi y}{a} e^{-\frac{m^2 \pi^2}{\lambda^2}} dy \left( -\int_{-\infty}^{\infty} \cos \frac{m \pi x}{a} e^{-\frac{m^2 \pi^2}{\lambda^2}} dx + \int_{-\infty}^{\infty} \cos \frac{m \pi x}{a} e^{-\frac{m^2 \pi^2}{\lambda^2}} dx \right)
\]

The H-plane is where \( \phi = 0^\circ \) cancelling all the \( \sin \phi \) factors. The simplified equations are:

\[
N_4(\theta) = \int_{-\infty}^{\infty} \left[ J_x(x,y) \cos \theta \right] e^{jk(x \sin \theta)} dx dy
\]

\[
= \int_{-\infty}^{\infty} \left[ H_x(x,y) \cos \theta \right] e^{jk(x \sin \theta)} dx dy
\]

\[
= j \frac{\gamma}{\omega \mu} E^0_{xy} \frac{m}{a} e^{\frac{2\pi i}{\lambda} \frac{a}{4L}} \int_{-\infty}^{\infty} e^{-\frac{\pi x^2}{\lambda^2}} \cos \theta e^{jk(x \sin \theta)} dx \int_{-\infty}^{\infty} \cos \frac{m \pi y}{a} e^{-\frac{m^2 \pi^2}{\lambda^2}} dy
\]

\[
N_4(\theta) = \int_{-\infty}^{\infty} \left[ J_y(x,y) \right] e^{jk(x \sin \theta)} dx dy
\]

\[
= \int_{-\infty}^{\infty} \left[ - H_y(x,y) \right] e^{jk(x \sin \theta)} dx dy
\]

\[
= -j \frac{\gamma}{\omega \mu} E^0_{xy} \frac{m}{a} e^{\frac{2\pi i}{\lambda} \frac{a}{4L}} \int_{-\infty}^{\infty} \cos \frac{m \pi x}{a} e^{-\frac{m^2 \pi^2}{\lambda^2}} e^{jk(x \sin \theta)} dx \int_{-\infty}^{\infty} e^{-\frac{\pi y^2}{\lambda^2}} dy
\]

\[
L_0(\theta) = \int_{-\infty}^{\infty} \left[ M_x(x,y) \cos \theta \right] e^{jk(x \sin \theta)} dx dy
\]

\[
= \int_{-\infty}^{\infty} \left[ - E_x(x,y) \cos \theta \right] e^{jk(x \sin \theta)} dx dy
\]

\[
= E^0_{xy} \frac{m}{a} e^{\frac{2\pi i}{\lambda} \frac{a}{4L}} \int_{-\infty}^{\infty} \cos \frac{m \pi x}{a} e^{-\frac{m^2 \pi^2}{\lambda^2}} \cos \theta \left[ e^{jk(x \sin \theta)} dx \int_{-\infty}^{\infty} e^{-\frac{\pi y^2}{\lambda^2}} dy \right]
\]

\[
L_4(\theta) = \int_{-\infty}^{\infty} \left[ M_y(x,y) \right] e^{jk(x \sin \theta)} dx dy
\]

\[
= \int_{-\infty}^{\infty} \left[ E_y(x,y) \right] e^{jk(x \sin \theta)} dx dy
\]

RADIOASTRONOMY INSTRUMENTATION
- THE DIAGONAL HORN -
\[ E_{\theta} = -\frac{jk e^{-jkr}}{4\pi\rho} \left( L_{\theta}(\theta) + Z_0 N_\theta(\theta) \right) \]

\[ = -\frac{jk e^{-jkr}}{4\pi\rho} \left[ E_{\phi}^0 m e^{j\frac{2\pi}{\lambda} \frac{x}{4L}} \int_{-\infty}^{\infty} e^{-j\frac{\pi^2 x^2}{\lambda^2}} e^{jksin\theta} dx \int_{-\infty}^{\infty} \cos \frac{m\pi y}{a} e^{-j\frac{\pi^2 y^2}{\lambda^2}} dy + \right. \]

\[ Z_0 (jE_{\phi}^0 \frac{\gamma m e^{j\frac{2\pi}{\lambda} \frac{x}{4L}}}{\omega k} \int_{-\infty}^{\infty} e^{-j\frac{\pi^2 x^2}{\lambda^2}} e^{jksin\theta} dx \int_{-\infty}^{\infty} \cos \frac{m\pi y}{a} e^{-j\frac{\pi^2 y^2}{\lambda^2}} e^{jksin\theta} dy + \left. j \int_{-\infty}^{\infty} \cos \frac{m\pi y}{a} e^{-j\frac{\pi^2 y^2}{\lambda^2}} dy \right] \]

\[ E_{\phi} = -\frac{jk e^{-jkr}}{4\pi\rho} \left( L_{\phi}(\theta) + Z_0 N_\phi(\theta) \right) \]

\[ = -\frac{jk e^{-jkr}}{4\pi\rho} \left[ E_{\phi}^0 m e^{j\frac{2\pi}{\lambda} \frac{x}{4L}} \int_{-\infty}^{\infty} \cos \frac{m\pi x}{a} e^{-j\frac{\pi^2 x^2}{\lambda^2}} \cos \phi e^{jksin\theta} dx \int_{-\infty}^{\infty} e^{-j\frac{\pi^2 x^2}{\lambda^2}} dy - \right. \]

\[ Z_0 (-jE_{\phi}^0 \frac{\gamma m e^{j\frac{2\pi}{\lambda} \frac{x}{4L}}}{\omega k} \int_{-\infty}^{\infty} \cos \frac{m\pi x}{a} e^{-j\frac{\pi^2 x^2}{\lambda^2}} e^{jksin\theta} dx \int_{-\infty}^{\infty} e^{-j\frac{\pi^2 x^2}{\lambda^2}} e^{jksin\theta} dy + \left. j \int_{-\infty}^{\infty} \cos \frac{m\pi x}{a} e^{-j\frac{\pi^2 x^2}{\lambda^2}} dx \right] \]

The D-plane equations cannot be simplified since these are integrated across the diagonal, meaning the integration occurs over the x and y directions.
Appendix C - Program

The program is designed so that it can be used as a tool when developing diagonal horn antennas or arrays. For the present time the program is for the analysis of diagonal horns. Uniquely, this program can be developed further for other types of horns by determining only the fields at the aperture, since all the calculations are developed from first principals as described in great detail throughout this thesis. One disadvantage that is mentioned by many researchers in this field is the time it takes to display a graph. When the computer does all the computing it takes significant time, but hopefully personal computers will be quicker in the years to come. And looking from the other point of view when the research starts from the basics, for every new horn, instead of emphasising the features of the new horn, a great deal of understanding is required in obtaining the correct process to obtain a far field pattern.

Subsequently, due to this reason, this program is divided into two parts, the aperture fields and radiation patterns. A third part, DFT testing was incorporated into the program to audit the DFT process. The first two parts are split into smaller functions, which will be described in this section. The aim of this section is to apply the theory from the other sections into the program.

Aperture Fields

The aperture fields in the program refer to the near field, which can be chosen to be electric or magnetic. Firstly, the program needs some inputs such as the horn details (i.e. frequency, length of the side, length of the flared edge and the mode), the near field details (i.e. aperture points is the number of points along one side of the horn), and the output details (i.e. display the graph along the x or y axis or view the whole aperture field and if the display should be a percentage ratio or a dB ratio or the phase). The display details will not be discussed as thoroughly as the inclusion of the theory.

The near fields can be displayed in two ways - one as a graph, which cuts through the centre of the horn, and the other displays the strength of the fields across the aperture of the horn. Each of these two displays can represent either the field measured in dBs or as a percentage of the field at a particular point.
compared to the maximum field on the aperture plane. The graph is displayed in black and white, which shows the precise values of the field. The field view shows the strengths of the field where white denotes the maximum, and black being the minimum. In both of these features the user is required to choose how many points across the length of the horn (i.e. more points more resolution). There is, however, a limit, the maximum number of points allowable is sixty.

![Figure C.1 - Points on the aperture plane.](image)

As required by the formulae some variables are required from the user. Since this program is uniquely only for diagonal horns, the flared edge length is required as well as the side length at the aperture plane. In this case, the assumption is that the pyramidal structure is of square nature. One more essential variable is the frequency at which the horn is to operate. The two lengths are expressed as millimetres while frequency has the unit GHz.

A two-dimensional array is set up with the origin at the centre. In the x-direction, x starts from -side/2 and ends at side/2. Similarly, with the y-direction. The increments between the points depend on the length of the side of the horn and the number of aperture points along the side. A loop is set up to determine the value for x and the value for y for each individual point. These values and the horn detail values are used in Equation 2.13 to give a complex field value. The last step is to process this near field value so the display shows what the user requires. When a graph is displayed only the point from one of the axis is used. All the points are used for the field view.
The aperture fields have the mode variables (i.e. $m$ and $n$) incorporated into them. Most text books tend to use only $m=1$ and $n=0$ mode but there has been some papers written suggesting that mode coupling effects cause part of the losses in the total radiation pattern. Note that TEM modes do not exist in the diagonal horn since it originates from the rectangular waveguide.

*Radiation pattern*

The radiation pattern refers to the electric field pattern produced by both magnetic and electric field sources. It is well known that the radiation pattern is a Fourier Transform of the aperture pattern. Because this thesis is based on this theory, the aperture field will be transformed to the far field pattern by a Fourier Transform, and thus all the variables which were required for the aperture field will again be required for the radiation pattern (i.e. frequency, length of the side, length of the flared edge and the mode; aperture points is the number of points along one side of the horn; percentage ratio or a dB ratio or the phase). The far field parameters required the far field details (i.e. copolarisation or cross-polarisation and the number of points to view the far field). One variation from the aperture field is, instead of viewing the plane in the x-direction or y-direction, in the far field the options are the E-plane, H-plane, D-plane and G-plane, as well as the two dimensional view. The radiation pattern points are transposed from the Cartesian coordinate system to the spherical coordinate system so the horizontal axis is varying theta. Again, emphasis will be placed on the theory in the program rather than the display functionality.

Prior to the far field radiation pattern, an explanation is due as to why the program performs the radiation pattern as it does. Initially, a Fast Fourier Transform was used to compute the radiation pattern but it was found that 4096x4096 array was required to give a reasonable resolution. That means the memory size for a floating point complex two dimensional array is 2 bytes (float) x2(complex) x4096x4096(two dimensional) = 67108864, approximately 67 Mbytes. This kind of memory size is quite a problem for a personal computer in 1997. At first it was thought that the memory problem could be overcome by using a Discrete Fourier Transform instead of the Fast Fourier Transform process. But it wasn’t the case. Finally, the Direct Fourier
Transform process was found to be the most suitable for this application since the field was calculated for each individual point. Although this process eliminates the memory problem it does take longer for the entire far field to be calculated. This experience has proven what many textbooks mention - that there is a trade off between memory size needed and time. The resolution from the latter process was found to be adequate, since the radiation pattern corresponds to Johansson and Whyborn's paper[33], 'The Diagonal Horn as a Sub-Millimeter Wave Antenna'.

Before any processing takes place all of the user's input variables are required. There is a limit on the number of viewpoints possible, which depends on the aperture points along one side and the frequency of the horn. If the user chooses a graph display, then the program performs a loop, which computes a viewpoint number of far field values along the chosen plane. On the other hand if the user chose a field view display, then the program will perform a loop that computes viewpoint-by-viewpoint number of far field values on the far field plane. In both cases the varying variables start from -view_point/2 and end at view_point/2. Within the loop the values for X, Y and Z are calculated for each individual point and passed to the far field function. The far field function performs a simple addition or subtraction (i.e. Equation 2.28) depending if the far field required is co-polarised or cross-polarised, respectively and calls the functions L_density and N_density. The L_density and N_density functions are similar. They both call the near_field function, then the DFT function, and again the near_field function and finally the DFT function. The difference between them is that the L_density function processes Equation 2.22 by incorporating Equation 2.21 while the N_density function processes Equation 2.22 by incorporating Equation 2.21. Basically, the appropriate near field values are placed in the two dimensional array which is used in the DFT function. The DFT function performs a two-dimensional summation using this array and returns a complex value that is multiplied by a factor and added or subtracted to the far_field_value.
Layout of program
The diagonal horn program layout is shown. It has two main parts - the near field and the far field.

Program Results
Johansson and Whyborn\cite{33} determined the radiation pattern by using a method called the Gaussian-Hermite mode. The radiation pattern shown in this paper is a 99GHz horn with an aperture side of 14mm and the flared edge is 55mm. This example was considered as the ideal example and was used as a reference.
during the debugging process. Again this horn dimensions will be used for comparison of the graphs given by this program and those shown in Johansson and Whyborn’s paper. The first question asked by the program is:

Field to be analysed . . . .

1. Near field

2. Radiation pattern

3. Test DFT

4. Exit

If the user chooses the first option, then the program will require all the details for the near field. See the aperture plane section. If the user chooses the second option, then the program will require all the far field details. See the radiation pattern section. Option three was used to debug the Fourier Transform and is not important when analysing the horn. The fourth option exits the program.

The remainder of this chapter will show all the field displays that are possible from this program.

*Aperture Planes*

At this stage all the near field details will be required by the user. These details are:

Details of horn . . . .

Frequency (GHz):

Side Length (mm):

Flared length (mm):

Mode of horn (m, n):
Points along the side:

Aperture field to view . . . .

1. Electric field in the x-direction
2. Electric field in the y-direction
3. Total electric field
4. Magnetic field in the x-direction
5. Magnetic field in the y-direction
6. Total magnetic field

Display . . . .

1. in x-direction
2. in y-direction
3. 2-dimensional field display

Display the . . . .

1. Real part
2. Imaginary part
3. Magnitude
4. Phase

View . . . .

1. Percentage
2. dB

The results of the program are on the following pages. The values entered into the program to obtain these results have been supplied with each graph.
Electric field in the x-direction - Equation 2.13; $E_{\text{normal}}(x,y)$

Figure C.3 - The x-direction aperture electric field along the y-axis (i.e. in the x-direction). User inputs are: 99, 14, 55, 10, 40, 1, 1, 3, 1.

Figure C.4 - The x-direction aperture electric field along the x-axis (i.e. in the y-direction). User inputs: 99, 14, 55, 10, 40, 1, 2, 3, 1.
Figure C.5 - The x-direction aperture electric field along the x-axis (i.e. in the y-direction). User inputs: 99, 14, 55, 10, 40, 1, 2, 3, 1.

Comment: The x-axis runs horizontally through the middle of this display and the y-axis runs vertically through the middle of this display.
Electric field in the y-direction - Equation 2.13; $E_{\text{cross-polarized}}(x,y)$

Figure C.6 - The y-direction aperture electric field along the y-axis (i.e. in the x-direction). User inputs are: 99, 14, 55, 10, 40, 2, 1, 3, 1

Figure C.7 - The y-direction aperture electric field along the x-axis (i.e. in the y-direction). User inputs are: 99, 14, 55, 10, 40, 2, 2, 3, 1
Figure C.8 - The y-direction aperture electric field in two-dimensional view. User inputs: 99, 14, 55, 10, 40, 2, 5, 3, 1.
Magnetic field in the $x$-direction - Equation 2.13; $H_{\text{normal}}(x, y)$

![Graph](image1)

Figure C.9 - The $x$-direction aperture magnetic field along the $y$-axis (i.e. in the $x$-direction). User inputs are: 99, 14, 55, 10, 40, 4, 1, 3, 1.

![Graph](image2)

Figure C.10 - The $x$-direction aperture magnetic field along the $x$-axis (i.e. in the $y$-direction). User inputs are: 99, 14, 55, 10, 40, 2, 3, 1.
Figure C.11 - The x-direction aperture magnetic field in two-dimensional view. User inputs: 99, 14, 55, 10, 40, 4, 5, 3, 1.
Magnetic field in the y-direction - Equation 2.13; $H_{\text{cross-polarised}}(x,y)$

Figure C.12 - The y-direction aperture magnetic field along the y-axis (i.e. in the x-direction). User inputs are: 99, 14, 55, 10, 40, 5, 1, 3, 1.

Figure C.13 - The y-direction aperture magnetic field along the x-axis (i.e. in the y-direction). User inputs: 99, 14, 55, 10, 40, 5, 2, 3, 1.
Figure C.14 - The y-direction aperture magnetic field in two dimensional view. User inputs: 99, 14, 55, 10, 40, 5, 5, 3, 1.


Radiation Pattern

At this stage all the far field details will be required from the user. These details are:

Details of horn . . . .

Frequency (GHz):

Side Length (mm):

Flared length (mm):

Mode of horn (m, n):

Points along the side:

Radiation pattern to view . . . .

1. Electric field co polarisation

2. Electric field cross polarisation

Number of points . . . .

1. To view radiation pattern:

Display . . . .

1. D-plane 0

2. G-plane 90

3. E-plane 45

4. H-plane -45

5. 2-dimensional field display
Display . . .

1. Real part
2. Imaginary part
3. Magnitude
4. Phase

View . . .

1. Percentage
2. dB

The results of the program are on the following pages. The values entered into the program to obtain these results have been supplied with each graph.
Co-polarisation - $E_0$ in Equation 2.28

![Figure C.15 - Co-polarisation D-plane radiation pattern. User inputs: 99, 14, 55, 10, 20, 1, 500, 1, 3, 2.]

![Figure C.16 - Co-polarisation two-dimensional radiation pattern. User inputs: 99, 14, 55, 10, 20, 1, 500, 5, 3, 2.]

Note: The plane views are different cross-sectional cuts that go through the centre of the two dimensional view. The D-plane is a cut where $\theta = 0^\circ$, that is along the x-axis (horizontal). The E-plane is cut where $\theta = -45^\circ$ (i.e. along one diagonal) and the H-plane is cut $\theta = 45^\circ$ (i.e. the other diagonal).
Figure C.17 - Co-polarisation E-plane radiation pattern. User inputs: 99, 14, 55, 10, 20, 1, 500, 3, 3, 2.

Figure C.18 - Co-polarisation H-plane radiation pattern. User inputs: 99, 14, 55, 10, 20, 500, 4, 3, 2.
Cross-polarised $E_\phi$ in Equation 2.28.

![Cross-polarisation D-plane radiation pattern](image1)

Figure C.19 - Cross-polarisation D-plane radiation pattern. Users inputs: 99, 14, 55, 10, 20, 2, 500, 1, 3, 2.

![Cross-polarisation two dimensional radiation pattern](image2)

Figure C.20 - Cross-polarisation two dimensional radiation pattern. User inputs: 99, 14, 55, 10, 20, 2, 500, 5, 3, 2.

Note: The plane views are different cross-sectional cuts that go through the centre of the two dimensional view. The D-plane is a cut where $\theta = 0^\circ$, that is along the x-axis.
Appendix D - Modes

A unique feature that was built into the program allowed other modes to be viewed. Modes where $m=n$ do not exist in the horn since it is of a square shaped cross section. These other modes do not affect the radiation pattern as much as the dominant mode. However it is interesting to see what kind of far fields are induced. Bird[^3], writes that minor variations in the radiation pattern seem to be caused by mode coupling. The following graphs show the aperture and radiation pattern for two different modes.
Mode $m=0, n=1$

Figure D.1 - The x-direction aperture electric field.

Figure D.2 - The y-direction aperture electric field.

Figure D.3 - The co-polarisation radiation pattern in the D-plane.
Mode $m=2, n=0$

Figure D.4 - The x-direction aperture electric field.

Figure D.5 - The y-direction aperture electric field.

Figure D.3 - The co-polarisation radiation pattern in the D-plane
Appendix E - Two dimensional Radiation Pattern

In this Appendix, there are step-by-step details of how the program produces two-dimensional radiation patterns with reference to the equations used.

Aperture fields

The radiation pattern is obtained by calculating the aperture field across an area for which the user specifies the size. The diagonal horn variables are as stated in Appendix C for the 12.75 GHz horn. For simplicity each side has eight points. The software, if required, increments the number of points so it is an odd value so diagrams can be drawn easily. Thus the aperture area has 9 x 9 points where each point is real and imaginary. The aperture diagrams shown in Chapter 6 are of the magnitude value of each point.

The following tables of values are those given by Equation 2.13 giving the following aperture field values:
### Electric field in the x-direction.

<table>
<thead>
<tr>
<th>0+j0</th>
<th>-2.4813 +j0.0838</th>
<th>-3.8082 +j3.2465</th>
<th>-4.1549 +j5.0505</th>
<th>-4.1684 +j5.7228</th>
<th>-4.1560 +j5.0531</th>
<th>-3.8129 +j3.2506</th>
<th>-2.4888 +j1.0871</th>
<th>-0.0089 +j0</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.4813 +j1.0838</td>
<td>0+j0</td>
<td>-1.0052 +j0.0048</td>
<td>-1.0463 +j3.6863</td>
<td>-0.9444 +j4.6491</td>
<td>-1.0472 +j3.6896</td>
<td>-1.0079 +j2.0705</td>
<td>-0.0056 +j0.0061</td>
<td>2.4731 +j0.0802</td>
</tr>
<tr>
<td>3.8082 +j3.2465</td>
<td>1.0052 +j0.0648</td>
<td>0+j0</td>
<td>0.0270 +j1.5352</td>
<td>0.1585 +j0.0271</td>
<td>-0.0010 +j0.0007</td>
<td>1.0016 +j0.0356</td>
<td>3.8014 +j3.24007</td>
<td>0+j0</td>
</tr>
<tr>
<td>4.1544 +j5.0505</td>
<td>1.0463 +j3.6863</td>
<td>-0.0270 +j1.5352</td>
<td>0.1352 +j0.5232</td>
<td>0.0007 +j0.0007</td>
<td>-0.0269 +j0.0243</td>
<td>1.0439 +j3.6873</td>
<td>4.1487 +j5.0436</td>
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<td>-0.1580 +j2.0634</td>
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<td>4.1632 +j5.7156</td>
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</tr>
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<td>1.0472 +j3.6896</td>
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<td>0.1343 +j0.5199</td>
<td>0.0269 +j0.0034</td>
<td>1.0449 +j3.6816</td>
<td>4.1509 +j5.0462</td>
<td>0+j0</td>
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</tr>
<tr>
<td>3.8129 +j3.2506</td>
<td>1.0079 +j0.0705</td>
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<td>3.8062 +j3.2448</td>
<td>0+j0</td>
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<td>2.488 +j1.0871</td>
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<td>0.1378 +j4.4260</td>
<td>-1.0439 +j0.5322</td>
<td>-1.0043 +j0.063</td>
<td>2.4888 +j1.0871</td>
<td>0+j0</td>
<td></td>
</tr>
<tr>
<td>0.0089 +j0.0802</td>
<td>-2.4731 +j3.2407</td>
<td>-3.8014 +j5.0435</td>
<td>-4.1487 +j5.7155</td>
<td>-4.1631 +j5.0462</td>
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<td>-3.8061 +j1.0835</td>
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</tbody>
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<th>3.7946 +j5.0367</th>
<th>4.1430 +j5.7084</th>
<th>4.1579 +j3.0393</th>
<th>4.1452 +j3.2390</th>
<th>3.7994 +j1.0799</th>
<th>2.4725 +j0.0890</th>
<th>-0.0089 +j0</th>
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</thead>
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<td>2.4731 +j1.0802</td>
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<td>4.1508 +j5.0462</td>
<td>3.8061 +j3.2448</td>
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<td>0+j0</td>
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</tbody>
</table>
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<th>0.0086-</th>
<th>0.0134-</th>
<th>0.0151-</th>
<th>0.0134-</th>
<th>0.0086-</th>
<th>0.0029-</th>
<th>0+ j0</th>
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</thead>
<tbody>
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<td>j0.0110</td>
<td>j0.0110</td>
<td>j0.0110</td>
<td>j0.0110</td>
<td>j0.0101</td>
<td>j0.0065</td>
</tr>
<tr>
<td>0.0029-</td>
<td>0.0105-</td>
<td>0.0183-</td>
<td>0.0236-</td>
<td>0.0253-</td>
<td>0.0236-</td>
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<tr>
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<td>0.0132-</td>
<td>0.0262-</td>
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<td>0.0340+</td>
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<td>0.0305+</td>
<td>0.0319+</td>
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<td>j0.0042</td>
<td>j0.0005</td>
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<td>j0.0005</td>
<td>j0.0042</td>
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<td>0.0151-</td>
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<td>j0.0065</td>
<td></td>
</tr>
</tbody>
</table>

### Magnetic field in the y-direction.

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<th>0.0086-</th>
<th>0.0134-</th>
<th>0.0152-</th>
<th>0.0134-</th>
<th>0.0086-</th>
<th>0.0028-</th>
<th>0+ j0</th>
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</thead>
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<td>j0.0110</td>
<td>j0.0110</td>
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<td>j0.0110</td>
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<td>0.0098-</td>
<td>0.0055-</td>
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<td>0.0055-</td>
<td>0.0058-</td>
<td>0.0098-</td>
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<td>-0.0132-</td>
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</table>
Appendix F - The Radiation Pattern Value

For every radiation pattern plotted the electric field value is calculated by adding or subtracting depending if co- or cross-polarisation is required. The variables used in this process are the results of specific DFT across the aperture of area of a specific aperture field with respect to each individual far field radiation pattern point. The aperture fields are given in the Cartesian coordinate system so that the DFT’s were performed in the Cartesian system, followed by a conversion to the spherical coordinate system, so that the radiation pattern will be in terms of the theta and phi directions. Here are the values obtained from a point with an expected value and a point with an expected value causing the null areas in the radiation pattern.

<table>
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<tr>
<th>Variable</th>
<th>Expected point</th>
<th>Unexpected point</th>
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<td>Nx - Equation 2.20</td>
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<td>-0.04563-j0.0203</td>
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<td>Ny - Equation 2.20</td>
<td>0.3236+j0.3059</td>
<td>0.0463+j0.0372</td>
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<tr>
<td>N - Equation 2.22</td>
<td>0.3118+j0.3006</td>
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<td>Lx - Equation 2.21</td>
<td>-113.6274+j120.1961</td>
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<td>Ly - Equation 2.21</td>
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<td>E - Equation 2.28</td>
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</table>

As can be seen, the unexpected points have a very small magnitude when compared to the expected value. Many aspects have been checked to obtain a realistic radiation pattern. Some issues have been eliminated such as the far field distance chosen is far enough from the aperture surface, vectorial addition, DFT processing properly and function correctness. Unfortunately, the inconsistency in the actual radiation pattern and that predicted by the software has not been fully pinpointed but it is suspected that it is caused by the values resulting from the DFT. This issue is left for future research.
REFERENCES


